

# Universal augmentation schemes for network navigability

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## Overcoming the $\sqrt{n}$ - barrier

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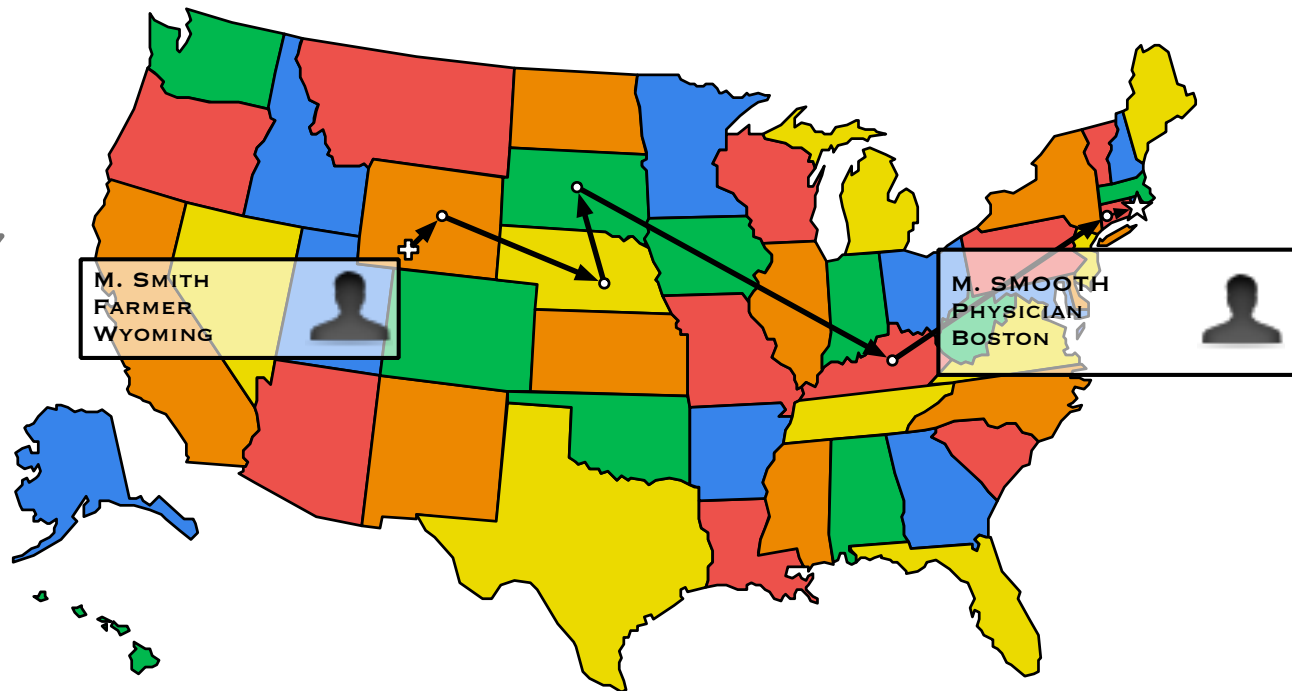
Adrian KOSOWSKI (U. Gdansk, Poland)

Emmanuelle LEBHAR (CNRS & U. Paris 7, France)

Zvi LOTKER (U. Ben Gurion, Israël)

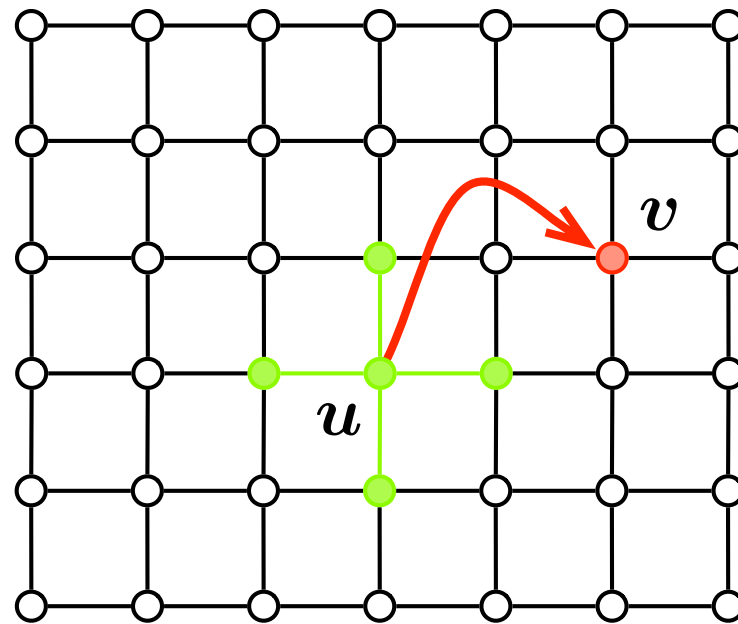
# Network navigability: the small world effect

Milgram 1967



- Very short paths exist.
- People are able to discover them locally.

# Kleinberg model (2000)



$$\Pr(u \rightarrow v) \propto 1/|u-v|^2$$

- Mesh: global geographical knowledge
- Red random arcs: local and private knowledge

# Navigability in Kleinberg model

A routing algorithm is claimed **decentralized** if:

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➔ Greedy routing computes paths of expected length  $O(\log^2 n)$  between any pair in this model.

# Augmented graphs $f(n)$ -navigability

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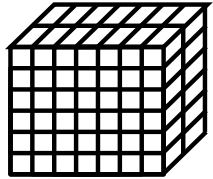
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## Problem:

- A graph  $G$  + one random link/node
  - Which graph and which distribution s.t. greedy routing computes paths of length  $f(n)$ ?
- ➡ Ex:  $d$ -dimensional meshes are  $O(\log^2 n)$ -navigable (with  $d$ -harmonic distribution of links).

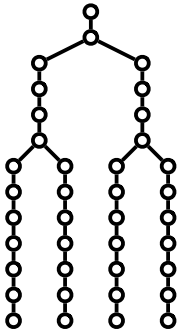
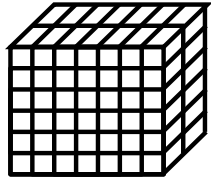
# Polylog(n)-navigability



- Bounded growth graphs [Duchon et al. 05]

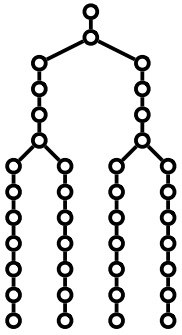
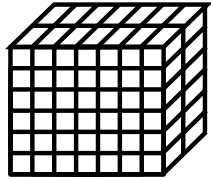


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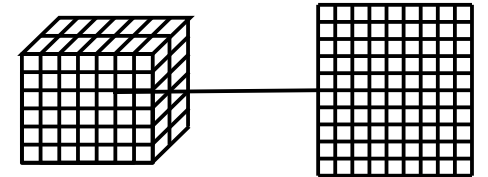


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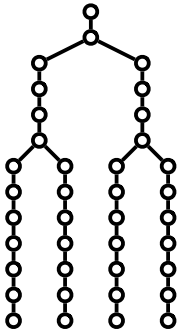
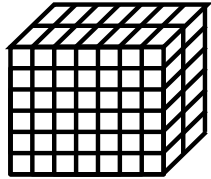
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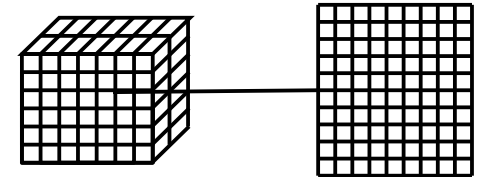
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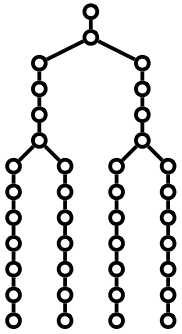
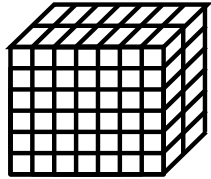
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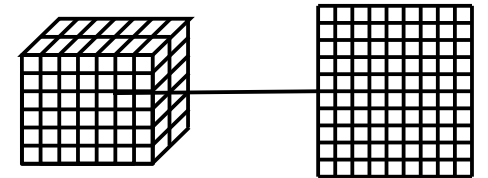
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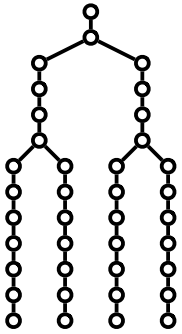
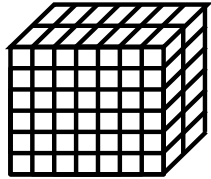


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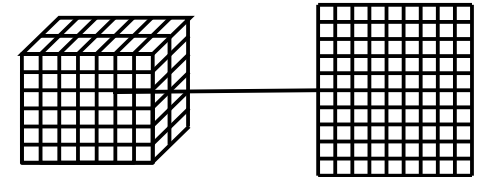


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BUT : not all graphs can be augmented.

For some graphs, greedy paths are of length at least  $\Omega(n^{1/\sqrt{\log n}})$  for any augmentation.

[Fraigniaud et al. 06]

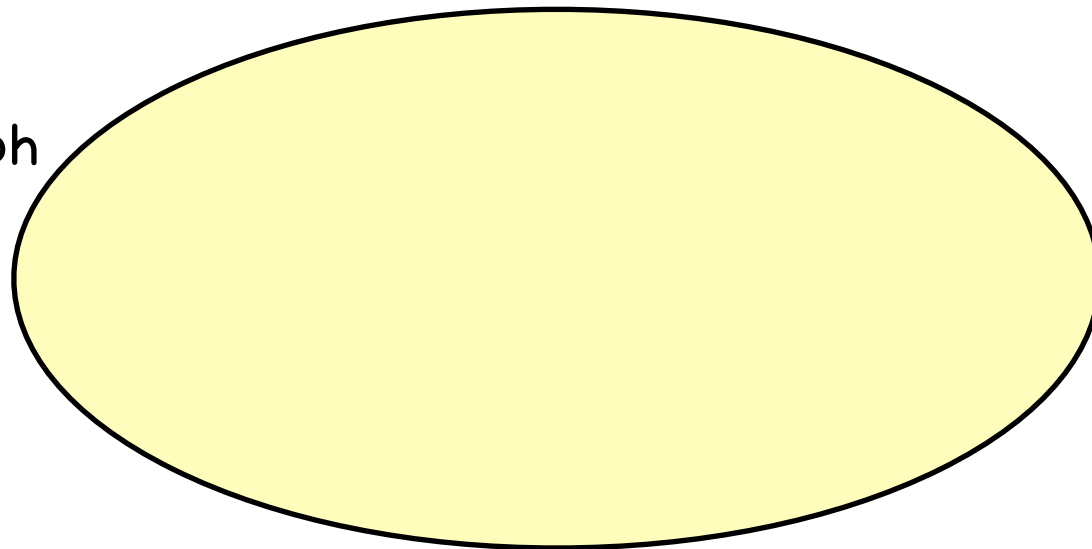
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- Lower bound:  $\Omega(n^{1/\sqrt{\log n}})$ .
- Upper bound:  $O(n^{1/2})$  with uniform augmentation.

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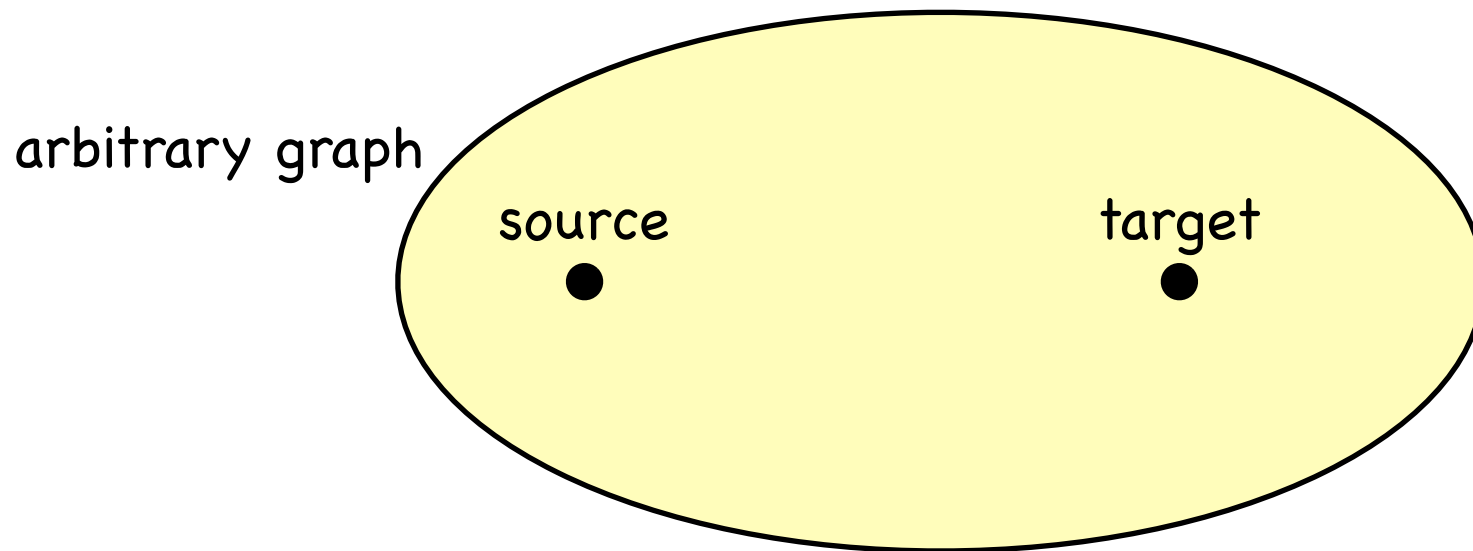
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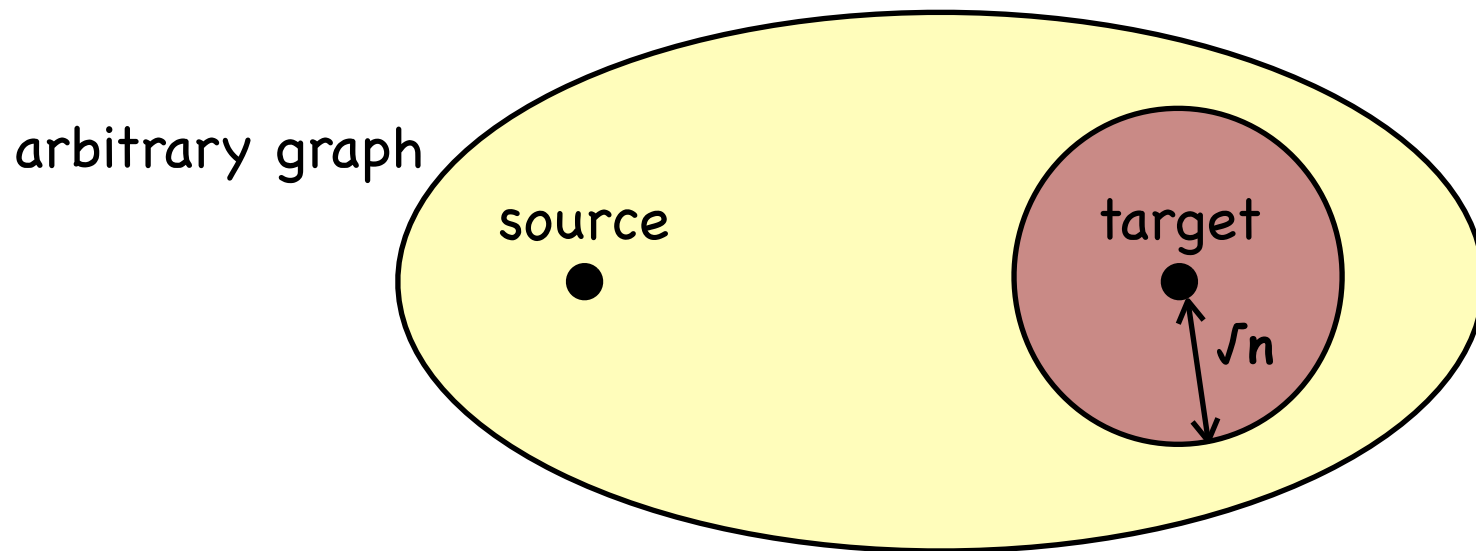
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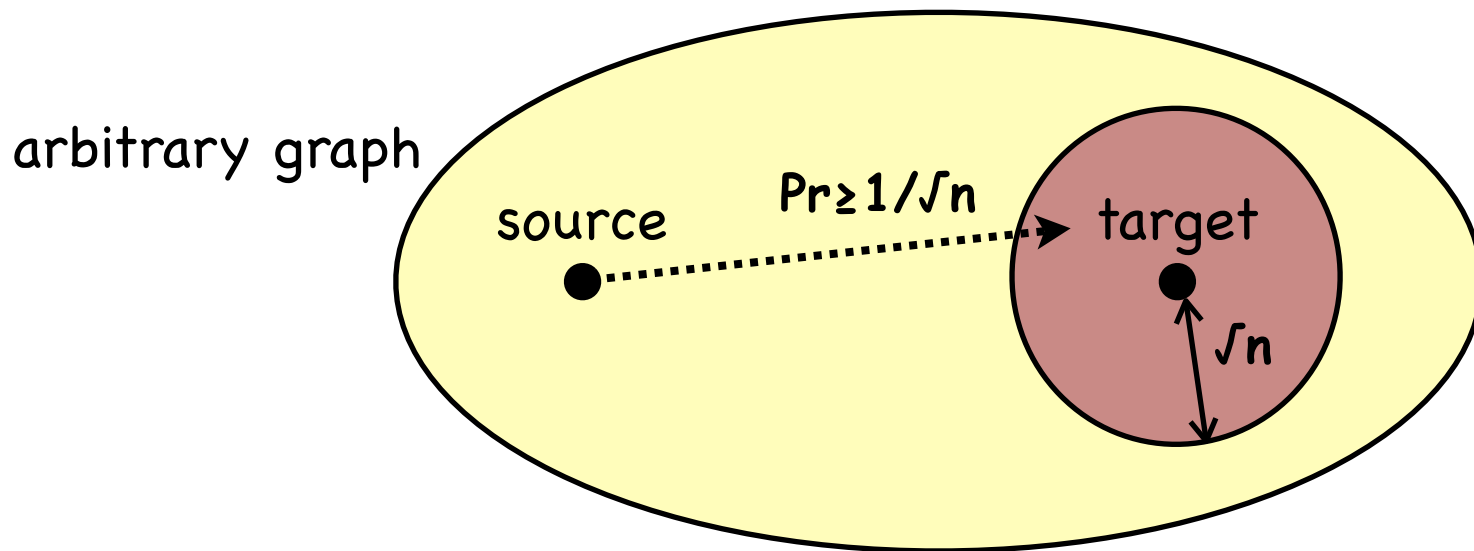
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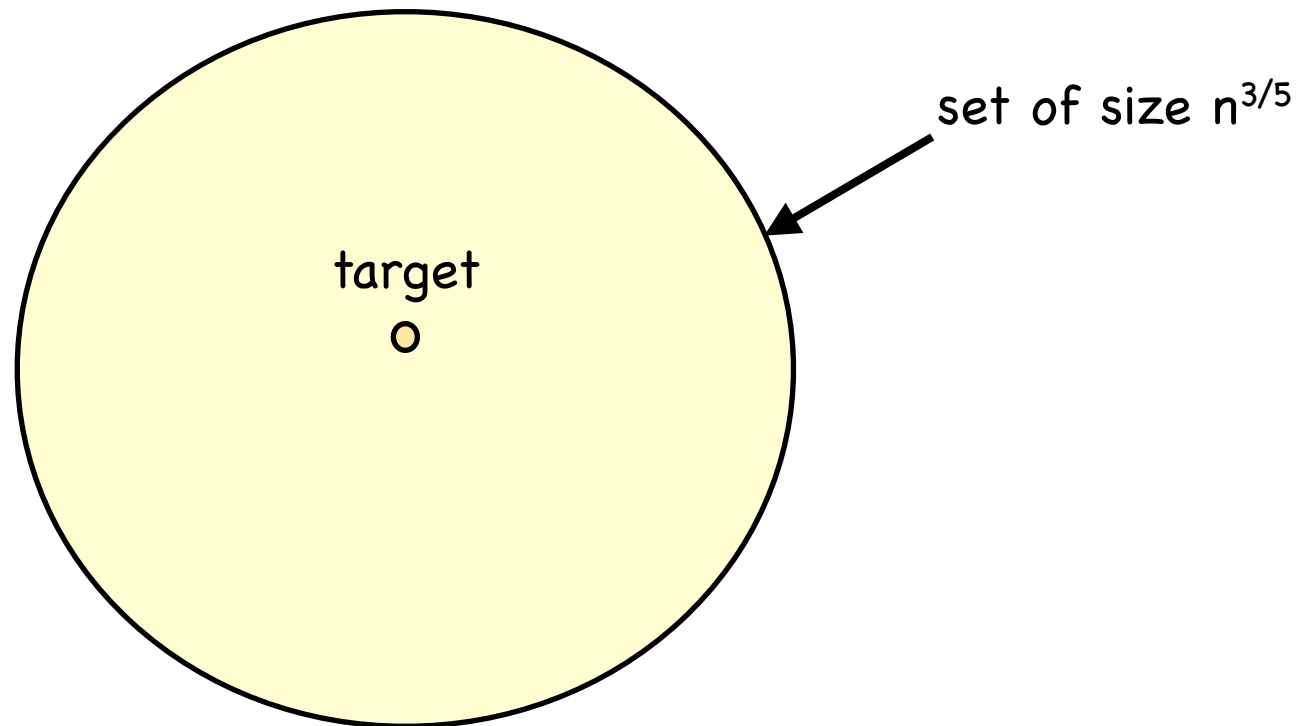
- Augmentation process:

1. Node  $u$  picks a level  $k$  in  $0 \dots \log n$  (u.a.r.)
2. Node  $u$  picks a node  $v$  in  $B(u, 2^k)$  (u.a.r.)

# Proof idea: $O(n^{2/5})$

Ex:

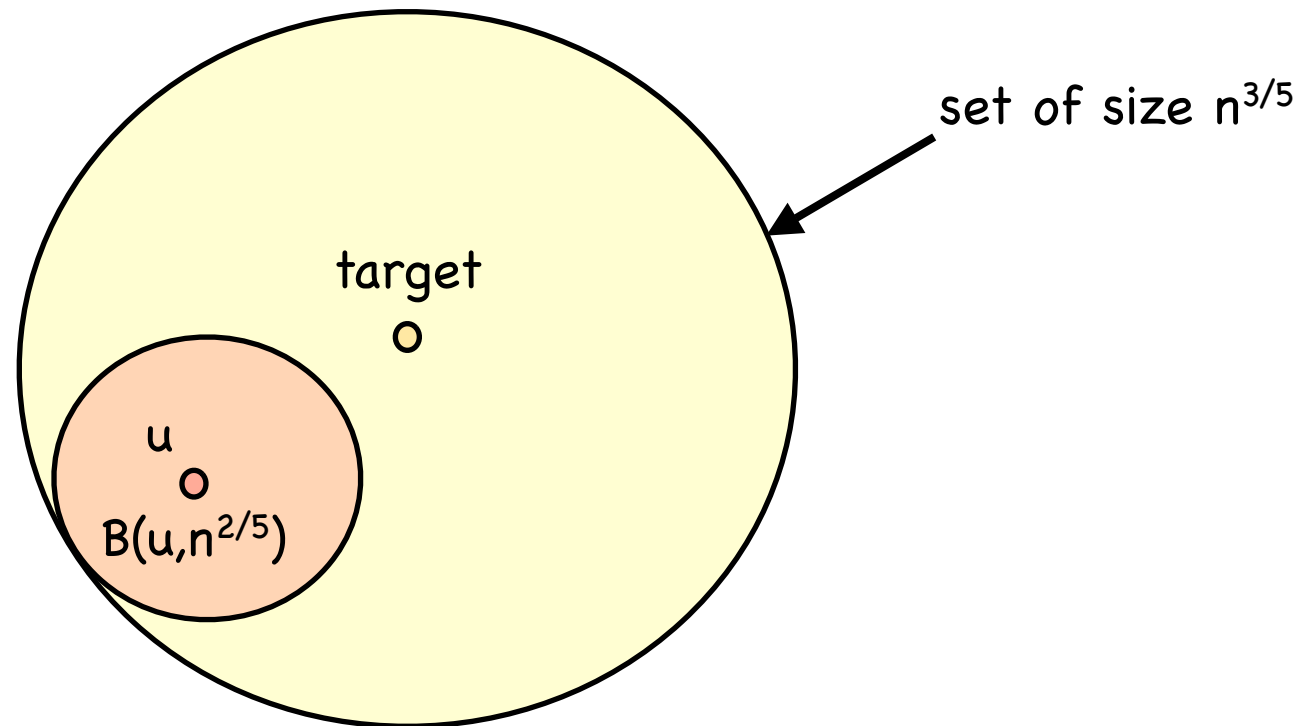
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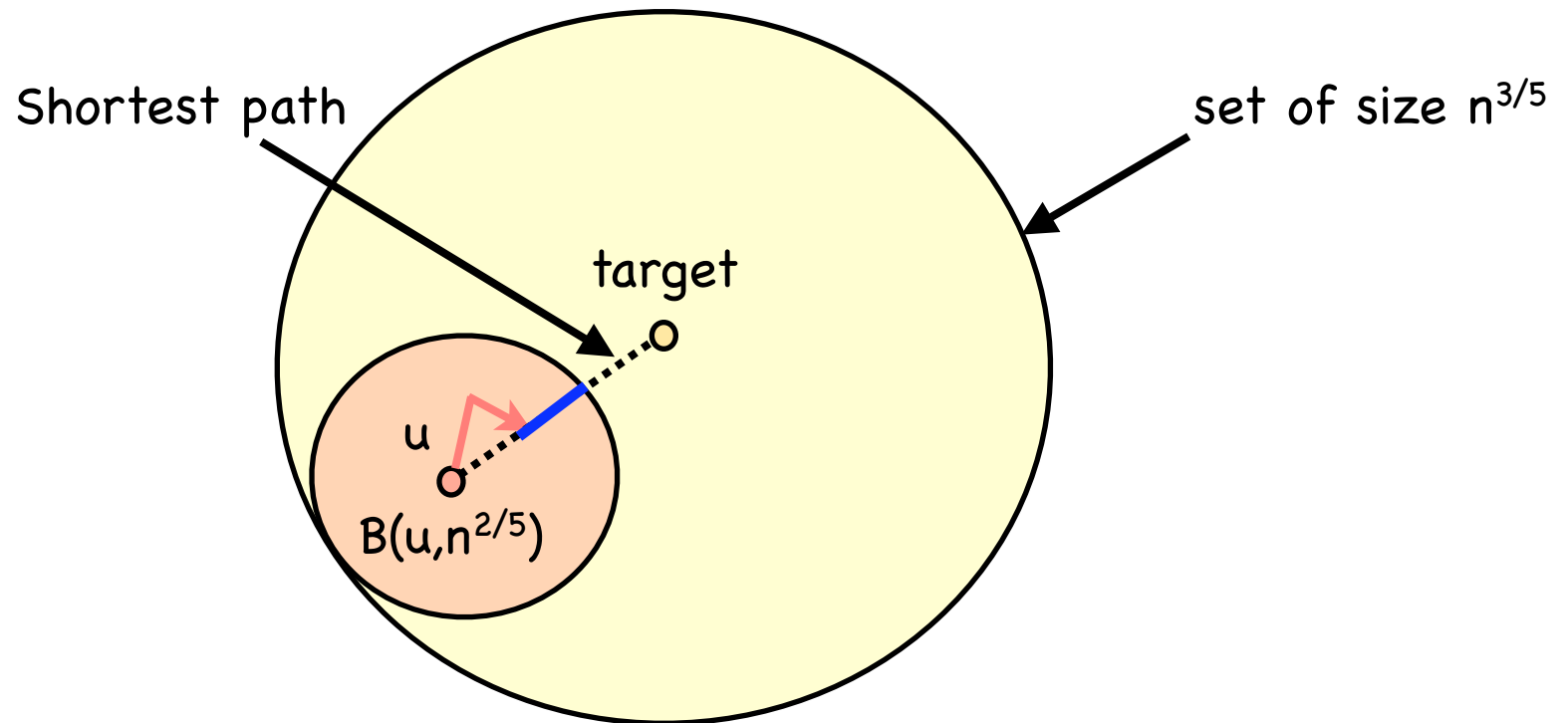
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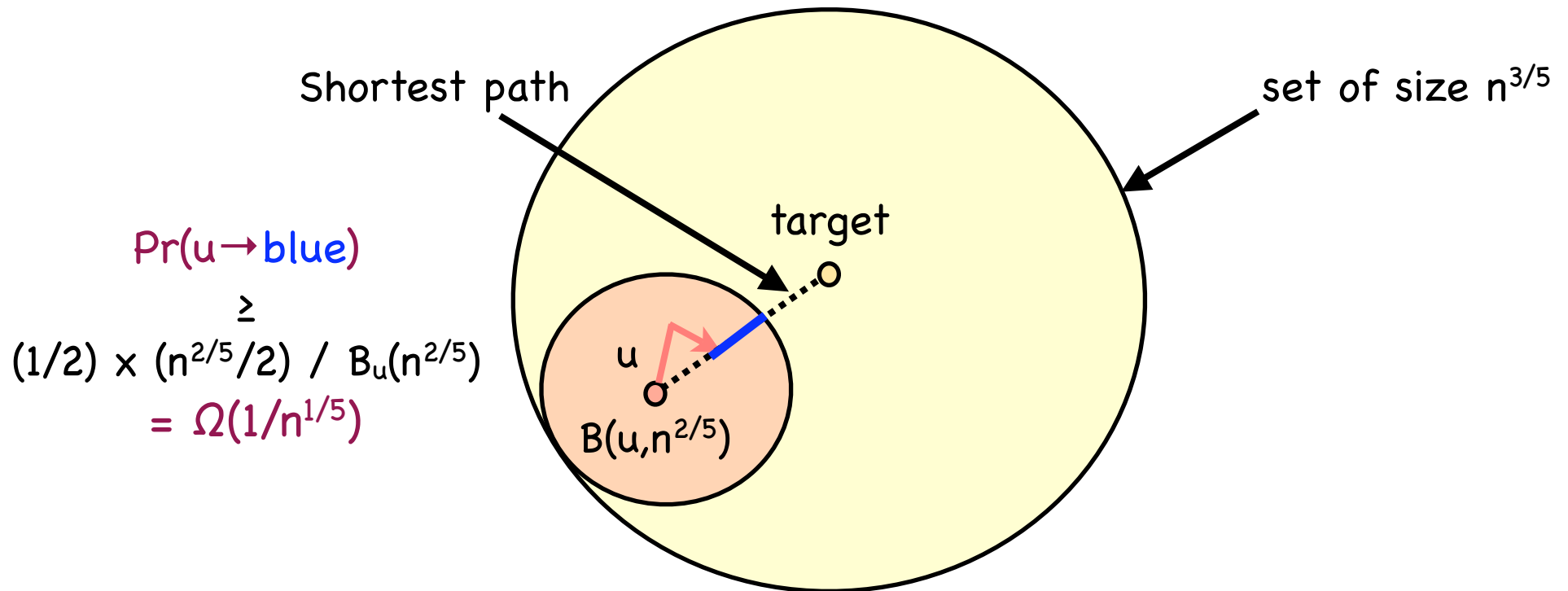
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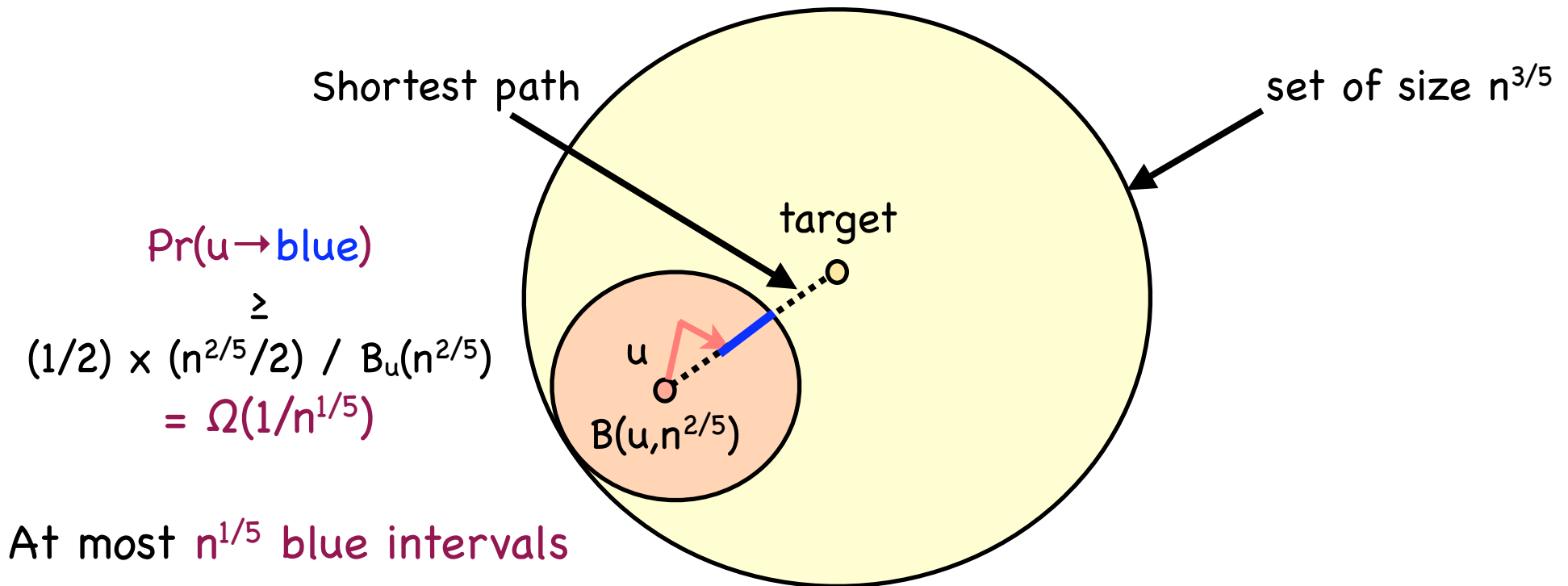




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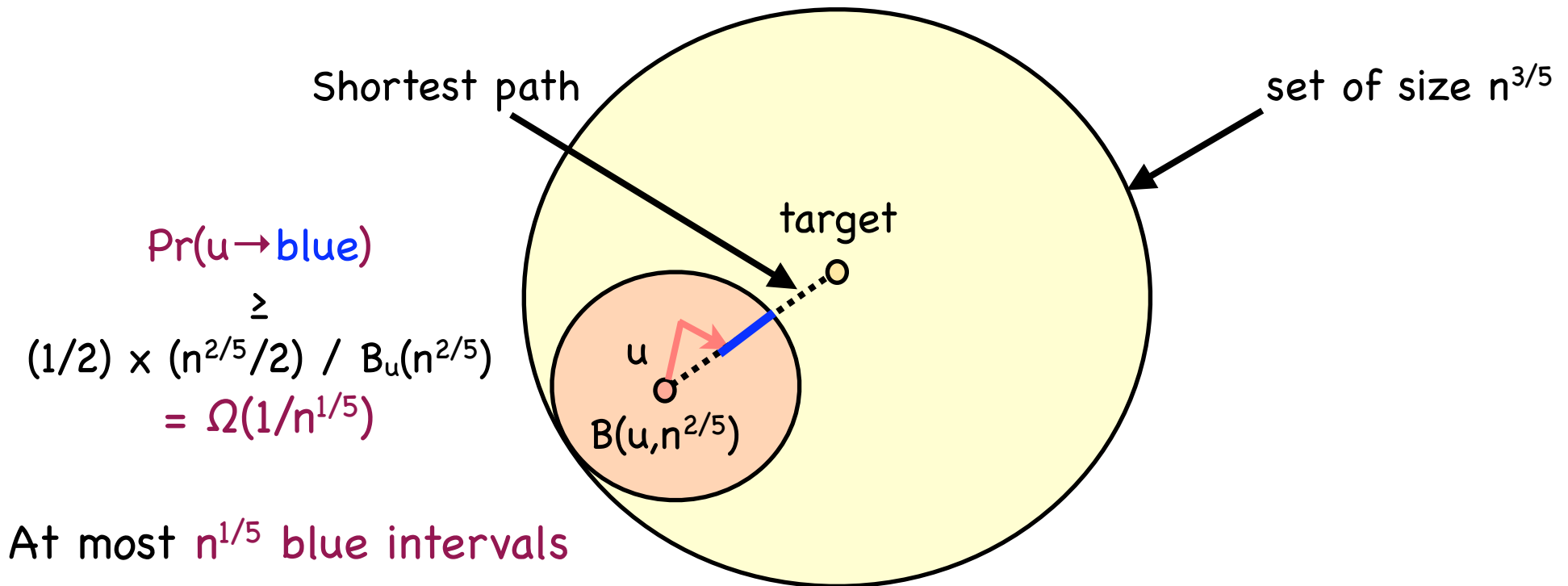
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$$E(\# \text{ steps}) \leq O(n^{2/5}) + O(n^{1/5}) \times n^{1/5} = O(n^{2/5}).$$

# Another perspective: matrix augmentation

- A **gap** remains between  $\tilde{O}(n^{1/3})$  and  $\Omega(n^{1/\sqrt{\log n}})$ .
- A new perspective to augment arbitrary graphs: a priori augmentation by giving a matrix of links distribution.

# An augmentation matrix

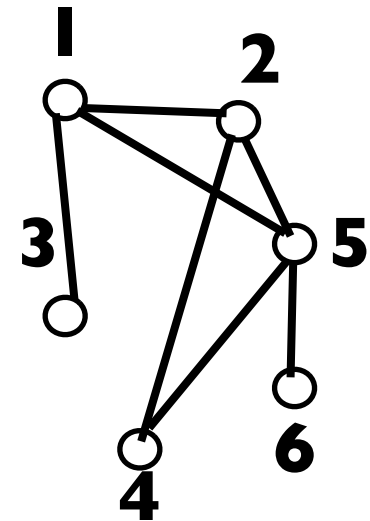
$p_{i,j}$  = probability that the link of node  $i$  is node  $j$

destinations of the links

Nodes

	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>1</b>	0	1/2	1/4	1/4	0	0
<b>2</b>	1/3	1/3	1/3	0	0	0
<b>3</b>	1/5	0	1/5	1/5	1/5	1/5
<b>4</b>	1/2	0	0	0	0	1/2
<b>5</b>	0	1/8	1/8	1/8	1/8	1/2
<b>6</b>	1/6	1/3	1/6	1/6	0	1/6

→  $\sum \leq 1$



$$M = (p_{i,j})_{1 \leq i,j \leq 6}$$

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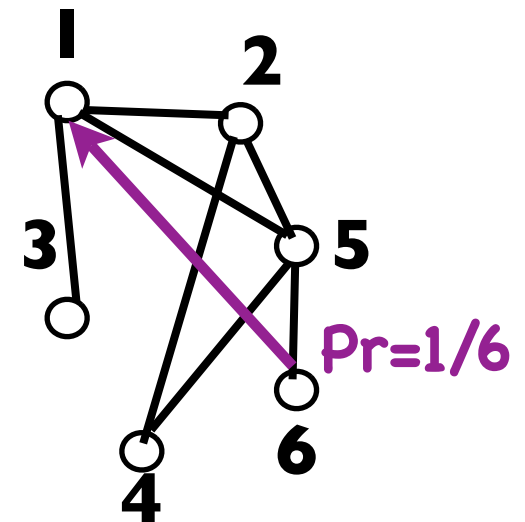
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- Distribution of links given by the matrix without looking at the graph.
- Without further information, what improvement can be hoped?

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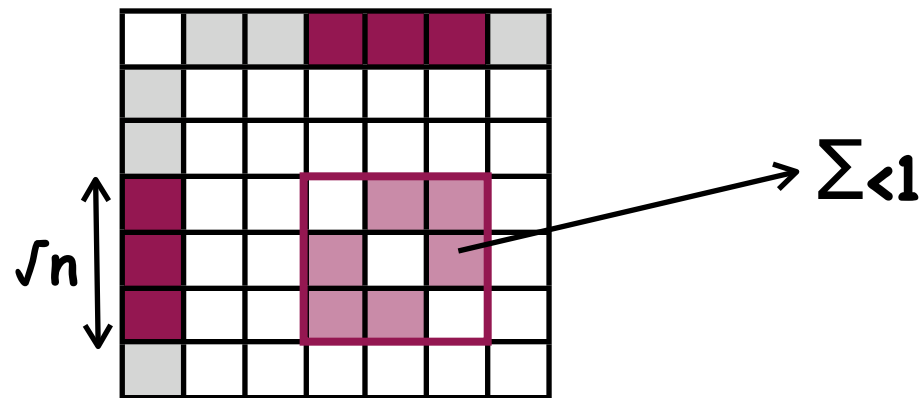
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## Theorem:

If the matrix is given independently from the graph labeling, the uniform matrix is optimal.

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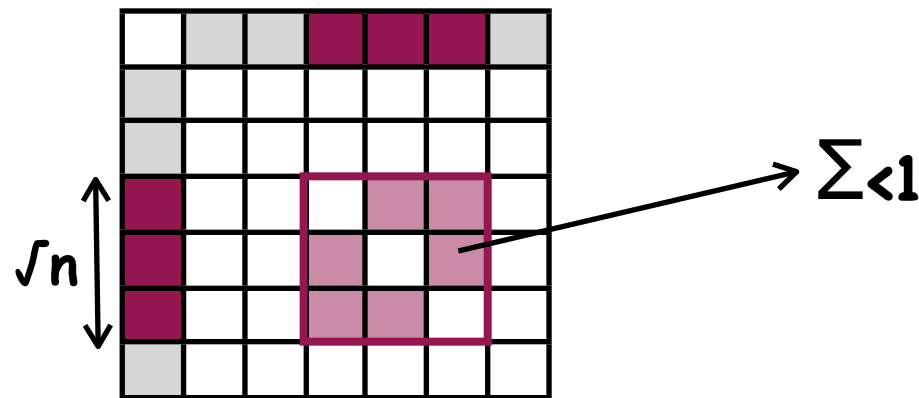
Lemma: in any augmentation matrix, there is a set of  $\sqrt{n}$  indices s.t.  $\sum p_{i,j} < 1$  on this set.





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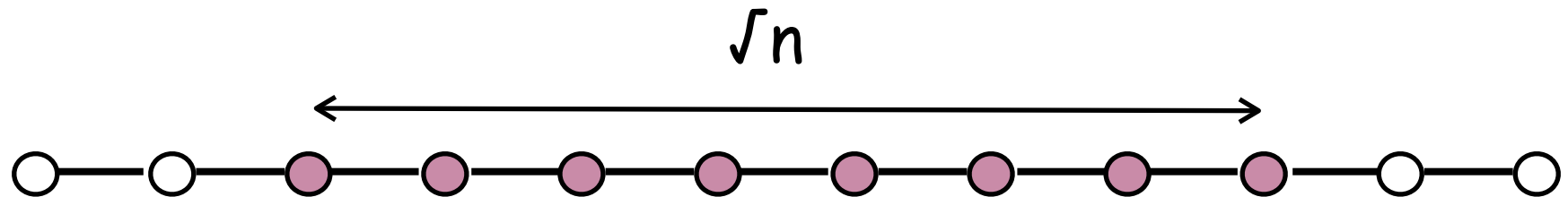
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- **BUT:**  $\sum p_{i,j}$  on a set of indices  $I$  is the expected number of links going out from  $I$  into  $I$ .

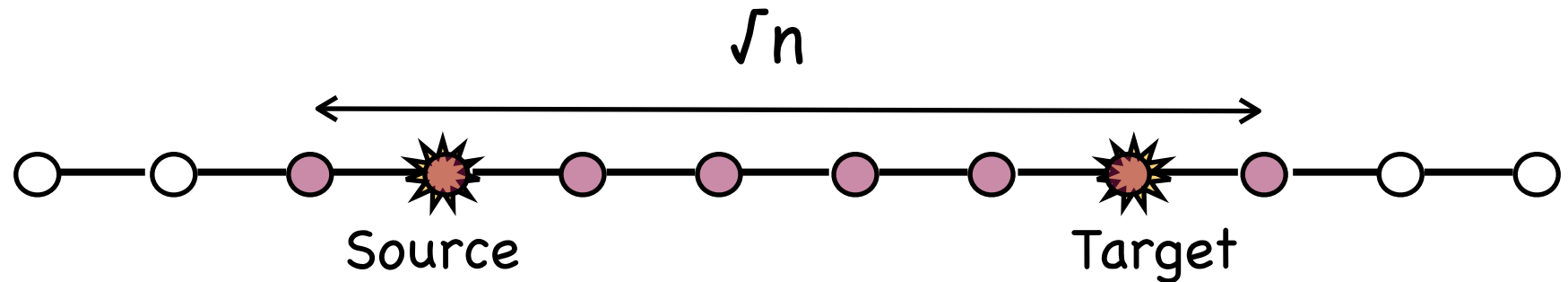
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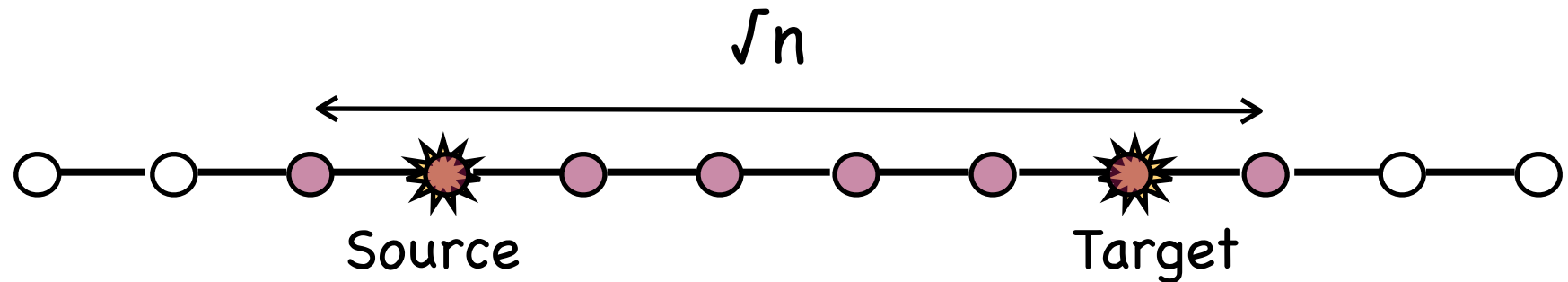
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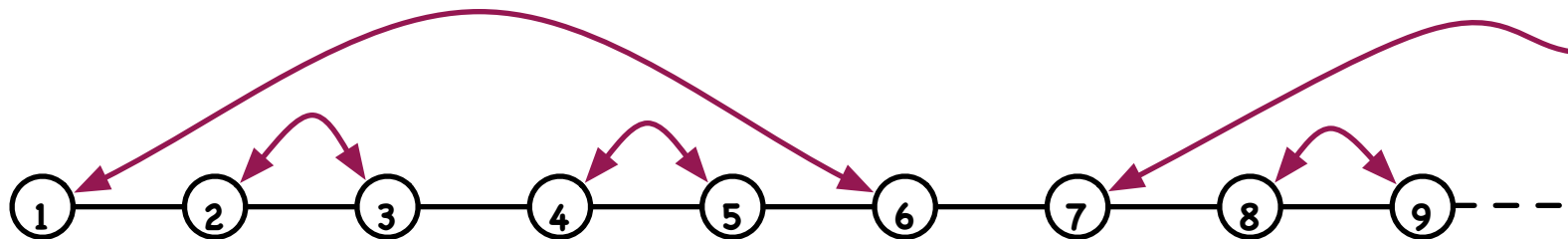
- An adversary can label an interval with the bad set of indices.
- The expected number of shortcuts is  $< 1$  inside the interval:  $\Omega(\sqrt{n})$  greedy steps.

# Matrix augmentation with labeling

- The idea: keep the a priori augmentation (given matrix) but **associate a proper labeling scheme**.

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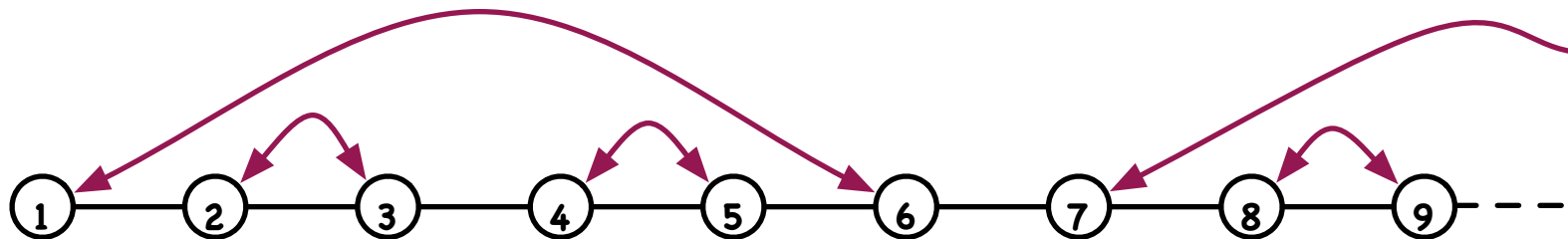
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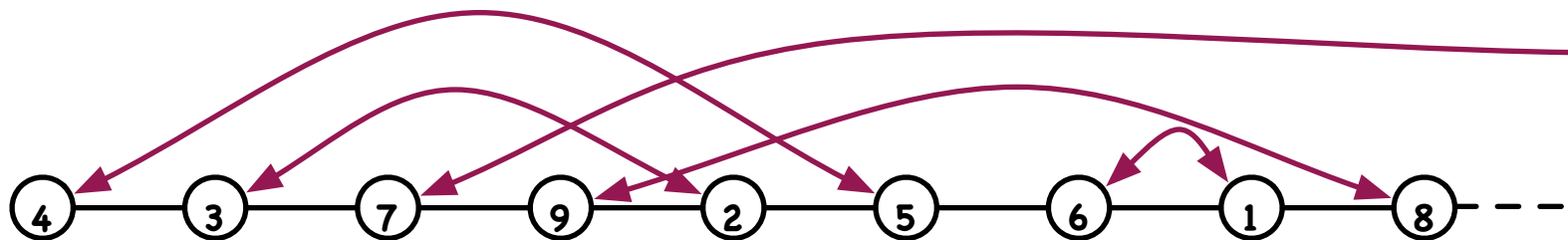
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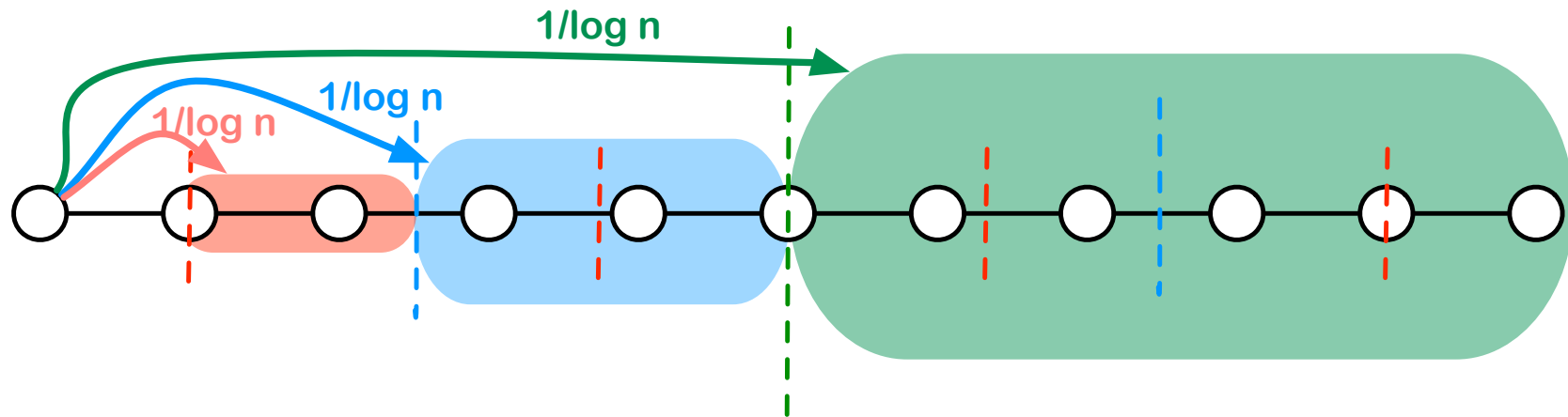
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Here: different labels, paths may rise up to  $\Omega(\sqrt{n})$

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- The key of efficient augmentations: find **good separators** in the graphs to distribute the links hierarchically.





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➔ Done through a path-decomposition.

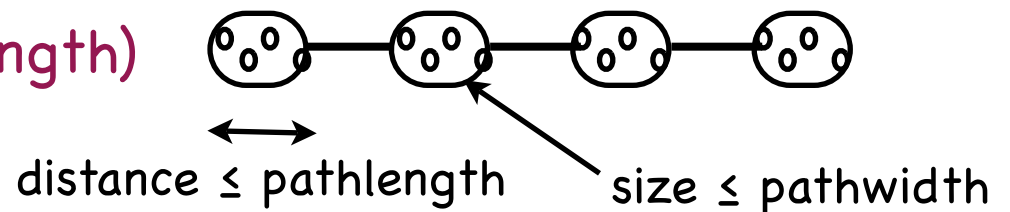
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- ➔ New  $O(\text{polylog } n)$ -navigable graphs: interval, AT-free...

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  - But raises big graph decomposition  $Q^\circ$ .
- **Matrix augmentation** : can we get rid of  $O(\sqrt{n})$  in the bound?