A Space Lower Bound for Routing in Trees

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STACS - March 2002
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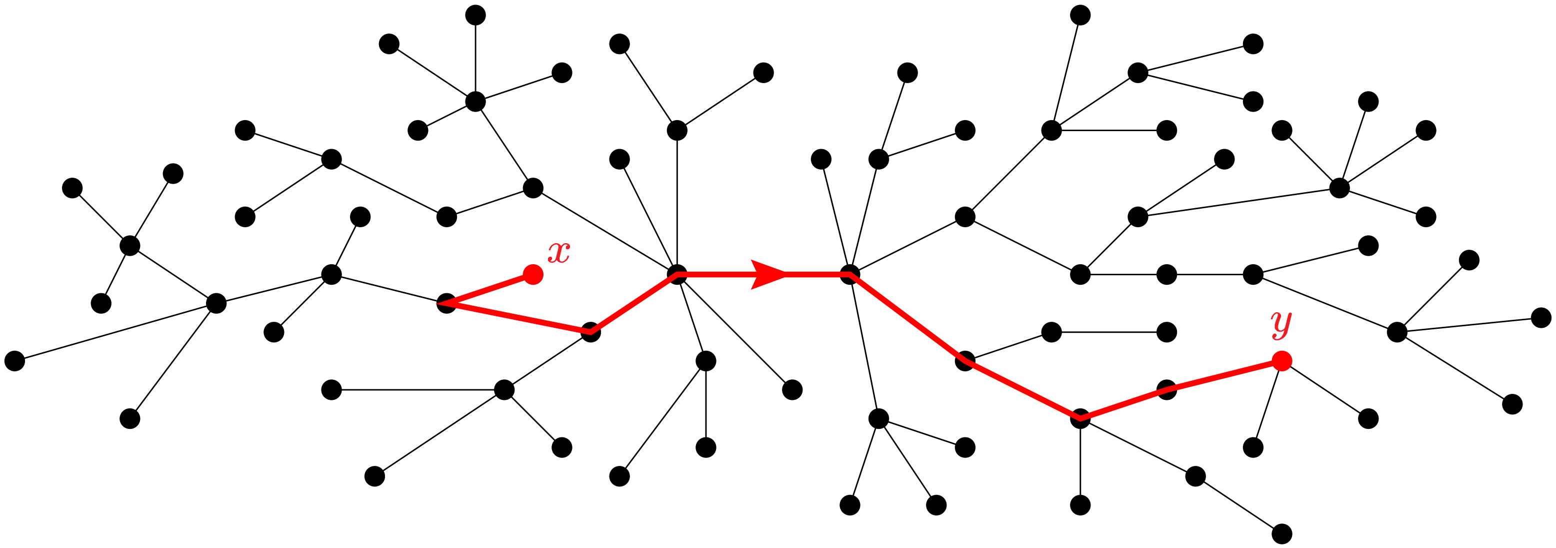
Problem

For all x, y, route a message from x to y in a tree network

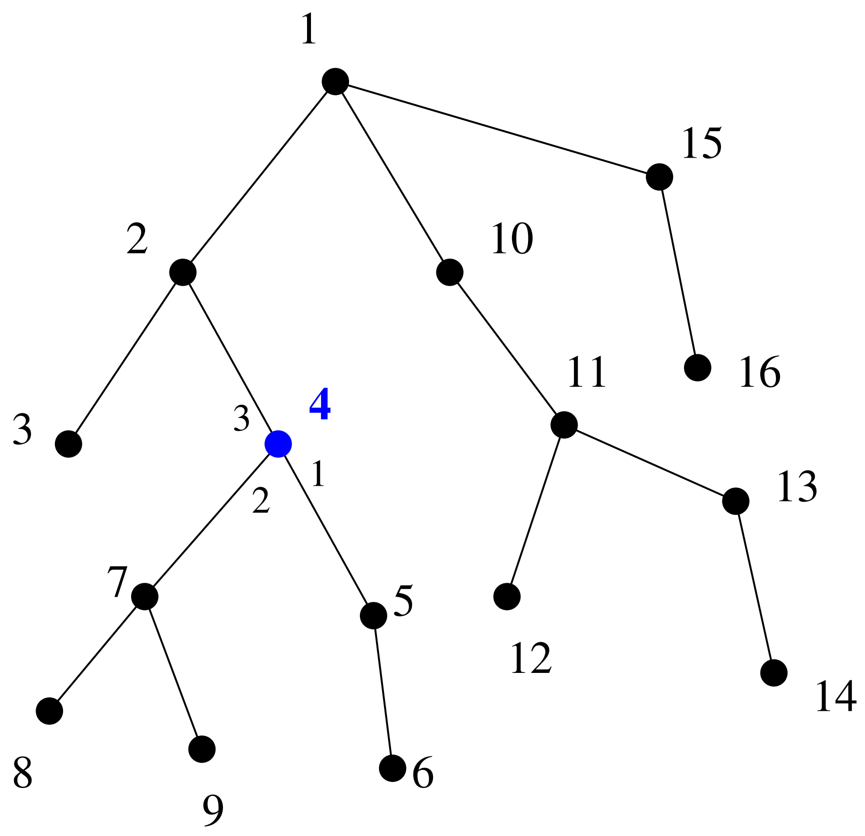


Routing with:

- local and compact data structures (compact routing tables);
- along shortest paths;
- fast routing decision.

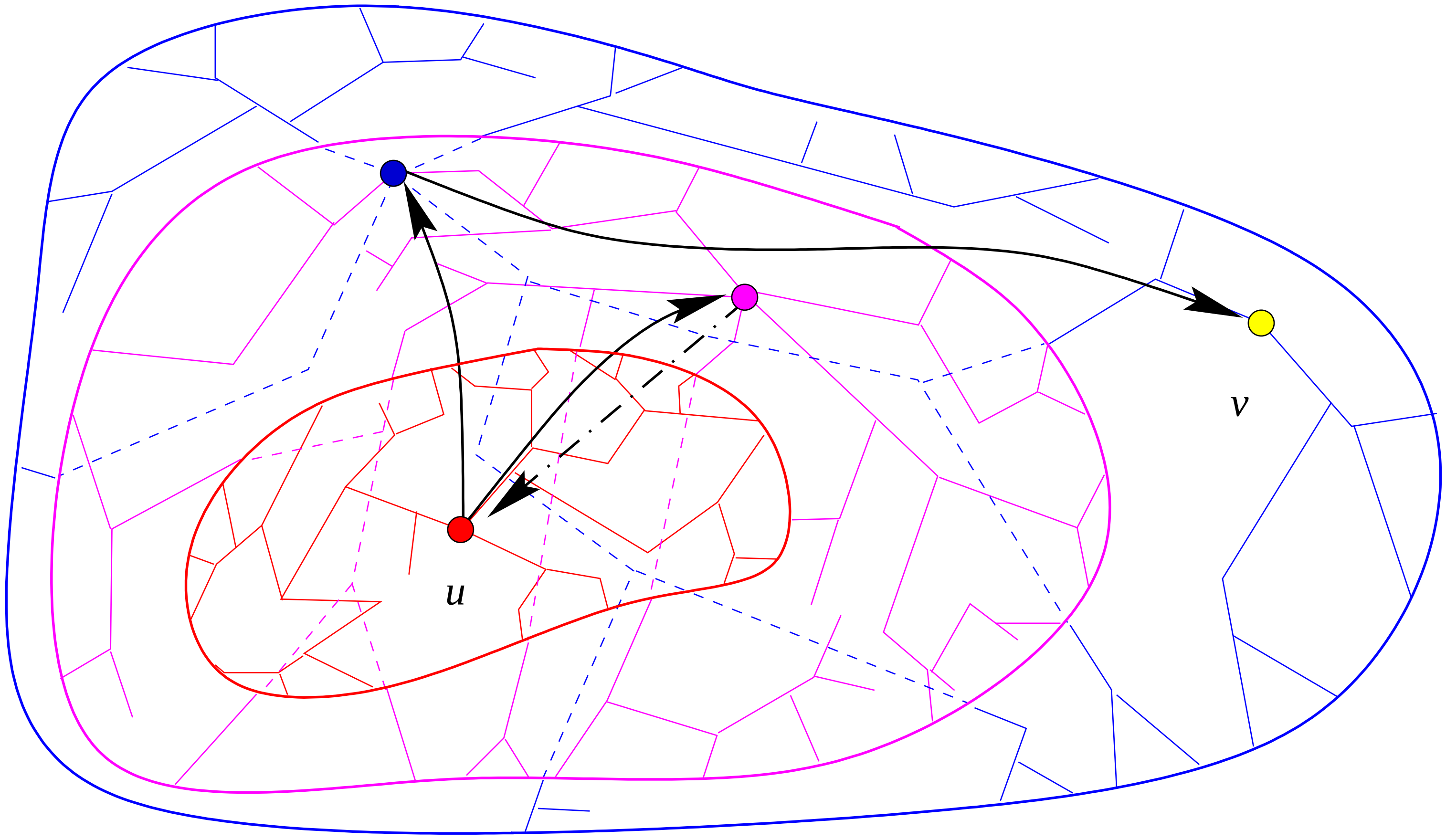
Example

4: [5,6],1
[7,9],2
-,3



Subroutine for universal routing schemes

Hierarchical routing schemes: applicable to **all** the networks



What is a routing scheme?

(for every graph of a given graph family $\mathcal{F}$)

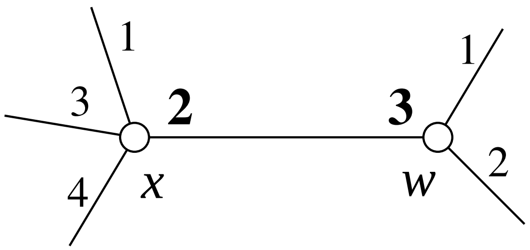
1. A pre-processing of G setting local data structures

$$\Rightarrow \quad \forall\, v \in G, \quad \text{address}(v) \text{ and } \text{data}(v)$$

2. A routing algorithm forwarding messages $\forall x, y \in G$

$$\text{send}(\text{data}(x), \text{address}(y)) \quad \text{that returns}$$

the label of an edge (x, w), $\text{port}(x, w) \in \{1, \ldots, \text{deg}(x)\}$.



Two models

- *Designer-port* model: edge labels (ports) are set at the pre-processing stage

- *Fixed-port* model: ports are fixed in advance (say by an adversary), and cannot be permuted

  $\Rightarrow$ multiple protocol routing schemes

Some complexity results for general graphs

$\|\text{data}(v)\|$ (in bits)	$\|\text{address}(v)\|$ (in bits)	approx. vs. shortest path	
$\tilde{O}(n^{1/k})$	$O(\log^2 n)$	$O(k^2)$	[Awerbuch,Peleg'92]
$\tilde{O}(n^{1/k})$	$O(k \log^2 n / \log\log n)$	$O(k)$	[Thorup,Zwick'01]
$\Omega(n^{1/k})$	$O(n^{1/k})$	$\Theta(k)$	[Peleg,Upfal'89]

For trees (shortest path)

$\lvert\text{data}(v)\rvert$	$\lvert\text{address}(v)\rvert$	
Designer-port model		
$O(d \log n)$	$\log n$	[Santoro,van Leeuwen'85]
$\Theta(\sqrt{n})$	$\log n$	[Eilam,G.,Peleg'98]
$O(\log n)$	$O(\log n)$	[FG01,TZ01]
Fixed-port model		
$\Omega(n)$	$\log n$	[FG01]
$O(\sqrt{n} \log n)$	$3 \log n$	[Cowen'01]
$O(\log^2 n / \log\log n)$	$O(\log^2 n / \log\log n)$	[FG01,TZ01]

This paper $\boxed{\lvert\text{data}(v)\rvert + \lvert\text{address}(v)\rvert = \Omega(\log^2 n / \log\log n)}$

Lower bound technique

Def: A *forwarding labeling scheme* for a graph family $\mathcal{F}$ is a pair of functions (L, R) s.t. $\forall G \in \mathcal{F}$:

- $L(v) =$ binary label asigned to each node $v \in G$;
- $R(L(x), L(y)) =$ port of an edge incident to x leading to y along a shortest path, $\forall x \neq y \in G$.

Fact: Since $\exists L, R$ with $|L(v)| \leqslant |\mathrm{data}(v)| + |\mathrm{address}(v)|$,
a lower bound on the forwarding label length
$\Rightarrow$ a lower bound on $|\mathrm{data}| + |\mathrm{address}|$

More precisely ...

Let $\mathcal{L}$ be the set of **all** the labels assigned by L for $\mathcal{F}$,

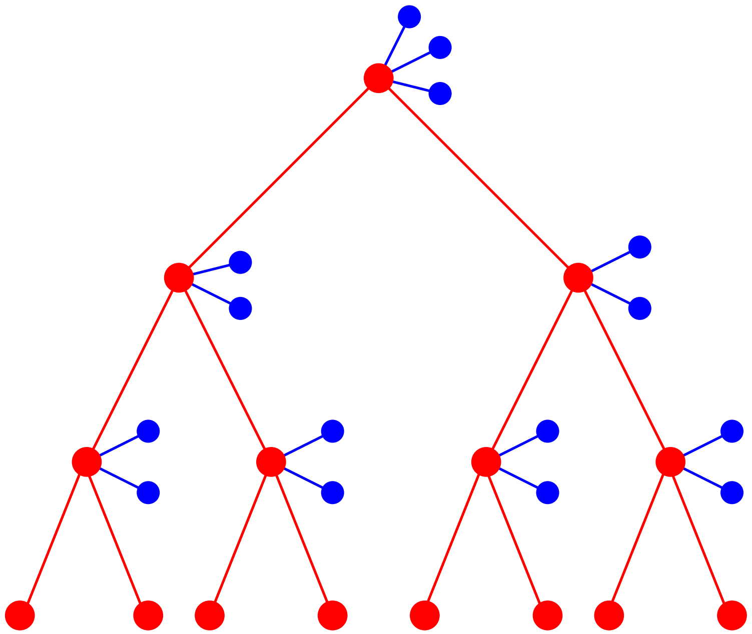
$$\mathcal{L} := \bigcup_{G \in \mathcal{F}} \{L(v) \mid v \in G\}$$

If $\mathcal{L}$ is of minimum cardinality over all fowarding labeling schemes on $\mathcal{F}$, then

$$\exists G_0 \in \mathcal{F},\ \exists v_0 \in G,\ |\mathsf{data}(v_0)| + |\mathsf{address}(v_0)| \geqslant \log |\mathcal{L}|$$

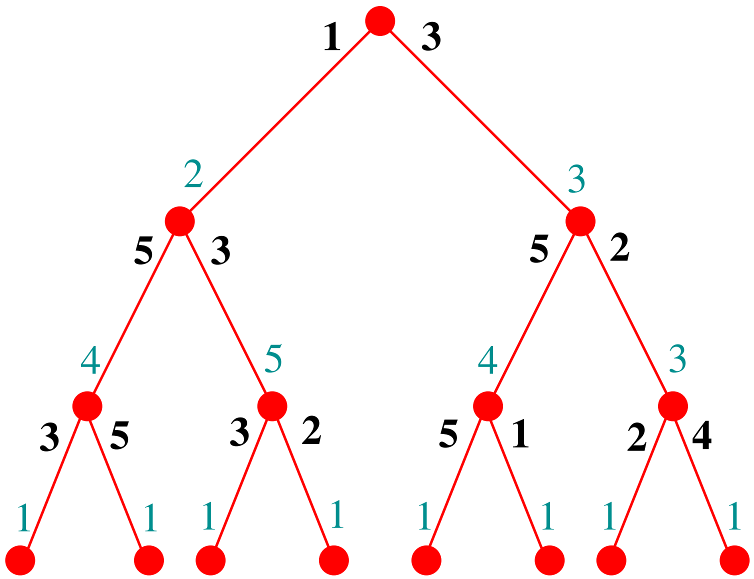
The tree family: $\mathcal{T}(h, t, d)$

(a set of port-labeled trees with $O(n)$ nodes)



$h = 3$ = height
$t = 2$ = arity of the red tree
$d = 5$ = degree (blue+red)

$$d \sim \sqrt{n}$$
$$t^h \sim \sqrt{n}$$
$$t \sim h \sim \Theta(\log n / \log\log n)$$



$$\begin{aligned}
\phi_x \quad &:= \quad \text{sequence of ports of the children of } x \\
\phi_x \in \mathcal{S}(d) \quad &:= \quad \left\{ (s_1, \ldots, s_t) \mid 1 \leqslant s_i \neq s_j \leqslant d \right\} \\
|\mathcal{S}(d)| \quad &= \quad t! \binom{d}{t} \ \geqslant \ (d/t)^t
\end{aligned}$$

Key point of the proof

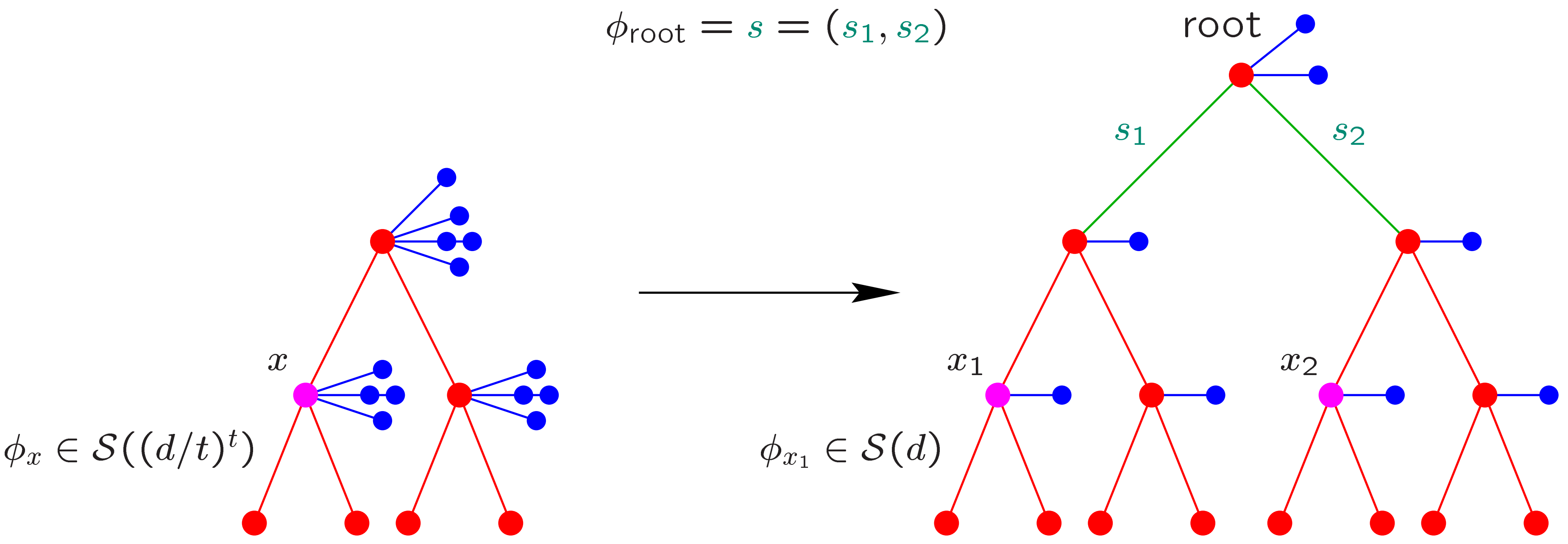
Let (L, R) be any forwarding labeling scheme on $\mathcal{T}(h, t, d)$ s.t. $|\mathcal{L}|$ is minimal. Set $f(h, d) := |\mathcal{L}|$.

<u>**Key point:**</u>
$\forall d \geqslant t$ where t divides d, $\forall h \geqslant 1$,

$$f(h, d)^{t+1} \geqslant |\mathcal{S}(d)| \cdot f(h-1, (d/t)^t)$$

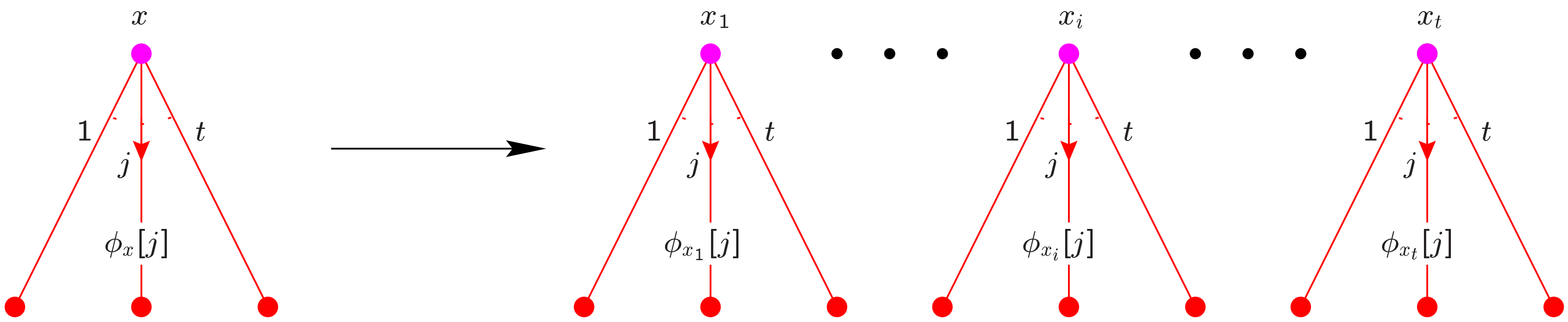
Injective mapping:

Let $\mathcal{T}_s := \{T \in \mathcal{T} \mid \phi_{\text{root}} = s\}$, for every sequence $s \in \mathcal{S}(d)$



$$T \in \mathcal{T}(h-1, t, (d/t)^t), s \quad \longrightarrow \quad T' \in \mathcal{T}_s(h, t, d)$$

Injective port-labeling



$$\phi_x[j] = 527 \quad \longrightarrow \quad \phi_{x_1}[j] = 5 \qquad \phi_{x_2}[j] = 2 \qquad \phi_{x_3}[j] = 7$$

$$\begin{aligned} L^*(x) &:= (L(\text{root}), L(x_1), \ldots, L(x_t)) \\ L^*(y) &:= (L(\text{root}), L(y_1), \ldots, L(y_t)) \end{aligned}$$

So, any (L, R) on $\mathcal{T}_s(h,t,d) \Rightarrow$ a labeling on $\mathcal{T}(h-1,t,(d/t)^t)$

$$\Rightarrow f(h,d)^{t+1} \geqslant |\mathcal{S}(d)| \cdot f(h-1,(d/t)^t)$$

... solving recursion

Recall $h = t = (\log \sqrt{n}\,) / \log\log \sqrt{n}$ and $d = \sqrt{n}$

$$\log |\mathcal{L}| \;\geqslant\; \left(1 - e^{-h/t}\right) \cdot t \cdot \log\left(d/t^2\right)$$
$$> \frac{(\log n)^2}{7 \log\log n}$$

QED

Conclusion – Open problem

- For trees, $\Theta(\log^2 n / \log\log n)$ bit routing labels.

- $O(k \log^2 n)$ bit routing labels for treewidth k graphs [Peleg'99]

- $O(k \log^2 n / \log\log n)$ bit routing labels, for every k? $k = 2$?

- Dynamic trees ...