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Sparse Additive Spanners for Bounded Tree-Length Graphs

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Sparse Additive Spanners for Bounded Tree-Length Graphs*

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Abstract

This paper concerns construction of additive stretched spanners with few edges for n-vertex graphs having a tree-decomposition into bags of diameter at most δ, i.e., the tree-length δ graphs. For such graphs we construct additive 2δ-spanners with $O(\delta n \log n)$ edges, and additive 6δ-spanners with $O(\delta n)$ edges. We also show a lower bound, and prove that there are graphs of tree-length δ for which every multiplicative δ-spanner (and thus every additive $(\delta-1)$-spanner) requires $\Omega(n^{1+1/\Theta(\delta)})$ edges.

Keywords: additive spanner, tree-decomposition, tree-length, chordality

1 Introduction

Let G be a connected graph with n vertices. A subgraph H of G is an (s,r)-*spanner* if $d_H(u,v) \leqslant s \cdot d_G(u,v) + r$ for all pair of vertices u,v of G. An $(s,0)$-spanner is also termed *multiplicative s-spanner*, and an $(1,r)$-spanner is termed *additive r-spanner*. An (s,r)-spanner is also an $(s+r,0)$-spanner (in particular, an additive r-spanner is a multiplicative $(r+1)$-spanner), but the reverse is false in general.

The main objective is to construct for a graph an (s,r)-spanner with few edges. There are many applications of spanners, for example, the complexity of a lot of distributed algorithms depends on the number of messages, itself depending on the number of edges [PU89a, Pel00]. Sparse spanners occur also in the efficiency, of compact routing schemes [PU89b]. Unfortunately, given an arbitrary graph G and three integers s, r and m, determine whether G admits an (s,r)-spanner with m or fewer edges, is NP-complete [PS89], even if we restrict $r = 0$ (see also [BDLL04, BH98, CC95, DK99, FK01] for complexity issue). Best known results on (s,r)-spanners for general graphs are summarized in the following table.

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stretch	edges	reference
$(1,2)$	$\Theta(n^{3/2})$	[DHZ00, EP01][1]
$(2k-1,0)$	$O(n^{1+1/k})$	[ADD+93, TZ01, BS03][2]
$(k-1,2k-4)$	$O(kn^{1+1/k})$	[EP01], $k \geqslant 4$ even
$(k-2+\epsilon, 2k-2-\epsilon)$	$O(\epsilon^{-1}kn^{1+1/k})$	[EP01], $k \geqslant 3$ odd
$(k-1+\epsilon, 2k-4-\epsilon)$	$O(\epsilon^{-1}kn^{1+1/k})$	[EP01], $k \geqslant 4$ even
$(1+\epsilon, \beta(\epsilon,k))$	$O(\beta(\epsilon,k)n^{1+1/k})$	[EP01][3], $k \geqslant 2$

An interesting question still left open is to know whether every graph has an additive $(2k-2)$-spanner with $O(n^{1+1/k})$ edges, for $k > 2$. In the affirmative, this would generalize the result of [DHZ00, EP01] ($k = 2$) and implies also the observation of [ADD$^+$93]. Another interesting question is to known whether the $O(n^{1+1/k})$ edge bound for multiplicative $(2k-1)$-spanner is tight or not. This bound directly relies to an 1963 Erdös Conjecture [Erd64] on the existence of graphs with $\Omega(n^{1+1/k})$ edges and girth at least $2k+2$. This has been proved only for $k = 1, 2, 3$ and $k = 5$.

Better bounds can be achieved if we restrict spanners to be trees, or if particular classes of graphs are considered: planar graphs [FK01], and more structured graphs (e.g., see [MVPR96] and [Soa92] for a survey). Among them, the class of *chordal* graphs is of particular interests [BDLL04, PS89, CDY03]. A graph is k-*chordal* if its induced cycles are of length at most k. Chordal graphs coincide with 3-chordal. Here below are summarized the best constructions for k-chordal graphs.

chordal	stretch	edges	reference
3	$(2,0)$	$\Theta(n^{3/2})$	[PS89]
3	$(3,0)$	$O(n \log n)$	[PS89]
3	$(1,3)$	$O(n \log n)$	[CDY03]
k	$(1,k+1)$	$\Theta(n)$	[CDY03], $k \geqslant 3$

Tree-decomposition is a rich concept introduced by Robertson and Seymour [RS86] and is widely used to solve various graph problems. In particular efficient algorithms exist for graphs having a tree-decomposition into subgraphs (or *bags*) of bounded size, i.e., for bounded *tree-width* graphs.

The *tree-length* of a graph G is the smallest integer δ for which G admits a tree-decomposition into bags of diameter at most δ. It as been formally introduced in [DG03], and extensively studied in [Dou03]. Chordal graphs are exactly the graphs of tree-length 1, since a graph is chordal if and only if it has a tree-decomposition in cliques (see [Die00]). Actually, [GKK$^+$01] showed that k-chordal graphs have tree-length at most $k/2$. So, for

[1] The bound of [DHZ00] was $O(n^{3/2}\log^{O(1)} n)$ edges but with a better running time, $O(n^{5/2})$ v.s. $O(n^2)$.

[2] This observation due to [ADD$^+$93] is based on the classical result (see [AHL02]) that every graph with at least $\frac{1}{2}n^{1+1/k}$ edges has a cycle of length at most $2k$, for every $k \geqslant 1$. [TZ01] and [BS03] gave respectively an $O(kmn^{1/k})$ and $O(km)$ time algorithm for the construction of such spanner.

[3] Actually $\beta(\epsilon,k) = k^{\max\{\log\log k - \log\epsilon, \log 6\}}$ for $2 \leqslant k \leqslant \log n$ and fixed $0 < \epsilon < 1$.

instance AT-free graphs and permutation graphs (that are 5-chordal) are of tree-length 2. However, there are graphs with bounded tree-length and unbounded chordality[4], like the wheel. So, bounded tree-length graphs is a larger class than bounded chordality graphs.

For several problems involving distance computation, like the design of approximate distance labeling schemes [GKK+01] or of near-optimal routing schemes [Dou03], tree-length δ graphs are a natural generalization of chordal graphs, and their tree-decomposition induced can be successfully used. In this paper we highlight a new property of bounded tree-length graphs: the design of sparse additive spanners. The following table summarizes the bounds we have obtained on the minimum number of edges of additive spanner.

tree-length	stretch	edges
δ	$(1, 2\delta)$	$O(\delta n \log n)$
δ	$(1, 6\delta)$	$O(\delta n)$
δ	$(\delta, 0)$	$\Omega(n^{1+\epsilon})$, $\epsilon \geqslant 1/\lceil \delta/2 \rceil$
		for[5] $\delta = 1, 2, 3, 4, 5, 6, 9, 10$

In this paper, we also compare our algorithm to the Chepoi-Dragan-Yan's algorithm (CDY) used successfully for k-chordal graphs [CDY03]. For small chordality k, our algorithm produces an additive $3k$-spanner (or $(3k-3)$-spanner of odd k) with $O(n)$ edges (recall that $\delta \leqslant k/2$), whereas CDY's algorithm constructs a $(k+1)$-additive spanner with $O(n)$ edges. However, we show in Section 4 that the CDY's algorithm cannot be used for bounded tree-length graphs of large chordality. More precisely, we construct a worst-case graph of tree-length 3 and chordality $\Omega(n^{1/3})$ for which the CDY's algorithm produces an $\Omega(n^{1/3})$-spanner with $O(n)$ edges. A generic algorithm would certainly combine both algorithms.

The lower bound shows that every additive $o(\delta)$-spanner requires $\Omega(n^{1+\epsilon})$ edges. However, combined with our two upper bounds, this naturally leads to the question of whether there exists, for every tree-length δ graph, a $O(\delta)$-spanner with $O(n \log n)$ or even with $O(n)$ edges.

The paper is organized as follows. After some definitions in Section 2, we present in Section 3 the first algorithm (Line 1 of the previous table). Section 4 presents the second algorithms (Line 2) and the CDY's algorithm, both based on a *layering-tree* of the graph. We conclude in Section 5 with the lower bound.

2 Preliminaries

We recall the notion of *tree-decomposition* used by Robertson and Seymour in their work on graph minors [RS86].

[4]The *chordality* is the smallest k such that the graph is k-chordal.

[5]The last line relies to the Erdös's Conjecture, but in any case $\epsilon \geqslant 1/\Theta(\delta)$ (cf. Table of Corollary 1).

Definition 1 *A tree-decomposition of a graph G is a tree T whose vertices, called* bags, *are subsets of $V(G)$ such that:*

1. $\bigcup_{X \in V(T)} X = V(G)$;
2. *for all $\{u,v\} \in E(G)$, there exists $X \in V(T)$ such that $u,v \in X$; and*
3. *for all $X,Y,Z \in V(T)$, if Y is on the path from X to Z in T then $X \cap Z \subseteq Y$.*

A tree-decomposition is *reduced* if any bag is contained in no other bags. A leaf of such decomposition contains necessarily a vertex contained in none other bags. Thus by induction the number of bags of a reduced tree-decomposition does not exceed $\max\{n-1,1\}$ for an n-vertex connected graph (cf. [Bod96]).

For every induced subgraph H of G, the *diameter of H in G* is $\mathrm{diam}_G(H) = \max_{x,y \in V(H)} d_G(x,y)$, where $d_G(x,y)$ denotes the distance in G between x and y. Observe that $\mathrm{diam}_G(H)$ can be constant even if H is not connected. By extension, $\mathrm{diam}_G(X) = \mathrm{diam}_G(G[X])$ for every subset $X \subseteq V(G)$, where $G[X]$ denotes the subgraph of G induced by X.

The *length* of tree-decomposition T of a graph G is $\max_{X \in V(T)} \mathrm{diam}_G(X)$, and the *tree-length of G* is the minimum, over all tree-decompositions T of G, of the length of T.

3 Additive 2δ-Spanner with $O(\delta n \log n)$ Edges

Theorem 1 *Every n-vertex graph of tree-length δ has an additive 2δ-spanner with $O(\delta n \log n)$ edges.*

The remaining of this section concerns the proof of Theorem 1. For this purpose we need several ingredients, and of two basic properties.

It is well known that every tree T has a vertex u, called *median*, such that each connected component of $T \setminus \{u\}$ has at most $\frac{1}{2}|V(T)|$ vertices. A *hierarchical tree* of T is then a rooted tree H defined as follows: the root of H is the median of T, u, and its children are the roots of the hierarchical trees of the connected components of $T \setminus \{u\}$. Observe that T and H share the same vertex set, and that the depth of H is at most[6] $\log|V(T)|$.

From now, G is a graph with n vertices and of tree-length δ. T denotes a tree-decomposition of G of length δ supposed reduced, and H denotes a hierarchical tree of T. So, the depth of H is at most $\log n$. We denote by $\mathrm{NCA}_H(U,V)$ the *nearest common ancestor* of U and V in H.

Property 1 *Let U,V be two vertices of T and let Q be the path in T from U to V, and let $Z = \mathrm{NCA}_H(U,V)$. Then, $Z \in Q$, and Z is an ancestor in H of all the vertices of Q.*

[6] All the logs are in base two.

Proof. By construction, the subtree induced by Z and its descendants in H is a connected component of T, say A. Thus, Z, U, V are in A, but U and V are in two different components of $T \setminus \{Z\}$. Thus in T, the path Q from U to V is wholly contained in A and intersects Z. So, $Z \in Q$ and Z is ancestor of all vertices of Q in H. $\square$

We assume that T is rooted. For every vertex u of G, the *bag* of u, denoted by $\mathcal{B}(u)$, is a bag X of T of minimum depth such $u \in X$. Observe that, once T has been fixed, $\mathcal{B}(u)$ is unique for each u.

Property 2 *For every edge $\{u, v\}$ of G, either*

1. *$\mathcal{B}(u)$ is an ancestor of $\mathcal{B}(v)$ in T and $u \in \mathcal{B}(v)$, or*
2. *$\mathcal{B}(v)$ is an ancestor of $\mathcal{B}(u)$ in T and $v \in \mathcal{B}(u)$.*

Proof. Let X be any bag of T with a vertex x. Since $x \in X \cup \mathcal{B}(x)$, by Rule 3 of Definition 1 and by minimality of the depth of $\mathcal{B}(x)$, we have that X is a descendant of $\mathcal{B}(x)$. Now, by Rule 2 there is a bag, say Y, containing u and v. It follows that Y is a descendant of $\mathcal{B}(u)$ and of $\mathcal{B}(v)$, i.e., either $\mathcal{B}(u)$ is an ancestor of $\mathcal{B}(v)$ or the reverse. If $\mathcal{B}(u)$ is an ancestor of $\mathcal{B}(v)$, then $\mathcal{B}(v)$ is on the path from $\mathcal{B}(u)$ to Y in T. By Rule 3, $u \in \mathcal{B}(v)$. Similarly if $\mathcal{B}(v)$ is an ancestor of $\mathcal{B}(u)$, then $v \in \mathcal{B}(u)$. $\square$

We associate with every bag X of T a shortest path spanning tree of G rooted at an arbitrary vertex $r_X \in X$. A vertex u of G is *good* w.r.t. S_X if $\mathcal{B}(u)$ is a descendant of X in H. Otherwise u is *bad* w.r.t. S_X. Clearly a vertex u can be good for some trees and bad for others, but as H is of depth at most $\log n$, u is good for at most $\log n$ trees.

For every tree S_X, we define S'_X as the tree obtained from S_X by removing recursively each bad leaf w.r.t. S_X. The spanner of G claimed by Theorem 1 is simply the graph defined by $G' := \bigcup_{X \in V(T)} S'_X$.

Lemma 1 *G' is an additive 2δ-spanner of G.*

Proof. Let u, v be two vertices of G, and let $Z = \mathrm{NCA}_H(\mathcal{B}(u), \mathcal{B}(v))$. By definition, u, v are good in S_Z, thus u, v are both in S'_Z. By Proposition 1, Z belongs to the path from $\mathcal{B}(u)$ to $\mathcal{B}(v)$ in T, thus there is a vertex $z \in Z$ such that $d_G(u,v) = d_G(u,z) + d_G(z,v)$. As S'_Z is a shortest path rooted at r_Z, we have $d_{G'}(u,v) \leqslant d_{S'_Z}(u,v) \leqslant d_G(u,r_Z) + d_G(r_Z,v)$. Note that $d_G(z,r_Z) \leqslant \delta$, and by the triangle inequality, we have:

$$\begin{aligned} d_{G'}(u,v) \leqslant d_{S'_Z}(u,v) &\leqslant d_G(u,z) + d_G(z,r_Z) + d_G(r_Z,z) + d_G(z,v) \\ &\leqslant d_G(u,z) + d_G(z,v) + 2d_G(z,r_Z) \\ &\leqslant d_G(u,v) + 2\delta \ . \end{aligned}$$

$\square$

The difficult part is to upper bound the number of edges.

Lemma 2 *G' has $O(\delta n \log n)$ edges.*

Proof. For every vertex u of G and every bag X of T, we define $\mathrm{cost}(u, X) = 1$ if $u \in S'_X$, and $\mathrm{cost}(u, X) = 0$ if $u \notin S'_X$. Clearly, the number of vertices of S'_X is $\sum_{u \in V(G)} \mathrm{cost}(u, X)$, and so the total number of edges of G' is

$$|E(G')| \leqslant \sum_{X \in T} \left(\left(\sum_{u \in V(G)} \mathrm{cost}(u, X) \right) - 1 \right) < \sum_{X \in T} \sum_{u \in V(G)} \mathrm{cost}(u, X)$$

The problem to estimate this upper bound is that neither $\sum_u \mathrm{cost}(u, X)$, nor $\sum_X \mathrm{cost}(u, X)$ are easy to calculate. To overcome this problem, we assume that each vertex u has a "charge" for each bag X, denoted by $\mathrm{charge}(u, X)$, that can be exchanged with its neighbors under the following condition. Initially, $\mathrm{charge}(u, X) = \mathrm{cost}(u, X)$, for all u and X. Then, while there is a vertex u and a bag X such that:

1. u is bad in S'_X, and
2. $\mathrm{charge}(u, X) > 0$,

$\mathrm{charge}(u, X)$ is decremented and $\mathrm{charge}(v, X)$ incremented, where v is one of the children of u in S'_X (observe that there is no leaf u in S'_X that is bad in S'_X). Such a procedure converges, and we denote by $\mathrm{charge}^*(u, X)$ the final charge of u w.r.t. X.

Clearly, $\sum_X \sum_u \mathrm{cost}(u, X) = \sum_X \sum_u \mathrm{charge}^*(u, X)$. On the other hand we have:

$$\sum_X \sum_u \mathrm{charge}^*(u, X) = \sum_u \sum_X \mathrm{charge}^*(u, X) \leqslant n \cdot \max_u \left\{ \sum_X \mathrm{charge}^*(u, X) \right\}$$

From the above procedure, if u is bad in S'_X then $\mathrm{charge}^*(u, X) = 0$. Say in other words, if u is good in t trees, then the sum $\sum_X \mathrm{charge}^*(u, X)$ has at most t non-null terms. We have seen that a vertex u is good in at most $\log n$ trees. Thus, $\sum_X \mathrm{charge}^*(u, X) \leqslant \max_X \{\mathrm{charge}^*(u, X)\} \cdot \log n$. To summarize,

$$|E(G')| < \sum_X \sum_u \mathrm{charge}^*(u, X) \leqslant n \log n \cdot \max_{u,X} \{\mathrm{charge}^*(u, X)\}$$

It remains to show that $\mathrm{charge}^*(u, X) = O(\delta)$, for all u and X. We assume that u is good in S'_X (otherwise we have seen that $\mathrm{charge}^*(u, X) = 0$). Observe that, in S'_X, u can receive the charge of v only if: 1) v is bad, 2) v is an ancestor of u, and 3) there is no good vertex on the path from u to v in S'_X. So, let v be the nearest ancestor of u that is good in S'_X (v is set to the root if such vertex does not exist). The charge received by u (w.r.t. X) is thus at most $d_{S'_X}(u, v)$, and thus its total charge is $\mathrm{charge}^*(u, X) \leqslant 1 + d_{S'_X}(u, v)$.

Let $P = x_0, x_1, \ldots, x_p$ be the path in S'_X from $u = x_0$ to $v = x_p$. P is a shortest path in G and $p = d_G(u, v) \geqslant 1$. Our aim is to show that $p \leqslant 3\delta$. Let Q be the path in T from $\mathcal{B}(u)$ to $\mathcal{B}(v)$.

Claim 1 *For every* $0 < i < p$, $\mathcal{B}(x_i) \notin Q$.

Let $Z = \mathrm{NCA}_H(\mathcal{B}(u), \mathcal{B}(v))$. By construction of P, every $x_i \in P \setminus \{x_0, x_p\}$ is bad, i.e., $\mathcal{B}(x_i)$ is not a descendant of Z. By Property 1, all the bags of Q are descendant of Z, so $\mathcal{B}(x_i) \notin Q$, for every $0 < i < p$, as claimed.

Let j be the largest index such that $x_j \in \mathcal{B}(u)$. Obviously, $j \leqslant \delta$ as x_0, x_j belongs to the same bag, $\mathcal{B}(x_0)$. W.l.o.g. $p > j$, since otherwise $p \leqslant \delta$ and we are done. Let $Y = \mathrm{NCA}_T(\mathcal{B}(u), \mathcal{B}(v))$.

Claim 2 $x_j \in Y$.

If $\mathcal{B}(u) = Y$, then we are done. Assume, $\mathcal{B}(u) \neq Y$. If $j = 0$, then we get a contradiction applying Property 2 on the edge $\{u, x_1\}$. Indeed, either $\mathcal{B}(x_1)$ is an ancestor in T of $\mathcal{B}(u)$ and $x_1 \in \mathcal{B}(u)$, contradicting $j = 0$, or $\mathcal{B}(u)$ is an ancestor of $\mathcal{B}(x_1)$. In this latter case, as the P intersects Y and as $\mathcal{B}(u) \neq Y$, the sub-path $x_1, \ldots, x_p$ must intersect $\mathcal{B}(u)$ in some x_i with $i \geqslant 1$, contradicting again $j = 0$. So we have $0 < j < p$, and thus $\mathcal{B}(x_j) \notin Q$. Now, $\mathcal{B}(x_j)$ must be an ancestor in T of $\mathcal{B}(u)$, as $x_j \in \mathcal{B}(u)$. Since $\mathcal{B}(x_j) \notin Q$, it follows that $\mathcal{B}(x_j)$ is an ancestor in T of all the ancestors of $\mathcal{B}(u)$ in Q. In particular x_j belongs to all the ancestors of $\mathcal{B}(u)$ in Q, so $x_j \in Y$ as claimed.

Let k be the largest index such that $x_k \in Y$. As $d_G(x_j, x_k) \leqslant \delta$, we have $k \leqslant 2\delta$. W.l.o.g. $k < p$, since otherwise $p \leqslant 2\delta$ and we are done. As $p > k$, the vertex x_{k+1} exists. Let $Y' = \mathrm{NCA}_T(\mathcal{B}(x_{k+1}), \mathcal{B}(x_p))$.

Claim 3 $Y' \in Q$, *and* $x_k \in Y'$.

First observe that for every $t > k$, $\mathcal{B}(x_t)$ is a descendant of Y. Otherwise, as the sub-path $x_t, \ldots, x_p$ must intersect Y, then Y would contain some x_i with $i > k$, contradicting the definition of k. In particular, $\mathcal{B}(x_{k+1})$ is a descendant of Y. As $\mathcal{B}(x_{k+1})$ is descendant of Y', it follows that Y' is a descendant of Y, and thus Y' is on the path in T from Y to $\mathcal{B}(x_p)$. I.e., $Y' \in Q$ as claimed. Let us apply Property 2 on the edge $\{x_k, x_{k+1}\}$. $\mathcal{B}(x_k)$ is ancestor of Y ($x_k \in Y$), and thus $\mathcal{B}(x_k)$ is ancestor of $\mathcal{B}(x_{k+1})$, and thus $x_k \in \mathcal{B}(x_{k+1})$. Thus x_k belongs to all the bags of the path in T from Y to $\mathcal{B}(x_{k+1})$. In particular, $x_k \in Y'$ completing the proof of the claim.

The sub-path $x_{k+1}, \ldots, x_p$ must intersect Y', say in some vertex x_ℓ. As $\ell > k$, $\mathcal{B}(x_\ell)$ is a descendant of Y. It is also an ancestor of Y'. Thus, $\mathcal{B}(x_\ell)$ belongs to the path in T from Y to Y', i.e., $\mathcal{B}(x_\ell) \in Q$. Since $\mathcal{B}(x_i) \notin Q$ for $i < p$, it turns out that $\ell = p$. We conclude with the fact that $d_G(x_k, x_\ell) \leqslant \delta$, thus $p = \ell \leqslant k + \delta \leqslant 3\delta$.

Therefore, $|E(G')| < (3\delta + 1)n \log n$, completing the proof of Lemma 2. $\square$

Theorem 1 directly follows from lemmas 1 and 2.

4 Additive Spanner with a Linear Number of Edges

We present in this section an algorithm to construct for any tree-length δ graph an additive 6δ-spanner with $O(\delta n)$ edges. As for every graph, the tree-length is at most half the chordality, it follows that this spanner is as well an additive $3k$-spanner (or $(3k-3)$-spanner of odd k) with $O(kn)$ edges if the graph is k-chordal. This latter construction is far from the optimal. Indeed, Chepoi et al [CDY03] have presented an algorithm which computes, for any graph G of chordality k, an additive $(k+1)$-spanner with $O(n)$ edges.

The CDY's algorithm, and our algorithm as well, is based on a *Layering-tree*, technique we present in Section 4.1 with the CDY's algorithm. Our variant is detailed in Section 4.2. In Section 4.3, we compare both algorithms and show that, there are tree-length 3 graph for which the CDY's algorithm returns an additive $\Omega(n^{1/3})$-spanner whereas our algorithm guarantees an additive 18-spanner[7] with $O(n)$ edges.

4.1 Layering-Tree and the CDY's Algorithm

Let G be a graph with a distinguished vertex s. For every integer $i \geqslant 0$, we define $L^i := \{u \in V(G) \mid d_G(s,u) = i\}$. A *layering partition* of G is a partition of each set L^i into $L^i_1, \ldots, L^i_{p_i}$ such that $u, v \in L^i_j$ if and only if there exists a path from u to v using only intermediate vertices w such that $d_G(s,w) \geqslant i$.

Let H be the graph whose vertex set is the collection of all the parts L^i_j. In H, two vertices L^i_j and $L^{i'}_{j'}$ are adjacent if and only if there exists $u \in L^i_j$ and $v \in L^{i'}_{j'}$ such that u and v are adjacent in G (see Fig. 1 for an example). The vertex s is called the *source* of H.

Lemma 3 [CD00] *The graph H, called* layering-tree *of G, is a tree and is computable in linear time.*

In each part X of H, a special vertex r_X is distinguished[8] (grayed vertices on Fig. 1(a)).

The CDY's spanner of G is composed of the following edges: (see Fig. 1(c)):

1. all the edges of a shortest path spanning tree S of G rooted at s, and
2. for every vertex u of a part X of H, the edge $\{x,y\}$ (if it exists) such that x is the nearest ancestor of u in S with a neighbor y ancestor of r_X.

4.2 Additive $O(\delta)$-Spanner with $O(\delta n)$ Edges

Theorem 2 *Every n-vertex graph of tree-length δ has an additive 6δ-spanner with $O(\delta n)$ edges.*

[7] Actually, on the counter-example, it produces an additive 3-spanner.

[8] In the CDY's algorithm r_X is fixed by a method we do not detail here. Here, we consider that any vertex of X can be chosen, it is simpler but more powerful.

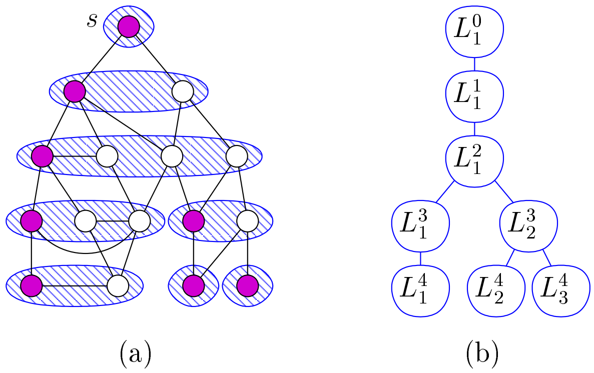

Figure 1: (a) A 5-chordal graph, (b) a layering-tree of it, and (c) the CDY's spanner.

The construction is also based on the layering-tree H of G, presented in Section 4.1, and on the following result:

Lemma 4 *Let H be a layering-tree of G. For every part L_j^i, $\mathrm{diam}_G(L_j^i) \leqslant 3\delta$. Moreover, for every δ, this bound is best possible.*

Proof. Let T be a tree-decomposition of G of length δ, W.l.o.g. T is supposed rooted at a bag containing s, the source of H. Let $u, v \in L_j^i$ be two vertices of G. Let us show that $d_G(u,v) \leqslant 3\delta$.

Let P_1, P_2 be two shortest paths from s to u and from s to v. Each on these paths intersect the bag $X = \mathrm{NCA}_T(\mathcal{B}(u), \mathcal{B}(v))$. Let x (resp. y) be the closest from s vertex in $P_1 \cap X$ (resp. in $P_2 \cap X$) (see Fig. 2(a).

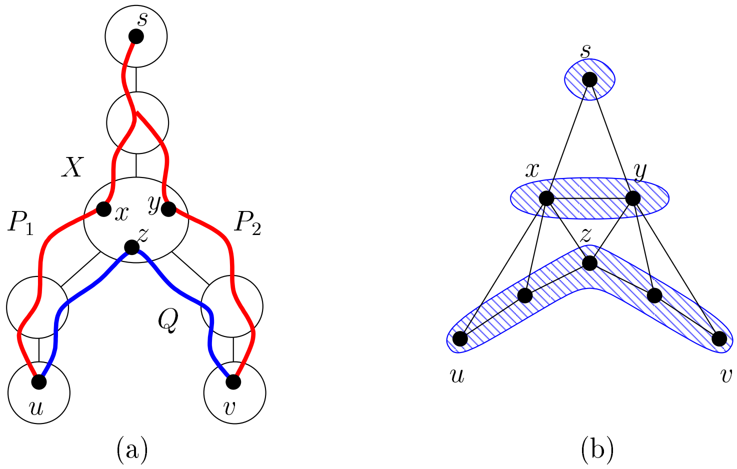

Figure 2: If u, v belong to a same partition then $d_G(u,v) \leqslant 3\delta$.

Moreover, as u, v are both in L_j^i, there exists a path Q from u to v using only intermediate vertices w such that $d_G(s,w) \geqslant i$. Q intersects X at a vertex z. Note that $d_G(s,u) = i = d_G(s,x)+d_G(x,u)$ and $d_G(s,z) \leqslant d_G(s,x)+\delta$. So, $d_G(s,z) \leqslant i+\delta-d_G(x,u)$. If $d_G(x,u) \geqslant \delta+1$ then $d_G(s,z) \leqslant i-1$: a contradiction since $z \in Q$. So, $d_G(u,x) \leqslant \delta$. Similarly, $d_G(v,y) \leqslant \delta$. As $d_G(x,y) \leqslant \delta$, by the triangle inequality, $d_G(u,v) \leqslant 3\delta$ as claimed.

This bound in best possible for each $\delta \geqslant 1$. For $\delta = 1$, the graph depicted on Fig. 2(b) is chordal, u, v belong to the same part and $d_G(u,v) = 3$. Replacing each edge by a path of length δ, the tree-length of this subdivision increases to δ, u, v still belong to the same part and are at distance 3δ. □

The spanner satisfying Theorem 2 is simply the graph defined by $G' := S \cup \bigcup_{X \in V(H)} S_X$, where S is a shortest path tree spanning G, H a layering-tree of G, and S_X a shortest path tree spanning X rooted at an arbitrary vertex $r_X \in X$.

Lemma 5 *G' is an additive 6δ-spanner of G.*

Proof. Let u, v be two vertices of G, and let U, V be the two parts of H containing respectively u and v. Let us show that every path from u to v must intersect the part $X = \mathrm{NCA}_H(U,V)$.

This clearly holds if $X = U$ or $X = V$. If $X \neq U$ and $X \neq V$, (so in particular $U \neq V$), then, by definition of H, every path intermediate vertex of a path from u to v must intersect an ancestor of U and of V. So, by induction, it must intersect the nearest common ancestor of U and of V, X.

So any shortest path from u to v decomposed into a shortest path from u to some $u' \in X$, a shortest path from u' to some $v' \in X$, and then a shortest path from v' to v: $d_G(u,v) = d_G(u,u') + d_G(u',v') + d_G(v',v)$. We observe that $d_G(u,u') = d_H(U,X)$. As G' contains a shortest path spanning tree of G rooted at s, it follows that $d_H(U,V) = d_{G'}(u,u')$, and finally, $d_G(u,u') = d_{G'}(u,u')$. Similarly, $d_G(v,v') = d_{G'}(v,v')$, and thus $d_{G'}(u,u') + d_{G'}(v,v') \leqslant d_G(u,v)$.

Using the tree S_X contained in G', and by Lemma 4, we have $d_G(u',v') \leqslant d_{G'}(u',r_X) + d_{G'}(r_X,v') \leqslant 6\delta$. Therefore, we obtain:

$$d_G(u,v) \leqslant d_{G'}(u,u') + d_G(u',v') + d_{G'}(v',v) \leqslant d_G(u,v) + 6\delta \ .$$

□

Lemma 6 *G' has $O(\delta n)$ edges.*

Proof. S has $n-1$ edges. Each vertex u of G belongs to exactly one vertex X of H, and the path from u to r_X is of length at most 3δ (Lemma 4). So S_X contains at most $|X|-1$ leaves and $3\delta(|X|-1)$ edges. Over all, the number of edges of G' is at most

$$n - 1 + \sum_{X \in V(H)} 3\delta(|X|-1) \ = \ O(\delta n) \ .$$

□

4.3 Counter-Example

Theorem 3 *There is a graph of tree-length* 3 *and with* $n + o(n)$ *vertices for which the CDY's algorithm, for any choice of the special vertices, constructs an additive* $\Omega(n^{1/3})$*-spanner with* $O(n)$ *edges.*

The remaining of this section is devoted to the proof of Theorem 3.

The *Cartesian product* of two graphs A and B is the graph denoted by $A \times B$ such that $V(A \times B) = \{(x, y) \mid x \in V(A), y \in V(B)\}$, and $E(A \times B) = \{((x, x'), (y, y')) \mid (x = x'$ and $(y, y') \in E(B))$ or $(y = y'$ and $(x, x') \in E(A))\}$. E.g., the mesh is the Cartesian product of two paths. Let K_t and P_t denote respectively the complete graph and the path with t vertices.

We set $D_p = K_p \times K_p$. The counter-example, denoted by G_0, is the graph $D_t \times P_{t-1}$, so composed of $t - 1$ copies $D_t^1, \ldots, D_t^{t-1}$ of D_t, with an extra vertex s connected to all the vertices of D_t^1 (see Fig. 3). To every vertex u of G_0, $u \neq s$, we denote by $P(u)$ the copy of the path P_{t-1} containing u. Hereafter, we set $t := \lceil n^{1/3} \rceil$, so that G_0 has $t^2(t-1) + 1 = n + O(n^{2/3})$ vertices.

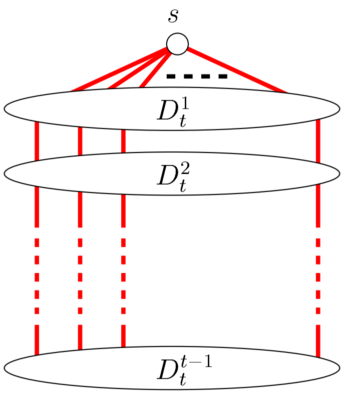

Figure 3: Counter-example G_0.

A subgraph H of a graph G is *isometric* if $d_H(x, y) = d_G(x, y)$, for all $x, y \in V(H)$. It is a natural generalization of induced subgraph (any isometric subgraph is clearly an induced subgraph). We have:

Lemma 7 [DG03] *The tree-length of any isometric subgraph of* G *is no more than the tree-length of* G*.*

Let u, v, w be three vertices of some D_t^i inducing a path of length two (since D_t is of diameter two, such vertices exists). We check that the graph induced by the vertices of the three paths $P(u)$, $P(v)$, $P(w)$ is a isometric mesh of G_0. This mesh has $t-1$ rows and 3 columns.

Lemma 8 *G_0 has chordality at least $2t = \Omega(n^{1/3})$, and tree-length 3 for $t \geqslant 5$.*

Proof. In a $(t-1) \times 3$ mesh, the perimeter is an induced cycle of the mesh of length $2t$. Since this mesh is an isometric subgraph of G_0, it follows that G_0 is of chordality at least $2t$.

It is proved in [DG03] that the tree-length of the mesh with p rows and q columns is $\min\{p, q\}$ if $p \neq q$ or p is even, and is $p-1$ otherwise. In particular, the $(t-1) \times 3$ mesh has tree-length 3 if $t \geqslant 5$. By Lemma 7, G_0 has tree-length at least 3 for $t \geqslant 5$.

We obtain a tree-decomposition of G_0 of length 3 by considering a path $X_0, X_1, \ldots X_{t-2}$ where $X_0 = \{s\} \cup V(D_t^1)$, and $X_i = V(D_t^i) \cup V(D_t^{i+1})$ for $i \geqslant 1$. □

A *dominating set* of a graph G is a set of vertices R such that for every vertex u of G either $u \in R$ or u is adjacent to a vertex of R.

Lemma 9 *If R is a dominating set of D_t, then $|R| \geqslant t$.*

Proof. The graph D_t is the union of two disjoint sets of cliques $\mathcal{K}_1$ and $\mathcal{K}_2$, each one composed of t disjoint copies of K_t, so that every edge belongs either to a clique of $\mathcal{K}_1$ or of $\mathcal{K}_2$. Every clique of $\mathcal{K}_1$ intersects each clique of $\mathcal{K}_2$ and vice et versa. Assume $|R| < t$. By the Pigeon Hole Principle, there is a clique $A \in \mathcal{K}_1$ with no vertices of R. Similarly, there is a clique $B \in \mathcal{K}_2$ with no vertices of R. The cliques A and B share exactly one vertex, say u (otherwise there would exists an edge that belongs to a clique of $\mathcal{K}_1$ and to a clique of $\mathcal{K}_2$). All the incident edges of u belongs either to A or to B. It follows that u is not adjacent to any vertex of R: a contradiction. □

Proof of Theorem 3. Let G' be the spanner obtained by CDY's algorithm applied on the source s of G_0. The parts of H are the set $L^0 = \{s\}$, and $L^i = V(D_t^i)$ for $i \geqslant 1$. The spanning tree S rooted at s used in G' contains exactly the edges incident to s and the edges of the paths P_{t-1}. No edge of any D_t^i is contained in S.

We assume that a special vertex r_i has been arbitrarily selected for each part L^i of H, and let $R = \{r_1, \ldots, r_{t-1}\}$.

Let $u \in L^i$ be a vertex of G_0, $i \neq 0$. Observe that if u and r_i are not adjacent in G_0, then there is no edge in G_0 (and thus in G') between $P(u)$ and $P(r_i)$. Let R' be the projection of R on D_t^{t-1}: $R' = \left\{u \in D_t^{t-1} \mid V(P(u)) \cap R \neq \varnothing\right\}$. $|R'| = |R| = t-1$, so R' is not a dominating set of D_t^{t-1} (Lemma 9).

Let v be a vertex of D_t^{t-1} with no neighbors in R', and let $r' \in R'$. From the above observation, in G', there is no edge between $P(v)$ and $P(r')$. During the second phase of the CDY's algorithm, only edges incident to r_i are added, for all $i \geqslant 1$. It follows that every vertex of $P(v)$ has no incident edges in G', excepted those of $P(v)$. So, $d_{G'}(v, r') \geqslant 2(t-1)+2 = \Omega(n^{1/3})$ whereas $d_{G_0}(v, r') = 2$. G' is an additive $\Omega(n^{1/3})$-spanner. □

5 Lower Bound

Let $m(n, g)$ be the maximum number of edges contained in a graph with n vertices and of girth at least g. It is clear that there exist at least an n-vertex graph for which every additive $(g-3)$-spanner (or multiplicative $(g-2)$-spanner) needs $m(n, g)$ edges. Indeed, any graph G of girth g and of $m(n, g)$ edges has no proper additive $(g-3)$-spanner: removing any edge $\{u, v\}$ of G implies that $d_H(u, v) \geqslant g - 1 = d_G(u, v) + g - 2$ and thus that H is not an additive $(g-3)$-spanner of G.

Theorem 4 *For each $\delta \geqslant 1$, there exists a graph of $n + 3\delta - 2$ vertices and of tree-length δ for which every multiplicative δ-spanner (and thus every additive $(\delta - 1)$-spanner) needs $m(n, \delta + 2) + 3\delta - 1$ edges.*

Proof. Consider a graph G with n vertices, a girth at least $\delta + 2$, and with $m(n, \delta + 2)$ edges. We have $\mathrm{diam}(G) \leqslant \delta$. Indeed, otherwise G has two vertices, say u and v, at distance $\delta + 1$. So augmenting G by the edge u, v would provide a graph with n vertices, a girth at least $\delta + 2$, and with $m(n, \delta + 2) + 1$ edges: a contradiction with the definition of $m(n, \delta + 2)$. So G is of tree-length at most $\mathrm{diam}(G) \leqslant \delta$.

Now we construct a graph G^* obtained from G by selecting an edge of G, say $\{u, v\}$, and by adding a path of length $3\delta - 1$, so that G^* contains a cycle C of length 3δ. The graph G^* has $n + 3\delta - 2$ vertices, a girth at least $\delta + 2$, and $m(n, \delta + 2) + 3\delta - 1$ edges. Again, G^* does not contain any proper multiplicative δ-spanner.

The tree-length of G^* is exactly δ observing that the tree-length of a graph composed of two subgraphs, say G and C, sharing a vertex or an edge is the maximum between the tree-length of G and the tree-length of C (because G and C are isometric subgraphs, and the common vertex or edge can be used to combined both optimal tree-decompositions). As shown in [DG03], the tree-length of a cycle of length $k = 3\delta$ is $\lceil k/3 \rceil = \delta$. □

An Erdös Conjecture [Erd64] claims existence of n-vertex graphs with $\Omega(n^{1+1/k})$ edges and of girth at least $2k + 2$. This has been proved only for $k = 1, 2, 3$ and $k = 5$. It is known however that there are graphs of girth at least $2k + 2$ with $\Omega(n^{1+1/(2k)})$ edges. From Theorem 4, we have:

Corollary 1 *For every constant $\delta \geqslant 1$, there are graphs with $O(n)$ vertices and tree-length δ for which every multiplicative δ-spanner requires $\Omega(n^{1+\epsilon})$ edges, where $\epsilon \geqslant 1/\lceil \delta/2 \rceil$ for*

$\delta \leqslant 6$. Moreover, for every δ, $\epsilon \geqslant 1/\Theta(\delta)$, where the best current lower bound on ϵ is given by the table below.

Proof. For each fixed integer $k \geqslant 1$, let $f(k)$ be the largest real such that there exists an n-vertex graph of girth at least $2k+2$ and with $\Omega(n^{1+f(k)})$ edges. We have $m(n, 2k+2) = \Omega(n^{1+f(k)})$.

Consider the worst-case graph G_δ given by Theorem 4. It has at most $5n/2$ vertices (recall that $\delta < n/2$ as the chordality of a graph is at most $n-1$), and at least $m(n, \delta+2)$. Note that $m(n, \delta+2) \leqslant m(n, \delta+1)$. So, G_δ has at least $m(n, 2\lceil \delta/2 \rceil + 2) = \Omega(n^{1+f(\lceil \delta/2 \rceil)})$ edges.

It is known that $f(k) = 1/k$ for all $k \geqslant 1$, if the Erdös's Conjecture holds. The following table summarizes the best known results on $f(k)$. Complete references can be found in [TZ01].

$k = \lceil \delta/2 \rceil$	$f(k)$
$1, 2, 3, 5$	$= 1/k$
4	$\geqslant 1/(k+1)$
$6, 7$	$\geqslant 1/(k+2)$
$k = 2r,\ r \geqslant 4$	$\geqslant 1/(3k/2 - 1)$
$k = 2r-1,\ r \geqslant 5$	$\geqslant 1/(3k/2 - 3/2)$

□

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