Tree-Decomposition of Graphs with small Diameter Bags*

Yon Dourisboure† Cyril Gavoille*

27th June 2003

Abstract

This paper deals with the *length* of a Robertson-Seymour's tree-decomposition. The *tree-length* of a graph denotes the largest distance between two nodes of a bag of a tree-decomposition, minimized over all tree-decompositions of the graph. The study of this invariant may be interesting in its own right as the class of bounded tree-length graphs includes (but is not reduced to) bounded chordality graphs (like chordal graphs, permutation graphs, AT-free graphs, etc.). For instance, we show that the tree-length of any outerplanar graph is $\lceil k/3 \rceil$, where k is the chordality of the graph.

The introduction of this parameter is motivated by the design of compact routing tables with near-optimal route length.

Keywords: tree-decomposition, tree-width, chordality, small separator.

1 Introduction

The notions of tree-decomposition and of tree-width are very rich concepts with many algorithmic implications. Many NP-complete problems are polynomial for graphs of bounded tree-width [1, 2]. In addition, the notion of

*Supported by the European Research Training Network COMBSTRU-HPRN-CT-2002-00278

†LaBRI, Université Bordeaux. {*Yon.Dourisboure,gavoille*}*@labri.fr*

tree-width is in the kernel of the graph minor theory [4]. Intuitively tree-width can be seen as a measure of "how far" a graph is from a tree. The smaller the tree-width, the more tree-like the graph is.

However many problems have still polynomial solutions on large class of graphs even when the size of the separator is large (or large tree-width), like interval graphs or more generally chordal graphs. Actually, the "shape" of the separator plays an important role. Introducing *well-separated* class of graphs Katz, Katz and Peleg [7] showed that the graphs of this class enjoy distance labeling with short labels, namely the property that all the nodes of the graph can be labeled with $O(\log^2 n)$ bit binary strings so that the distance between two nodes can be retrieved from the two node labels, without any other source of information [8]. They showed that interval and permutation graphs fall into that class. The same phenomenon occurs for bounded clique-width graphs, for which Courcelle and Vanicat [3] showed that $O(\log^2 n)$ bit distance labeling scheme exists, whereas the tree-width of these graphs is unbounded in general.

An application different from distance computation which is also related to global structure of the graph is the field of routing with succinct routing tables. Tree-width k graphs have shortest-path routing scheme with addresses and routing tables of $O(k \log^2 n)$ bits per nodes [9]. However, when diameter of the separator is small better results exist. For instance [5] proposed for chordal graphs (a class including interval graphs, and whose all separators are cliques) a routing scheme independent of separator size (unbounded for these graphs). Addresses and routing tables are of $O(\log^3 n / \log\log n)$ bits per nodes. This scheme guarantees a one-sided error of 2 on the route length. Decreasing the error or reducing the memory requirements is still an open problem even for this class of graphs.

1.1 Our results

The *tree-length* of a graph denotes the largest distance between two nodes of a bag of a tree-decomposition, minimized over all Robertson-Seymour's tree-decompositions of the graph. In Section 2 we discuss about the gap between the tree-length and the chordality of a graph. We prove that the tree-length of any outerplanar graph of chordality k is exactly $\lceil k/3 \rceil$ and we calculate the tree-length of the mesh. A natural question is: "for every graph G does there exist a tree-decomposition T of G such that the length and the width of T are close to the tree-length and the tree-width of G respectively?". More formally a tree-decomposition T is α-optimal if the length and the width of T are both at most α times the tree-length and the tree-width of the graph. In Section 3 we show that there are graphs with n nodes for which every α-optimal tree-decomposition requires $\alpha = \Omega(n^{1/5})$. In the long version, we show a polynomial 3-approximation algorithm running in linear time.

1.2 Definition

We need the notion of *tree-decomposition* used by Robertson and Seymour in their work on graph minors [10].

Definition 1 *A tree-decomposition of a graph G is a tree T whose nodes, called* bags*, are subsets of $V(G)$ such that:*

1. $\bigcup_{X \in V(T)} X = V(G)$;
2. *for all $\{u, v\} \in E(G)$, there exists $X \in V(T)$ such that $u, v \in X$; and*
3. *for all $X, Y, Z \in V(T)$, if Y is on the path from X to Z in T then $X \cap Z \subseteq Y$.*

For every induced subgraph H of G, the *diameter of H in G* is $\mathrm{diam}_G(H) = \max_{x,y \in V(H)} d_G(x, y)$, where $d_G(x, y)$ denotes the distance in G between x and y. Observe that $\mathrm{diam}_G(H)$ can be constant even if H is not connected. By extension, $\mathrm{diam}_G(X) = \mathrm{diam}_G(G[X])$ for every subset $X \subseteq V(G)$, where $G[X]$ denotes the subgraph of G induced by X.

Let T be a tree-decomposition of a graph G. The *width of T* is $\mathrm{width}(T) = \max_{X \in T} \{|X| - 1\}$. The *length of T* is $\mathrm{length}(T) = \max_{X \in T} \mathrm{diam}_G(X)$. The *tree-width* and the *tree-length of G* are respectively $\min_T \mathrm{width}(T)$ and $\min_T \mathrm{length}(T)$, where the minimum is taken over all tree-decompositions of G.

A graph is k-chordal if the length of the longest induced cycle is at most k. The *chordality of G* is the smallest integer k such that G is k-chordal. Graphs of tree-length 1 are exactly 3-chordal graphs.

2 Tree-length and chordality

An *isometric subgraph* H of G is a connected subgraph of G such that $d_H(x, y) = d_G(x, y)$, for all $x, y \in V(H)$. It is a natural generalization of induced subgraph (any isometric subgraph is clearly an induced subgraph). In particular, in the case H is a cycle of length k, then k must be at least the girth of G and at most the chordality of G. Isometric subgraph is a tool for tree-length as minor subgraph is for tree-width.

Proposition 1 *For every isometric subgraph H of G, the tree-length of H is no more than the tree-length of G.*

Proposition 2 *Every cycle of length k has tree-length $\lceil k/3 \rceil$.*

More generaly we have:

Theorem 1 *Every outerplanar graph of chordality k has tree-length $\lceil k/3 \rceil$. Moreover, a tree-decomposition of width 2 and of length $\lceil k/3 \rceil$ can be constructed in linear time.*

The gap between the chordality and the tree-length can be large. For instance, a $p \times q$ mesh (p rows, q columns) has chordality $\Omega(pq)$. However,

Theorem 2 *The tree-length of the mesh with p rows and q columns is $p - 1$ if $p = q$ is odd, or is $\min\{p, q\}$ otherwise.*

However chordality is an upper bound for the tree-length:

Theorem 3 [**6**] *Every k-chordal graph has tree-length at most $k/2$.*

3 Optimal tree-decomposition

Let us define an *α-optimal* tree-decomposition for a graph G as a tree-decomposition of G whose width is at most αk and whose length is at most $\alpha\delta$, where k and δ are respectively the tree-width and the tree-length of G. Theorem 1 says nothing else that every outerplanar graph has an 1-optimal tree-decomposition, that is a tree-decomposition optimizing both parameters, the width and the length.

At this step, an interesting question is to know whether every graph has a $O(1)$-optimal tree-decomposition? We answer this question in the negative.

Theorem 4 *There exists a graph with at most n nodes on which any α-optimal tree-decomposition requires $\alpha \geqslant \frac{1}{2}n^{1/5} - \frac{1}{2}$.*

An interesting question would be to know whether an $o(n)$ bound on α exists for every n-node graphs. Before proving Theorem 4, let us denote by $M(p,q,r)$ the graph composed of a $p \times q$ mesh (p rows, q columns), $p \leqslant q$, in which every edge of a column is replaced by a path of length r. The number of nodes of $M(p,q,r)$ is $pqr - q(r-1)$.

Lemma 1 *Let T be any tree-decomposition of $M(p,q,r)$. Then, if* $\mathrm{length}(T) < (p-1)r$*, then* $\mathrm{width}(T) \geqslant q$.

By choosing $p = n^{1/5}$ and $q = r = n^{2/5}$, Lemma 1 implies Theorem 4.

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