Interval Routing for Generalized Hypercube-Like Graphs

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In this article we study trade-offs between the size of the routing tables and the length of the routes for several Hypercube-like topologies: generalized Butterfly, generalized Wrapped-Butterfly, and generalized Cube-Connected Cycles.

Keywords: compact routing tables, Hypercube-like topologies

1 Generalities

Given a parallel or distributed system, the interconnection network ensures the communication between the processors, the terminal nodes. Each intermediate node has a router, a dedicated co-processor which forwards the messages between processors through the links of the underlying topology. The routers run a distributed algorithm which specifies the way to go from a node of the network to another. This algorithm is described by a routing function.

Once a router receives a message, it looks at its header and checks the destination of the message, and finds the output port that will be used to forward the message towards to next intermediate node up to its destination. The output port is a number local to each router and associated to each link between routers. A standard way to implement such algorithms is to use a *routing table*. To find the output port, the router consults a table which is kept in its local memory. For each destination, this table returns the output port number through which the message can be forwarded.

A simple method to organize this table is to associate to each destination the output port number which can serve it. This method is simple, but it is very memory expensive. It requires $O(n\log d)$ bits to maintain the routing table in each node of degree d, n being the number of nodes of the underlying graph representing the network.

For a large or growing network, this method is not feasible. It is interesting to look for another method in order to reduce the size of the data structure stored by the routers, and used for the routing task. In the field of compact routing, several methods and strategies were introduced to reduce the router memory size, as separator-based strategies routing schemes [FJ86, FJ90], hierarchical routing schemes [PU89, ABNLP89], prefix routing [BTvL90], boolean routing [FG97], and interval routing [SK85, vLT87]. We focus our work on this last technique that offer a more compact data structure for routing tables.

2 Interval Routing Scheme

The interval routing was introduced by Santoro and Khatib in [SK85], and then extended in [vLT87] by van Leeuwen and Tan. It has been intensively studied in recent years, and an overview can be found in [Gav00b]. This kind of routing supports a hardware implementation, for instance on the C104 routing chip [Inm91] of INMOS. This method consists to find a global labeling of the nodes with integers taken from $\{1,\dots,n\}$, and, given a routing table, to group in the smallest set of intervals the destination labels using the same output port in each node. An interval means a set of consecutive integers, the addresses 1 and n being considered as consecutive. If there exists a routing table such that each set of destination labels using a same output port can be grouped with at most k intervals, we deal with a k-interval routing scheme for this network, k-IRS for short. A k-IRS can be implemented with $O(kd\log n)$ bits per node by storing the interval boundaries of the destinations. Actually, this naive coding can be compressed into $O(kd\log(n/k))$ bits [Gav00b]. In a sense, interval routing is a compact implementation of routing tables. One can hope to store only $O(k)$ integers per node for bounded degree networks using k-IRS, whereas standard routing tables require $O(n)$ integers. The parameter k is called *compactness* of a routing table.

3 Motivations and Related Works

Many works try to determine routing tables with minimum compactness under several assumptions on the quality of the routing measured in term of length of the routes: *shortest path* routing [FvLMS98, GG98, GP99, KK96, KRŠ00] (all the routes are shortest paths), bounded *stretched* routing [EGP98, FGS94, NO97] (the ratio between the length of the route and the optimal length), bounded *dilation* routing [EMZ99, Gav00a, KRŠ00, NO97, TL97] (the length of the longest route for the network), etc. See [Gav00b].

Actually, several works show that there is no hope, for general graphs, to design routing tables providing simultaneously shortest paths and low compactness. And this is partially true even for specific graphs. For instance, in [KRŠ00], it is shown that binary Butterfly and binary Cube-Connected Cycles, that are strongly structured graphs, have no shortest path k-IRS for $k \le \sqrt{n/\log_2 n}$. On the other side the same authors showed that every graphs of diameter D have a k-IRS with $k \le \sqrt{n\ln n} + O(1)$ and with a dilation at most $\lceil 1.5D \rceil$. In [EGP98], it is shown that every graph has a k-IRS with $k \le 2.0\sqrt{n\log_2 n}$ and with a stretch factor at most 5 and less than 3 on the average[†], and with still a dilation at most $\lceil 1.5D \rceil$.

However, all these results involve a quite large compactness, $O(\sqrt{n\log n})$ intervals. If we insist on constant compactness IRS many problems are left open concerning the length of the routes. For instance, there is no known construction of a graph on which every $O(1)$-IRS has unbounded stretch factor. In [KRŠ00], it is left open the question to prove that every planar graph of diameter D has a $O(1)$-IRS with dilation at most $\lceil (1+\varepsilon)D \rceil$, for every constant $\varepsilon > 0$. A weaker version of this question has been proposed in [Gav00b]: does every planar graph of diameter D support some $O(1)$-IRS with dilation bounded by $\lceil 1.5D \rceil$? All these problems are still open and rather difficult (the latter question is related to the size of the $\lceil D/2 \rceil$-dominating set in a planar graph, and is also open, cf. [CV00, GRS00]).

4 Our Results

In this paper we concentrate our attention on IRS of low compactness (at most 3) for: generalized Butterfly (BF_d^δ), generalized Wrapped-Butterfly (WBF_d^δ), and generalized Cube-Connected Cycle (CCC_d^δ) graphs. Roughly speaking, they have $n = \Theta(d\delta^d)$ vertices, a diameter $D = \Theta(d)$, and the degree of every vertex is $\Theta(\delta)$. Here we consider the generalized form of these graphs (they are defined on a larger alphabet than the binary one). In [KRŠ00], only the binary case (i.e., $\delta = 2$) was considered.

Denoted by BF_d^δ, the Butterfly graph of arity δ and dimension d consists of $d+1$ levels, each level containing δ^d vertices. The vertex set consists of the set $\{(\alpha, X) \mid 0 \le \alpha \le d, X \in \mathcal{A}^d\}$ where $\mathcal{A} = \{0, \ldots, \delta - 1\}$ is the alphabet. A vertex (α, X) with $X = x_1x_2\ldots x_d$ is adjacent to the vertex (β, Y) with $Y = y_1y_2\ldots y_d$ if $\beta = \alpha + 1$ and $x_i = y_i$ for every $i \ne \alpha$. The WBF_d^δ is defined similary excepted that vertices of levels 0 end d are identified, that is $(0, X)$ and (d, X) are the same vertex for every $X \in \mathcal{A}^d$. Denoted CCC_d^δ, the Cube Connected Cycle of arity δ and dimension d has the same set of vertices than WBF_d^δ, and two vertices (α, X) and (β, Y) are connected if either $\beta = \alpha + 1$ and $X = Y$, or $\beta = \alpha$ and $x_i = y_i$ for every $i \ne \alpha$. A Butterfly BF_2^3 and a Cube Connected Cycle CCC_3^2 are depicted in Figure 1. In this representation, a vertex (α, X) is located at the column of level α, and at the row labeled by $x_d \ldots x_1$, if $X = x_1x_2\ldots x_d$. (In other words, the least significant digit, x_1, is written first in X.)

All these graphs are Hypercube-like: we obtain a d-dimensional δ-ary Hypercube (a cartesian product of d copies of K_δ) if all the edges of the same rows are contracted, cf. [HPS96]. From the binary case, it is

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[†] The average of the stretch factor taken over all pairs of vertices.

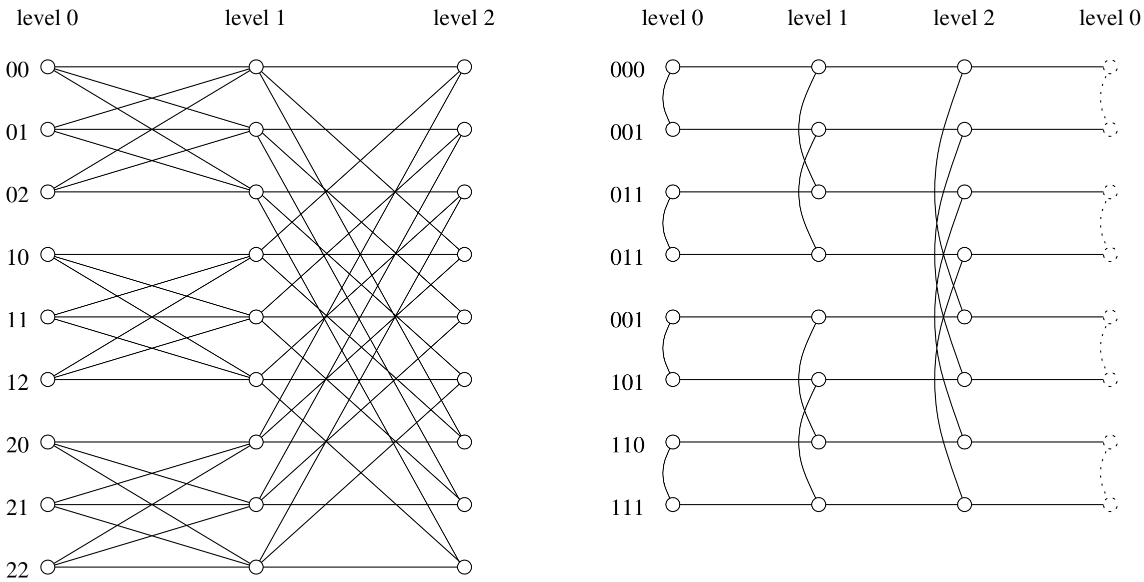


Figure 1: A ternary Butterfly (BF^3_2) and a binary Cube-Connected Cycle (CCC^2_3).

easy to extend the diameter to the δ-ary case. The diameter of BF^δ_d, WBF^δ_d, and CCC^δ_d are respectively $2d$, $d+\lfloor d/2 \rfloor$, and $2d+\max\{2,\lfloor d/2\rfloor\}-2$ that is $2d+\lfloor d/2\rfloor-2$ for $d>3$.

Our results concern trade-offs between compactness and dilation, and are summarized in Table 1. (Recall that the compactness denote the maximum number of intervals per edge, and the dilation is the length of the longest route generated by the routing scheme.). Note that it is possible to construct for every graph of diameter D a 1-IRS with dilation $2D$ by the use of a spanning tree, cf. [SK85]. So, it is natural to express the dilation by a function of the diameter, D.

Graph	Compactness	Dilation
BF^δ_d	$k=2$	$\le \frac{3}{2}D$
WBF^δ_d	$k=1$	$\le \frac{5}{3}D-\frac{2}{3}$
WBF^δ_d	$k=2$	$\le \frac{4}{3}D-\frac{1}{3}$
CCC^δ_d	$k=1$	$\le \frac{8}{5}D+1$
CCC^δ_d	$k=2$	$\le \frac{7}{5}D+1$
CCC^δ_d	$k=3$	$\le \frac{6}{5}D+2$
CCC^2_d	$k=1$	$\le \frac{7}{5}D+1$
CCC^2_d	$k=2$	$\le \frac{6}{5}D+2$

Table 1: Upper bounds on the dilation (as a function of the diameter) for constant compactness.

Moreover we have extended the lower bounds of [KRŠ00] for shortest path k-IRS. We show that for BF^δ_d, WBF^δ_d, and CCC^δ_d, every shortest path k-IRS requires $k=\Omega(\sqrt{n/\log_\delta n})$. On the other side, for every integer $s\ge 1$ and for these three kinds of graphs, we construct a k-IRS with $k\le n^{O(1/s)}$ and with a stretch factor bounded by s, i.e., the length of the route is at most s times the optimal length. Because lack of space, the proof of this last result, the results concerning the CCC_d^δ, and the lower bounds will appear in the full version of the paper.

5 Some Schemes with Low Compactness and Low Dilation

5.1 Preliminaries and notations

An *interval labeling scheme* R on a graph $G=(V,E)$ of n vertices is a routing scheme consisting of a pair $(\mathcal{L},I)$, generated in the preprocessing step, where $\mathcal{L}$ is a *vertex-labeling*, $\mathcal{L}:V\to\{1,\ldots,n\}$, and I is an *arc*[‡]*-labeling*, $I:E\to 2^{\mathcal{L}(V)}$, that satisfy the following condition. For any vertex u, the collection of sets that label all the outgoing arcs of u forms a partition of the name range (possibly excluding u itself[§]). Formally, for every $u\in V$ (hereafter E_u denotes the set of arcs incident to u),

1. $\displaystyle\bigcup_{e\in E_u} I(e)\cup\mathcal{L}(u)=\{1,\ldots,n\}$,
2. $I(e_1)\cap I(e_2)=\emptyset$, for every two distinct arcs $e_1,e_2\in E_u$.

A labeling scheme is said *valid* if for all pairs u,v of vertices of G, there is path $u=u_1,u_2,\ldots,u_t,u_{t+1}=v$ from u to v, such that $\mathcal{L}(v)\in I(u_i,u_{i+1})$ for every $i\in\{1,\ldots,t\}$. The length of the route traversed by a message from u to v according to the valid routing scheme R is denoted by $\rho_R(u,v)=t$. The *dilation* of a routing scheme R is the maximal route length of a path traversed by a message, formally $\max_{u\neq v}\{\rho_R(u,v)\}$.

Given an integer n and a subset $I\subseteq\{1,\ldots,n\}$, define the *compactness of I w.r.t. n* as the smallest integer k such that I can be represented by the union of k intervals $[a,b]$ of consecutive integers from $\{1,\ldots,n\}$, with n and 1 being considered as consecutive (cyclically). The *compactness* of an interval routing scheme $R=(\mathcal{L},I)$ on G is the maximum, over all arcs $e\in E$, of the compactness of the set $I(e)$ labeling e.

For BF_d^δ and WBF_d^δ we use the following notational convenience: $u=(\alpha,X)\equiv x_1\ldots x_\alpha{}^\triangledown x_{\alpha+1}\ldots x_d$ where $X=x_1x_2\ldots x_d$. The neighbors of u is this representation are the vertices obtained by moving the cursor to the left (resp. to the right) and replacing x_α (resp. $x_{\alpha+1}$) with any letter $\ell\in\mathcal{A}$. The edges traversed by a left moving are denoted by $e_\ell^-(u)$ (resp. $e_\ell^+(u)$). Note the moves are done cyclically, so that the vertex represented by $x_1x_2\ldots x_d{}^\triangledown$ is the same than ${}^\triangledown x_1x_2\ldots x_d$ (both correspond to the vertex $(0,X)$).

In all the remaining of the paper, we use upper case letters to denote a sub-word (a sub-sequence of letters, potentially the empty word), and lower case letters to denote a single letter. We denote by $\preceq$ the lexicographic order of words on $\mathcal{A}$, and by $|X|$ the number of letters of the word X on $\mathcal{A}$.

5.2 Vertex-Labeling

The vertex-labeling we use is similar for the three kinds of graphs. It consists to label consecutively the vertices of a same row, from lower level to higher level vertices. The rows are labeled then from the lowest row label (the word 0^d) to the highest row label (the word $(\delta-1)^d$) w.r.t. the relation $\preceq$. It is denoted $\mathcal{L}_w$ for WBF_d^δ and CCC_d^δ, and $\mathcal{L}$ for BF_d^δ. More precisely, $\mathcal{L}_w(\alpha+1,X)=\mathcal{L}_w(\alpha,X)+1$ and $\mathcal{L}(\alpha+1,X)=\mathcal{L}(\alpha,X)+1$. Moreover, if Y is the immediate successor of X for the relation $\preceq$, $\mathcal{L}_w(\alpha,Y)=\mathcal{L}_w(\alpha,X)+d$ and $\mathcal{L}(\alpha,Y)=\mathcal{L}(\alpha,X)+d+1$.

5.3 Interval Routing with Compactness One for WBF_d^δ

Let $u=(\alpha,x_1x_2\ldots x_d)\in V(\mathrm{WBF}_d^\delta)$. We define:

$$\begin{aligned}\mathcal{I}_\ell(u) &= \left\{(\beta,Y\ell x_{\alpha+1}\ldots x_d)\mid \beta\in\{1,\ldots,d\}\text{ and }Y\in\mathcal{A}^{\alpha-1}\right\}\\ \mathcal{K}(u) &= V(\mathrm{WBF}_d^\delta)\setminus\bigcup_{\ell\in\mathcal{A}}\mathcal{I}_\ell(u)\ .\end{aligned}$$

‡ In this paper, graphs are considered as undirected. However, for notational convenience, each edge is seen as the pair of two symmetric arcs.

§ A labeling excluding u from its arc-labels is termed *strict*. In this paper we use strict and non-strict labeling.

The interval routing $\mathcal{R}_1 = (\mathcal{L}_w, I_1)$ on WBF_d^δ is defined by:

$$I_1(e_\ell^-(u)) = \begin{cases} \mathcal{J}_\ell(u) \cup \mathcal{K}(u) & \text{if } \ell = 0 \text{ and } \alpha \le d/2 \\ \mathcal{J}_\ell(u) & \text{if } \ell \in \mathcal{A} \setminus \{0\} \text{ or } \alpha > d/2 \end{cases}$$
$$I_1(e_\ell^+(u)) = \begin{cases} \mathcal{K}(u) & \text{if } \ell = 0 \text{ and } \alpha > d/2 \\ \emptyset & \text{if } \ell \in \mathcal{A} \setminus \{0\} \text{ or } \alpha \le d/2 \end{cases}$$

Observe that $\bigcup_{e\in E_u} I_1(e) = \mathcal{K}(u) \cup \bigcup_{\ell\in\mathcal{A}} \mathcal{J}_\ell(u) = V(\mathrm{WBF}_d^\delta)$, and $\forall e_1, e_2 \in E_u, e_1 \neq e_2 \Rightarrow I_1(e_1) \cap I_1(e_2) = \emptyset$. So, $\mathcal{R}_1$ is a correct interval labeling scheme.

Let us define $\alpha \ominus 1 \equiv \alpha - 1 + d \bmod d$, the precedessor of α modulo d.

Lemma 1 *Let $u = (\alpha, X)$ and $v = (\beta, X)$ be two vertices of* WBF_d^δ. *Then, $\rho_{\mathcal{R}_1}(u,v) \le d-1$.*

Proof. Let $X = x_1 x_2 \ldots x_d$. Observe that $v \in \mathcal{J}_{x_\alpha}(u)$. So the first routing step uses $e_{x_\alpha}^-(u)$ to reach a vertex $u' = (\alpha', X)$, with $\alpha' = \alpha \ominus 1$. By induction of the level, the vertex $v = (\beta, X)$ is reached after crossing at most $d-1$ edges. ∎

Lemma 2 *Let u, v be two vertices of* WBF_d^δ *such that $v \notin \mathcal{K}(u)$. Then, $\rho_{\mathcal{R}_1}(u,v) \le 2d-1$.*

Proof. Let Z be the largest common prefix of the row label of u and of v. Since $v \notin \mathcal{K}(u)$, then $v \in \mathcal{J}_\ell(u)$, for some $\ell \in \mathcal{A}$. So, $u = ZY^\triangledown X$, and $v = (\beta, ZY'\ell X)$. Let $Y = y_1 y_2 \ldots y_t$. The first routing step uses $e_\ell^-(u)$ to get a vertex $u' = Zy_1y_2\ldots y_{t-1}{}^\triangledown \ell X$. Observe that $v \notin \mathcal{K}(u')$ (actually, $v \in \mathcal{J}_{y_t}(u')$). So by induction, using some edges e^-, we get the vertex $w = Z^\triangledown Y'\ell X = (\alpha', ZY'\ell X)$ in at most $t \le d$ steps. By Lemma 1, $\rho_{\mathcal{R}_1}(w,v) \le d-1$, thus $\rho_{\mathcal{R}_1}(u,v) \le \rho_{\mathcal{R}_1}(u,w) + \rho_{\mathcal{R}_1}(w,v) \le 2d-1$. ∎

Lemma 3 *Let u, v be two vertices of* WBF_d^δ *such that $v \in \mathcal{K}(u)$. Then, $\rho_{\mathcal{R}_1}(u,v) \le \frac{5}{2}d - 1$.*

Proof. Let X be the largest common suffix of the row label of u and of v. Let $u = Z^\triangledown YX = (\alpha, ZYX)$ and $v = (\beta, Z'Y'X)$, with $|Z| = |Z'| = \alpha$, $Z = z_1 z_2 \ldots z_\alpha$ and $Y = y_1 y_2 \ldots y_t$.

If $\alpha \le d/2$, then $v \in I_1(e_0^-(u))$. So after the first step, we reach the vertex $u' = z_1 \ldots z_{\alpha-1}{}^\triangledown 0YX = (\alpha - 1, z_1 \ldots z_{\alpha-1} 0YX)$. By induction on the level of u', we reach the vertex $w = (0, 0^\alpha YX)$ of level 0 in at most $\alpha \le d/2$ steps. Because $\mathcal{K}(w) = \emptyset$, we can conclude this case by applying Lemma 2, proving that $\rho_{\mathcal{R}_1}(u,v) \le d/2 + 2d - 1$.

If $\alpha > d/2$, then $v \in I_1(e_0^+(u))$. So after the first step, we reach the vertex $u' = Z0^\triangledown y_2 \ldots y_t X$. By induction on the level of u', we reach the vertex $w = Z0^{t\triangledown} X$ after at most $t < d/2$ steps because $\alpha = |Z| > d/2$. The routing from w to v is covered by Lemma 2, proving that $\rho_{\mathcal{R}_1}(u,v) < d/2 + 2d - 1$. ∎

Lemma 4 *The interval labeling $\mathcal{R}_1$ is of compactness* 1.

Proof. Let $u = (\alpha, X)$ a vertex of WBF_d^δ, $X = x_1 x_2 \ldots x_d$, and let $\ell \in \mathcal{A}$.

We have $\mathcal{J}_\ell(u) = \{(\beta, Y) \mid Y = y_1 \ldots y_{\alpha-1} \ell x_{\alpha+1} \ldots x_d\}$. In particular,

$$0^{\alpha-1} \ell x_{\alpha+1} \ldots x_d \preceq Y \preceq (\delta-1)^{\alpha-1} \ell x_{\alpha+1} \ldots x_d .$$

Thus $\mathcal{L}_w(\mathcal{J}_\ell(u))$ consists of a linear interval.

Moreover for $\ell < \delta - 1$ we have that $0^{\alpha-1}(\ell+1)x_{\alpha+1}\ldots x_d$ is the successor of $(\delta-1)^{\alpha-1}\ell x_{\alpha+1}\ldots x_d$ for the relation $\preceq$. Thus, $\mathcal{L}_w(\mathcal{J}_\ell(u) \cup \mathcal{J}_{\ell+1}(u))$ is a linear interval as well.

Consequently the set $\mathcal{L}_w(\bigcup_{\ell\in\mathcal{A}} \mathcal{J}_\ell(u))$ and $\mathcal{L}_w\left(\bigcup_{\ell=1}^{\delta-1} \mathcal{J}_\ell(u)\right)$ are linear intervals, and $\mathcal{L}_w(\mathcal{K}(u))$ and $\mathcal{L}_w(\mathcal{K}(u) \cup J_0(u))$ are circular intervals, completing the proof. ∎

Therefore,

Theorem 1 *Every* WBF_d^δ *of diameter D supports an* 1-IRS *with dilation at most $\frac{5}{2}d - 1 \le \frac{5}{3}D - \frac{2}{3}$.*

Proof. The diameter of WBF_d^δ is $D = d + \lfloor d/2 \rfloor$. For $d = 2k$, we have $D = 3k$, whereas the dilation is $\lfloor 5d/2 - 1 \rfloor = 5k - 1 = 5D/3 - 1$. For $d = 2k+1$, $D = 3k+1$ and the dilation is $\lfloor 5d/2 - 1 \rfloor = 5k+1 = 5D/3 - 2/3$. It follows that, in any case, the dilation is bounded by $5D/3 - 2/3$. ∎

5.4 Interval Routing with Compactness Two for WBF_d^δ

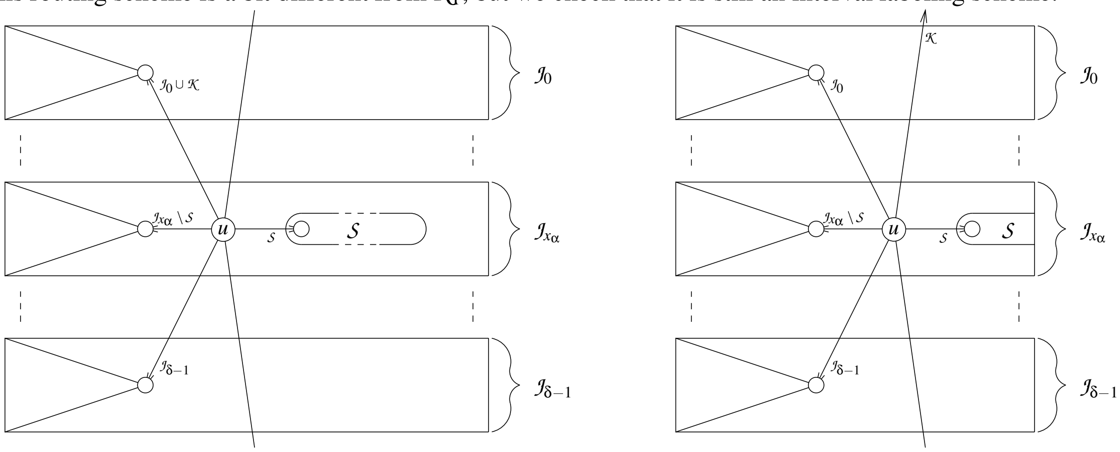
For every vertex $u = (\alpha, X) = x_1 \dots x_\alpha {}^\triangledown x_{\alpha+1} \dots x_d$ of WBF_d^δ, let us denote by $\mathcal{S}(u) = \left\{ (\beta, X) \mid \beta \in \left[\alpha, \min\left\{\alpha + \left\lfloor \frac{d}{2} \right\rfloor, d\right\}\right] \right\}$. Informally, $\mathcal{S}(u)$ contains all the vertices of u's row whose level is a bit larger than u's level (at most $\frac{d}{2}$).

The interval routing $\mathcal{R}_2 = (\mathcal{L}_w, I_2)$ on WBF_d^δ is defined by (summarized on the figure bellow):

$$I_2(e_\ell^-(u)) = \begin{cases} \mathcal{J}_\ell(u) \cup \mathcal{K}(u) \setminus \mathcal{S}(u) & \text{if } \ell = 0 = x_\alpha \text{ and } \alpha \le d/2 \\ \mathcal{J}_\ell(u) \cup \mathcal{K}(u) & \text{if } \ell = 0 \ne x_\alpha \text{ and } \alpha \le d/2 \\ \mathcal{J}_\ell(u) \setminus \mathcal{S}(u) & \text{if } \ell = x_\alpha \ne 0 \text{ or } (\ \ell = 0 = x_\alpha \text{ and } \alpha > d/2\ ) \\ \mathcal{J}_\ell(u) & \text{if } \ell \in \mathcal{A} \setminus \{0, x_\alpha\} \text{ or } (\ \ell = 0 \ne x_\alpha \text{ and } \alpha > d/2\ ) \end{cases}$$

$$I_2(e_\ell^+(u)) = \begin{cases} \mathcal{K}(u) & \text{if } \ell = 0 \ne x_{\alpha+1} \text{ and } \alpha > d/2 \\ \mathcal{K}(u) \cup \mathcal{S}(u) & \text{if } \ell = 0 = x_{\alpha+1} \text{ and } \alpha > d/2 \\ \mathcal{S}(u) & \text{if } \ell = x_{\alpha+1} \ne 0 \text{ or } (\ \ell = x_{\alpha+1} = 0 \text{ and } \alpha \le d/2\ ) \\ \emptyset & \text{if } \ell \in \mathcal{A} \setminus \{0, x_{\alpha+1}\} \text{ or } (\ \ell = 0 \ne_{\alpha+1} \text{ and } \alpha \le d/2\ ) \end{cases}$$

This routing scheme is a bit different from $\mathcal{R}_1$, but we check that it is still an interval labeling scheme.



Case: $\alpha \le d/2$ Case: $\alpha > d/2$

Lemma 5 *Let $u = (\alpha, X)$ and $v = (\beta, X)$ be two vertices of WBF_d^δ. If $\alpha \le d/2$, then $\rho_{R_2}(u,v) \le d/2$, and $\rho_{R_2}(u,v) \le d-1$ otherwise.*

Proof. Assume that $\alpha > d/2$. If $\beta > \alpha$, then $v \in \mathcal{S}(u)$. So in the first step we use the edge $e^+_{x_{\alpha+1}}(u)$ to reach the vertex $u' = (\alpha+1, X)$. Step by step we reach v in at most $\beta - \alpha < d/2 \le d-1$ steps. If $\beta < \alpha$, then $v \notin \mathcal{S}(u)$. So in the first step we use the edge $e^-_{x_\alpha}(u)$ to reach the vertex $u' = (\alpha - 1, X)$. Step by step we reach v in at most $\alpha - \beta \le d-1$ steps.

Assume now that $\alpha \le d/2$. If $\alpha < \beta \le \alpha + d/2$, then $v \in \mathcal{S}(u)$. So in the first step we use the edge $e^+_{x_{\alpha+1}}(u)$ to reach the vertex $u' = (\alpha+1, X)$. We have again $v \in \mathcal{S}(u')$, so by induction we reach v in at most $\beta - \alpha \le d/2$ steps. If $\beta < \alpha$, then $v \notin \mathcal{S}(u)$ so in the first step we use the edge $e^-_{x_\alpha}(u)$ to reach the vertex $u' = (\alpha - 1, X)$. Step by step we reach v in at most $\alpha - \beta \le d/2$ steps. Finally, if $\beta > \alpha + d/2$, then by definition $v \notin \mathcal{S}(u)$. So in the first step we use the edge $e^-_{x_\alpha}(u)$ to reach the vertex $u' = (\alpha \ominus 1, X)$. Step by step we reach $w = (0, X)$ in at most α steps, and then v in $d - \beta$ more steps. So $\rho_{R_2}(u,v) \le d - \beta + \alpha \le d - (\alpha + d/2) + \alpha \le d/2$. ∎

Lemma 6 *Let u, v be two vertices of WBF_d^δ such that $v \notin \mathcal{K}(u) \cup \mathcal{S}(u)$. Then, $\rho_{R_2}(u,v) \le 3d/2 - 1$.*

Proof. Let u,v such that $v \notin \mathcal{K}(u) \cup \mathcal{S}(u)$. We have $u = ZY^{\triangledown}X$ and $v = (\beta, ZY'X)$ where Z is the largest common prefix of the row of u and of v. Let $t = |Y|$ and $\alpha = |Z|$.

Clearly we reach $w = Z^{\triangledown}bY'X$ in at most t steps. If $\alpha > d/2$, then $t \leq d/2$. Thanks to Lemma 5, $\rho_{R_2}(u,v) \leq t + (d-1) \leq 3d/2 - 1$. If $\alpha \leq d/2$, then thanks to Lemma 5, $\rho_{R_2}(u,v) \leq t + d/2 \leq 3d/2 - 1$. ∎

Lemma 7 *Let u,v be two vertices of* WBF_d^{δ} *such that $v \in \mathcal{K}(u)$. Then, $\rho_{R_2}(u,v) \leq 2d-1$.*

Proof. Appling the same method that in the case with one interval we obtain that $\rho_{R_2}(u,v) \leq d/2 + (3d/2 - 1) \leq 2d - 1$. ∎

Lemma 8 *The interval labeling $\mathcal{R}_2$ is of compactness* 2.

Proof. By definition of $\mathcal{S}(u)$ and $\mathcal{L}_w$, we know that $\mathcal{L}_w(\mathcal{S}(u))$ is a linear interval. Compared to $\mathcal{R}_1$, two differences occurs:

1. We remove $\mathcal{S}(u)$ to $e^-_{x_\alpha}(u)$, as $\mathcal{L}_w((\mathcal{S}(u))$ is a linear interval, it costs at most one interval more;
2. We put $\mathcal{S}(u)$ to $e^+_{x_{\alpha+1}}(u)$, it costs at most one interval more too.

∎

Therefore,

Theorem 2 *Every* WBF_d^{δ} *of diameter D supports an* 2-IRS *with dilation at most* $2d - 1 \leq \frac{4}{3}D - \frac{1}{3}$.

Proof. The diameter of WBF_d^{δ} is $D = d + \lfloor d/2 \rfloor$. For $d = 2k$, we have $D = 3k$, whereas the dilation is $2d - 1 = 4k - 1 = 4D/3 - 1$. For $d = 2k+1$, $D = 3k+1$ and the dilation is $4k+1 = 4D/3 - 1/3$. It follows that, in any case, the dilation is bounded by $4D/3 - 1/3$. ∎

5.5 Interval Routing with Compactness Two for BF_d^{δ}

For every vertex $u = (\alpha, X) = x_1 \ldots x_\alpha{}^{\triangledown} x_{\alpha+1} \ldots x_d$ of BF_d^{δ}, we denote by $\mathcal{S}(u) = \{(\beta, X) \mid \beta \in [\alpha, d]\}$.

The interval routing $\mathcal{R}_3 = (\mathcal{L}_w, I_3)$ on WBF_d^{δ} is defined by:

$$I_3(e^-_\ell(u)) = \begin{cases} \mathcal{I}_\ell(u) \setminus \mathcal{S}(u) & \text{if } \ell = x_\alpha \\ \mathcal{I}_\ell(u) & \text{if } \ell \in \mathcal{A} \setminus \{x_\alpha\} \end{cases}$$

$$I_3(e^+_\ell(u)) = \begin{cases} \mathcal{K}(u) & \text{if } \ell = 0 \neq x_{\alpha+1} \\ \mathcal{K}(u) \cup \mathcal{S}(u) & \text{if } \ell = 0 = x_{\alpha+1} \\ \mathcal{S}(u) & \text{if } \ell = x_{\alpha+1} \neq 0 \\ \emptyset & \text{if } \ell \in \mathcal{A} \setminus \{0, x_{\alpha+1}\} \end{cases}$$

This routing labeling $\mathcal{R}_3$ is similar to the scheme $\mathcal{R}_2$ except in two points:

- $\mathcal{S}(u)$ contains all the vertices with same row label but with a level superior as u;
- $\mathcal{K}(u)$ is always assigned to $e^+_0(u)$.

Therefore the proof of Lemmas 9, 10 and 11 are similar to the previous cases. We check easily that $\mathcal{R}_3$ has compactness 2.

Lemma 9 *Let $u = (\alpha, X)$ and $v = (\beta, X)$ be two vertices of* BF_d^{δ}*, then $v \in \mathcal{S}(u)$, and $\rho_{R_3}(u,v) \leq d$.*

Lemma 10 *Let u,v be two vertices of* BF_d^{δ} *such that $v \notin \mathcal{K}(u) \cup \mathcal{S}(u)$, then $\rho_{R_3}(u,v) \leq 2d$.*

Lemma 11 *Let u,v be two vertices of* BF_d^{δ} *such that $v \in \mathcal{K}(u)$, then $\rho_{R_3}(u,v) \leq 3d$.*

Since the diameter of BF_d^{δ} is $2d$ and the dilation of $\mathcal{R}_3$ is $3d$, we have:

Theorem 3 *Every* BF_d^{δ} *of diameter D supports an* 2-IRS *with dilation at most* $2d \leq \frac{3}{2}D$.

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