Graph state preparation on arbitrary hardware

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Motivations

Introduction: quantum computing

- Some tasks are easier for a quantum computer;
- some quantum circuits are hard to simulate on a classical computer;
- high entanglement is a key ingredient for quantum speedup.

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- some quantum circuits are hard to simulate on a classical computer;
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$\Rightarrow$ graph state framework

nearly hard to simulate; good measure of entanglement: rank-width

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Stabilizer state represented by an undirected graph G on n vertices.

Recipe to prepare $|G\rangle$ on your hardware H:

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$\Rightarrow$ you used $|E(G)|$ two-qubit gates.

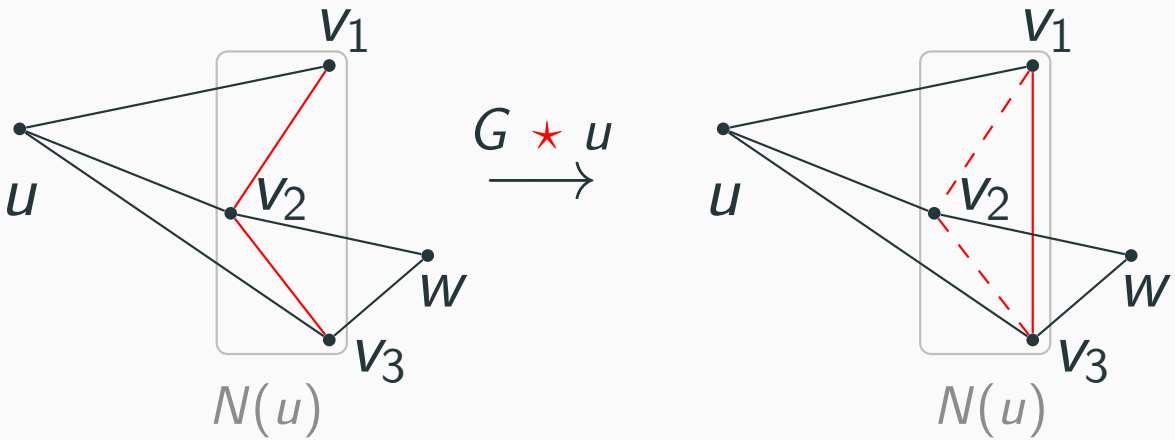
Given all-to-all connectivity

Local complementation(s)

Local complementation: as a graph operation

Definition

The *local complementation* of G according to u, written $G \star u$, is a graph which has the same vertices as G, but all the neighbors $v_1, v_2, \ldots$ of u are connected in $G \star u$ if and only if they are not connected in G. All other edges are unchanged.



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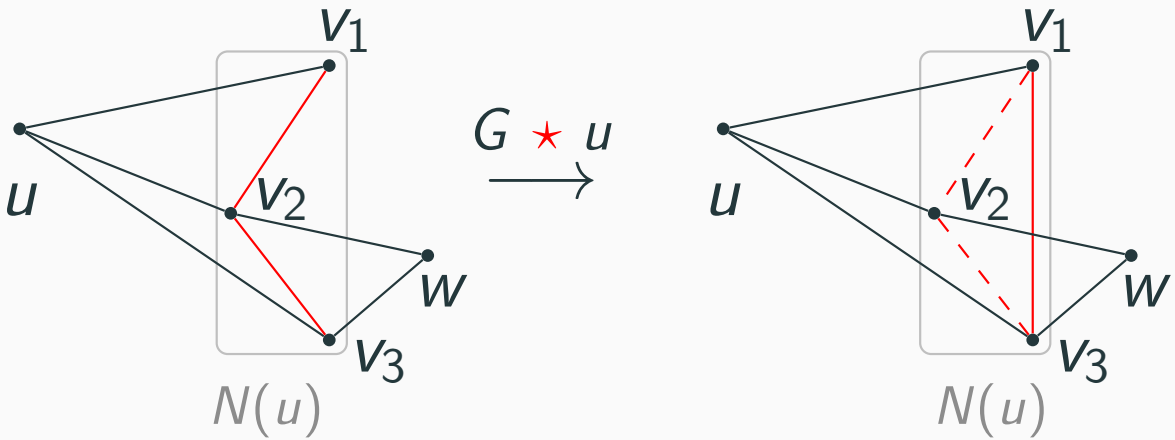


$$G \star u \star u = G$$

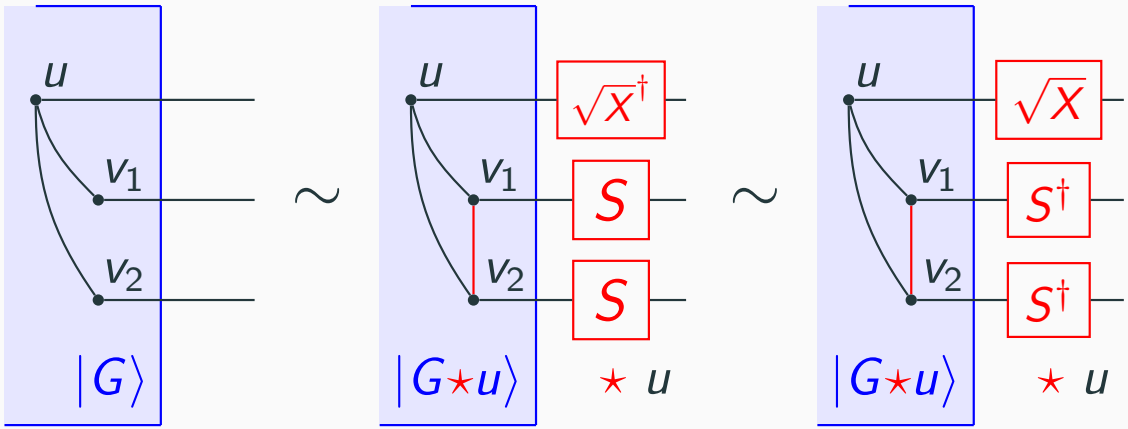
Note: this preserves rank-width (*i.e.* entanglement).

Local complementation: as a circuit operation

Proposition

Let $|G\rangle$ be a graph state, u a vertex of G and N_u the set of neighbors of u in G. Performing a local complementation on $|G\rangle$ according to u is done by:

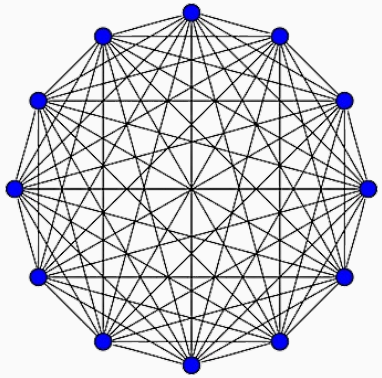
- *adding a $\sqrt{X}^\dagger$ gate on the qubit u,*
- *adding a S gate on every qubit in N_u.*



Replacing CZ by single qubit gate = less error

Local complementation: in practice

Your hardware: H, with all-to-all connectivity.



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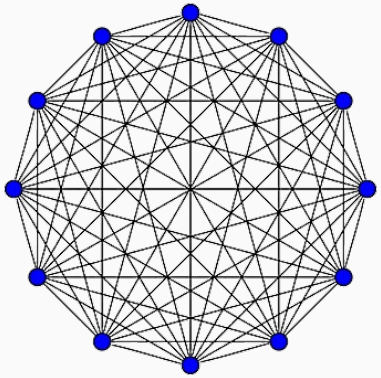

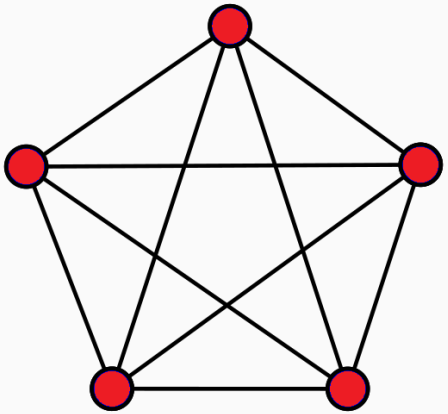

n=5

You want to prepare $|G\rangle$ with $G = K_5$

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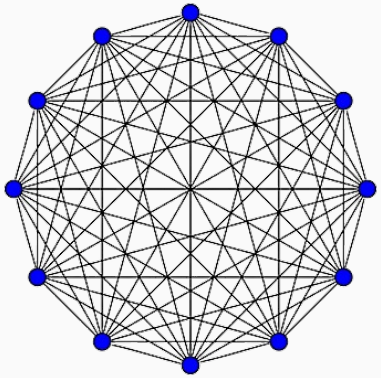

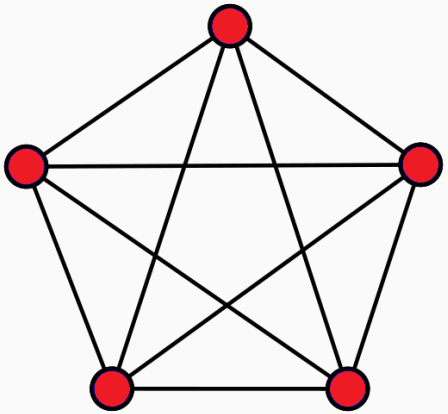

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You want to prepare $|G\rangle$ with $G = K_5$

Naively uses $\theta(n^2)$ CZ gates.

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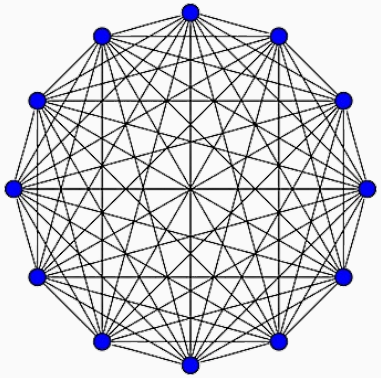

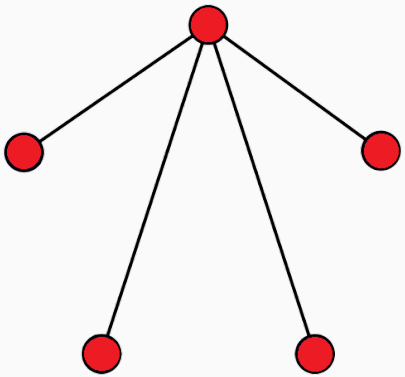

Notice that $K_5 \sim_{loc} K_{1,4}$

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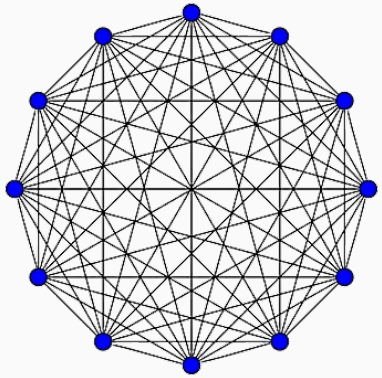

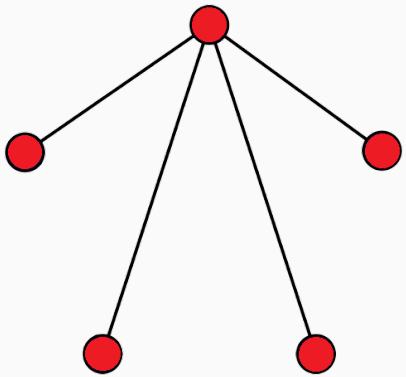

Notice that $K_5 \sim_{loc} K_{1,4}$

Preparing $|K_{1,4}\rangle$ uses $\mathcal{O}(n)$ CZ gates, then apply one local complementation to get $|G\rangle$.

One local complementation emulated 6 "virtual" CZ.

Local complementation: in practice

$\Rightarrow$ Local complementation enables optimizations in gate count

BUT...

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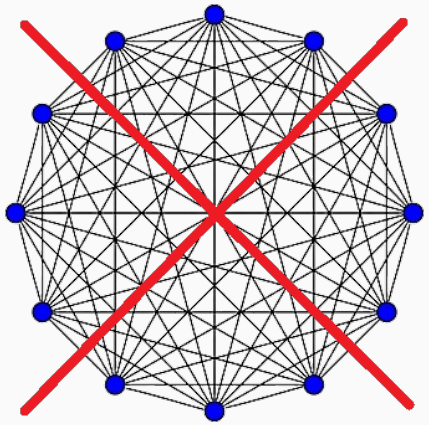
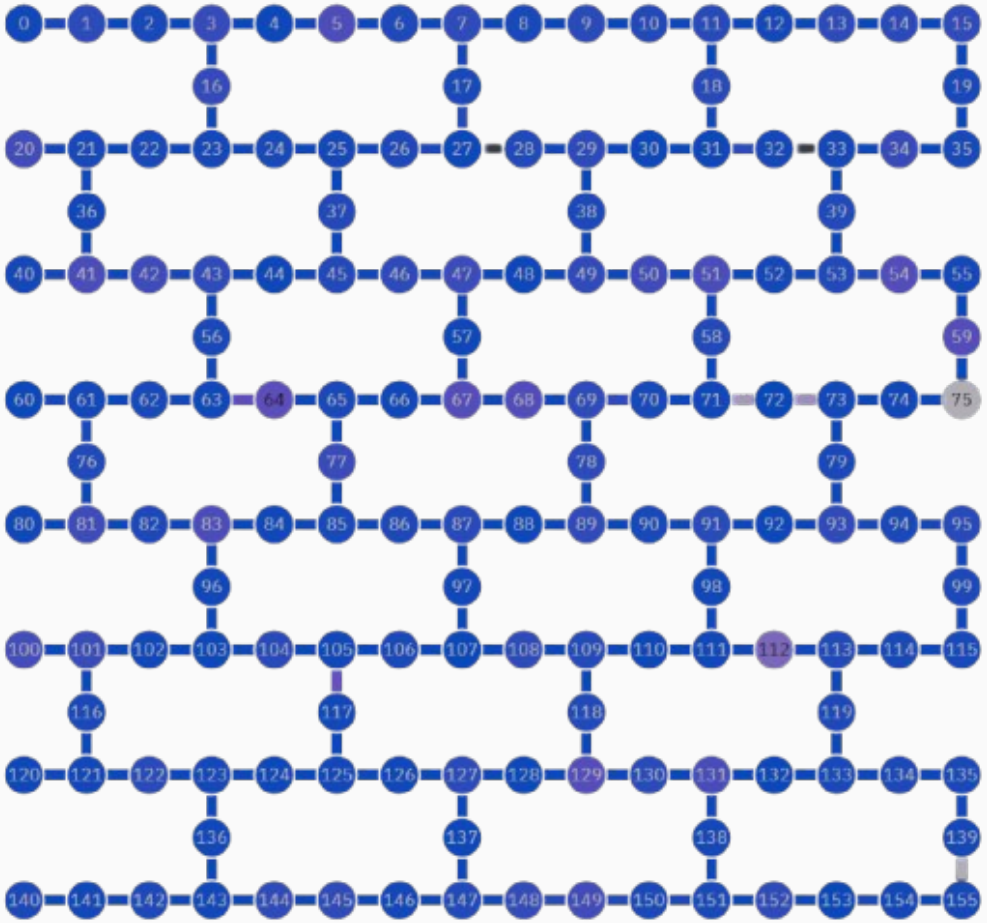


(your hardware may not be a complete graph.)

How about on arbitrary hardware?

Arbitrary hardware

Given a graph H : your hardware.



IBM's 156 qubit processor, Heron r2

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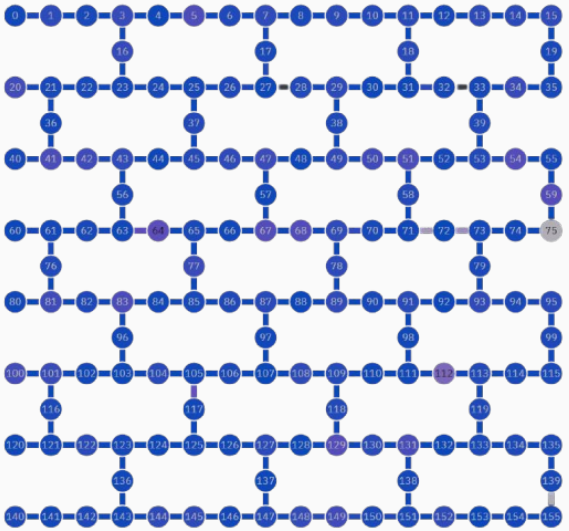

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Which graph states can be prepared on H by only:

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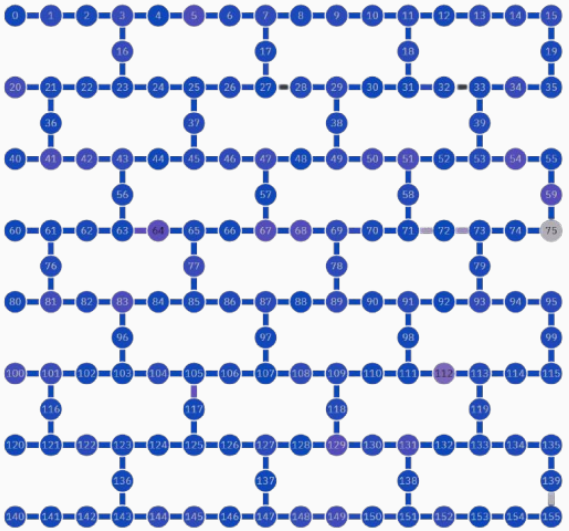

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Which graph states can be prepared on H by only:

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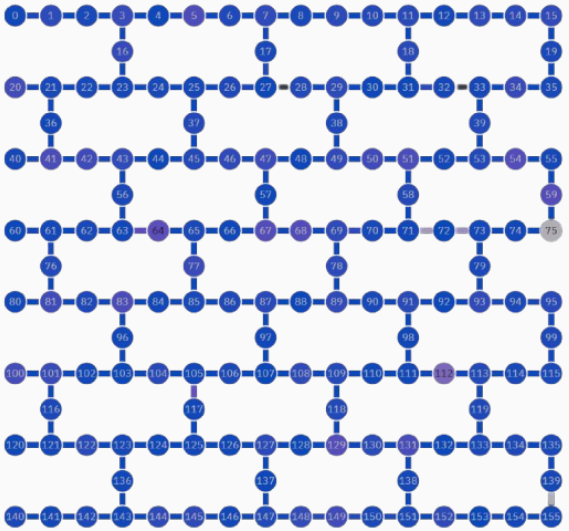


Problem

Which graph states can be prepared on H by only:

- adding CZ along edges of H (toggling),
- local complementations on vertices of H.

Our results I

Answer: all of them.

Theorem

Let H be a connected graph on n vertices.
Let G be any graph on n vertices.
The state $|G\rangle$ can be prepared by a circuit consisting of CZ gates along edges of H and local complementations.

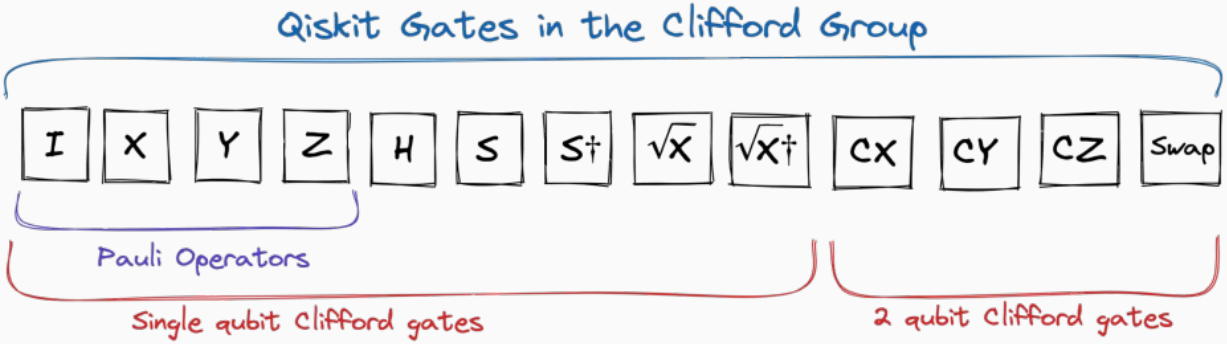
Our method

Useful fact

Proposition (Folklore)

Let H be the connected hardware graph.
Then any Clifford circuit can be implemented with local Clifford gates and CZ along edges of H.

i.e. any Clifford circuit is "compatible" with H if we allow all types of local Cliffords (naively using a $\mathcal{O}(n^3)$ overhead on CZ count).



Goal

C from Proposition,
Contains {CZ, Local Cliffords}.

C' from our method,
Contains {CZ, $\underbrace{S, \sqrt{X}^\dagger}_{\text{loc. comp.}}$}.

polynomial time algorithm

$$C\,|+\rangle \quad = \quad |G\rangle \quad = \quad C'\,|+\rangle$$

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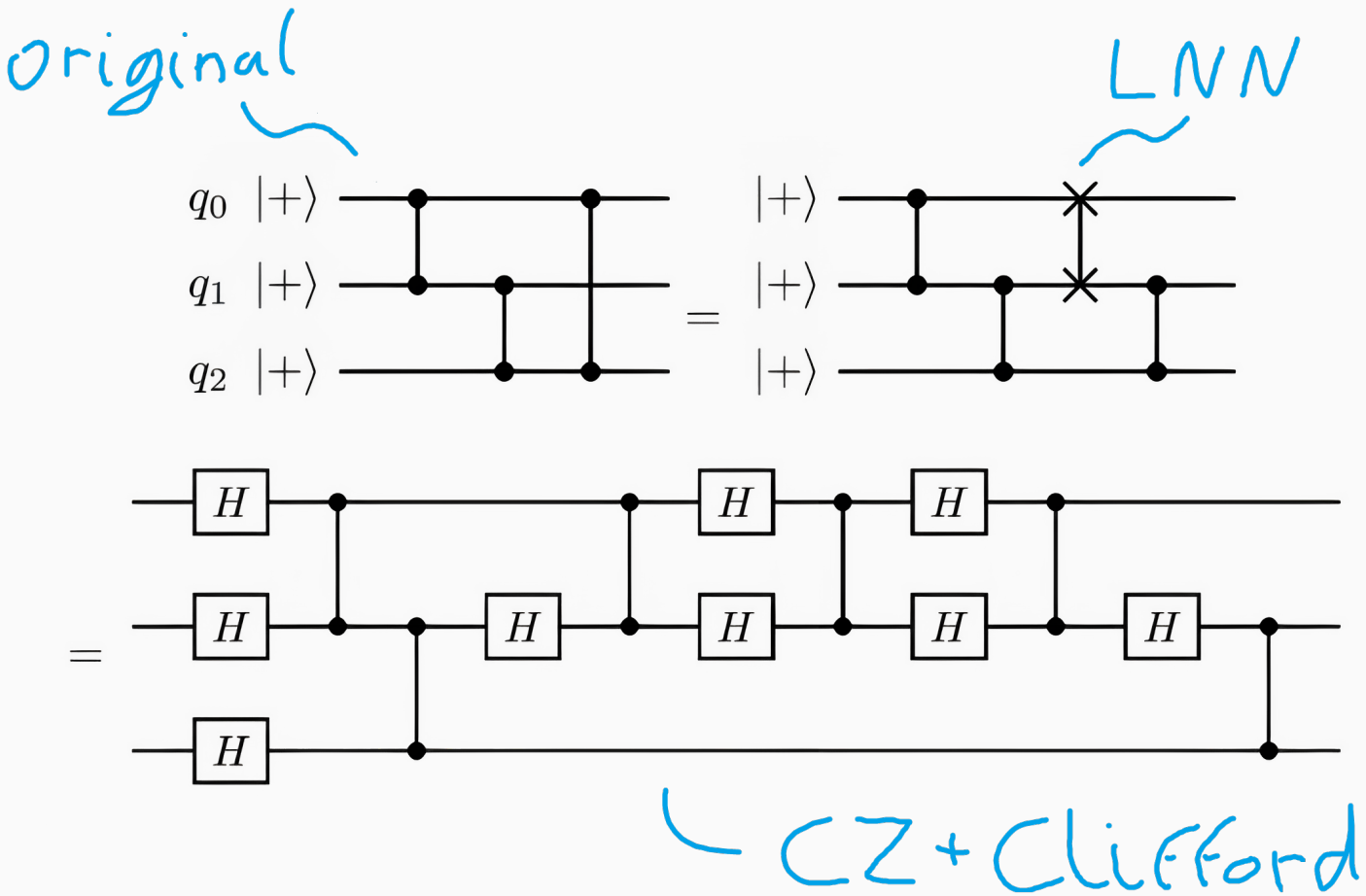
$$C\,|+\rangle \quad = \quad |G\rangle \quad = \quad C'\,|+\rangle$$

$$\#CZ(C) \quad = \quad m \quad \geq \quad \#CZ(C')$$

Overview


original
LNN
q_0 $|+\rangle$
q_1 $|+\rangle$
q_2 $|+\rangle$
=
$|+\rangle$
$|+\rangle$
$|+\rangle$
=
H
CZ + Clifford

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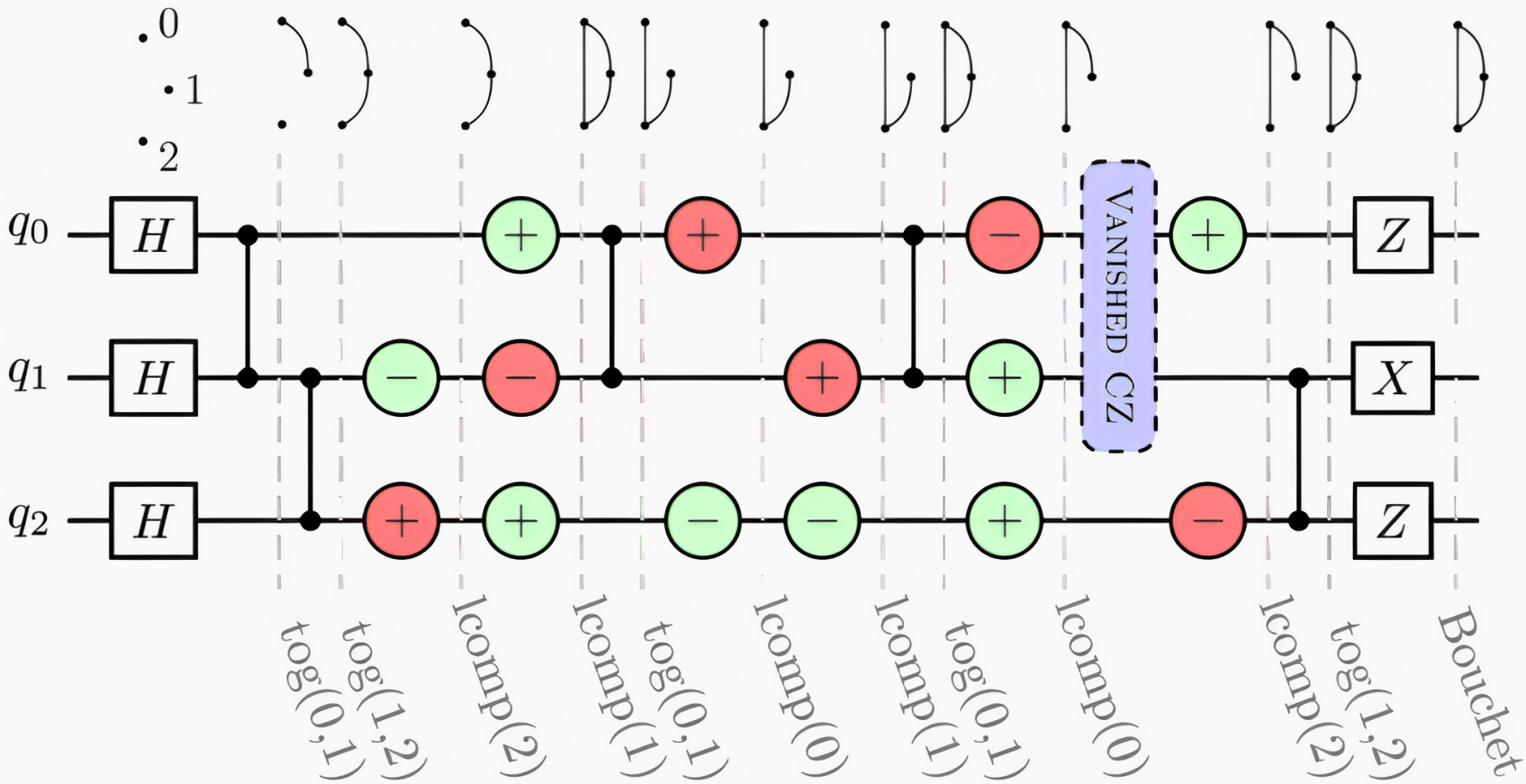
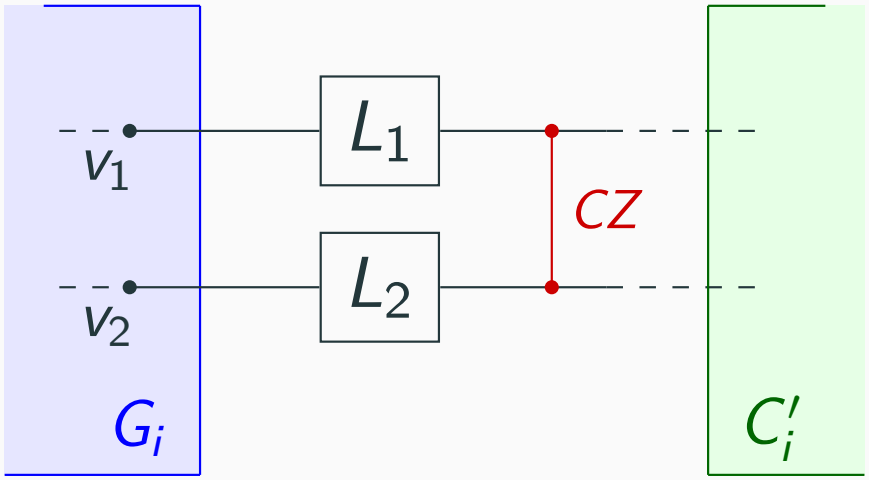


where $\oplus = \boxed{S}$, $\ominus = \boxed{S^\dagger}$, $\oplus = \boxed{\sqrt{X}}$, $\ominus = \boxed{\sqrt{X}^\dagger}$.

Our construction



Sketch of the i-th step.

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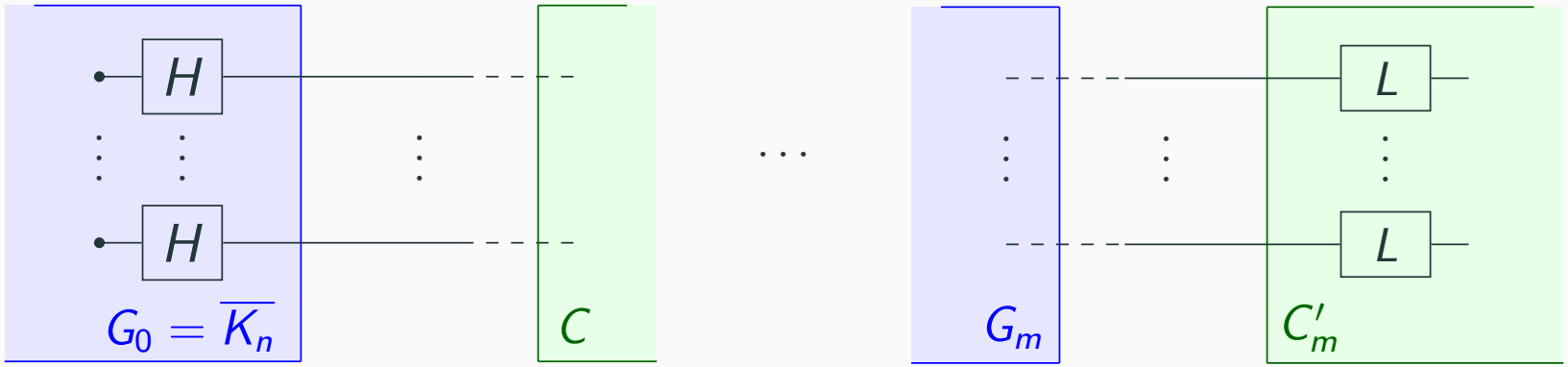
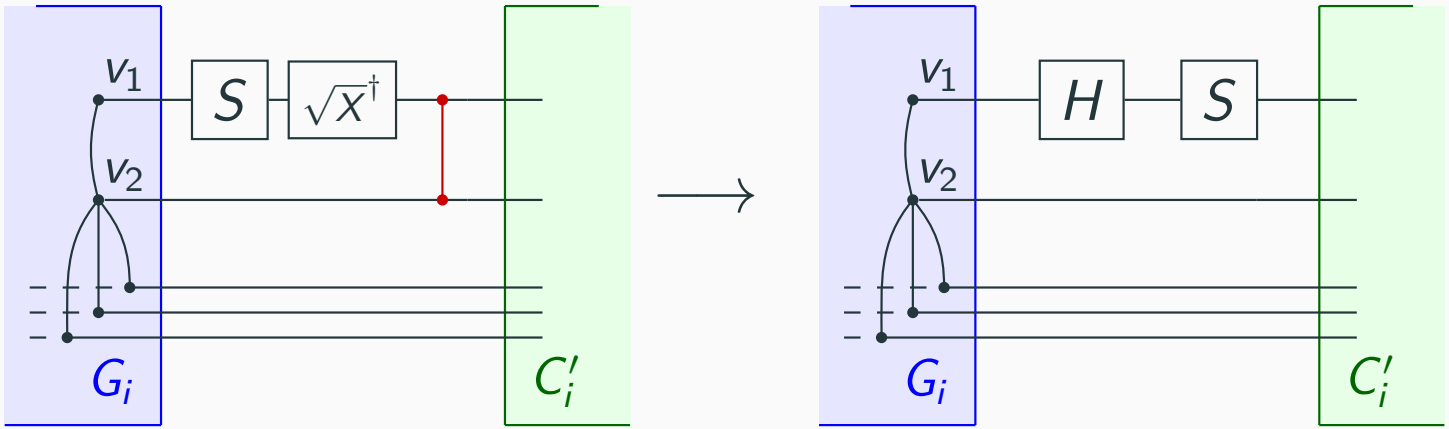


First step

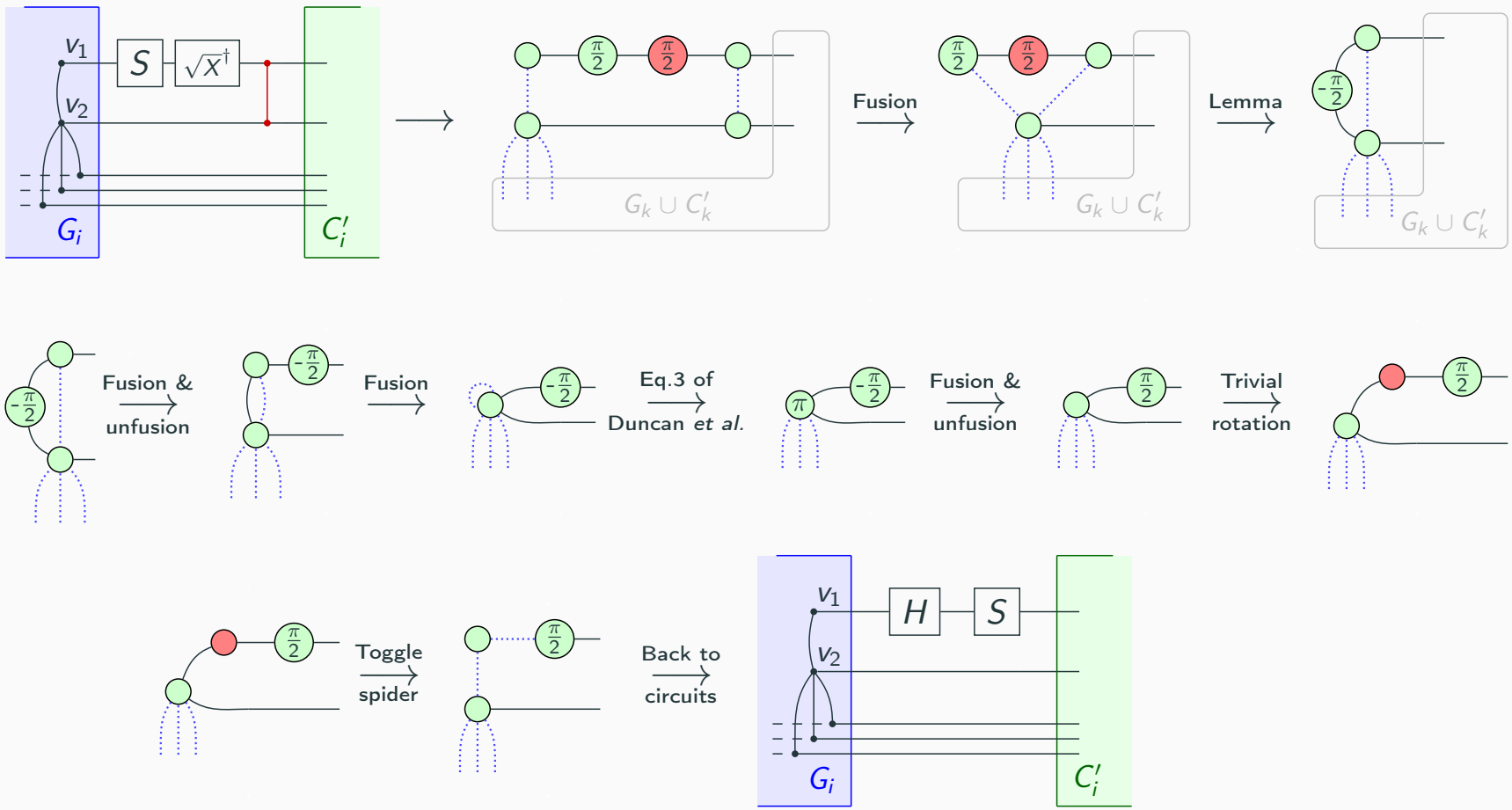
Last step

Preliminary Lemma 1: vanishing CZ

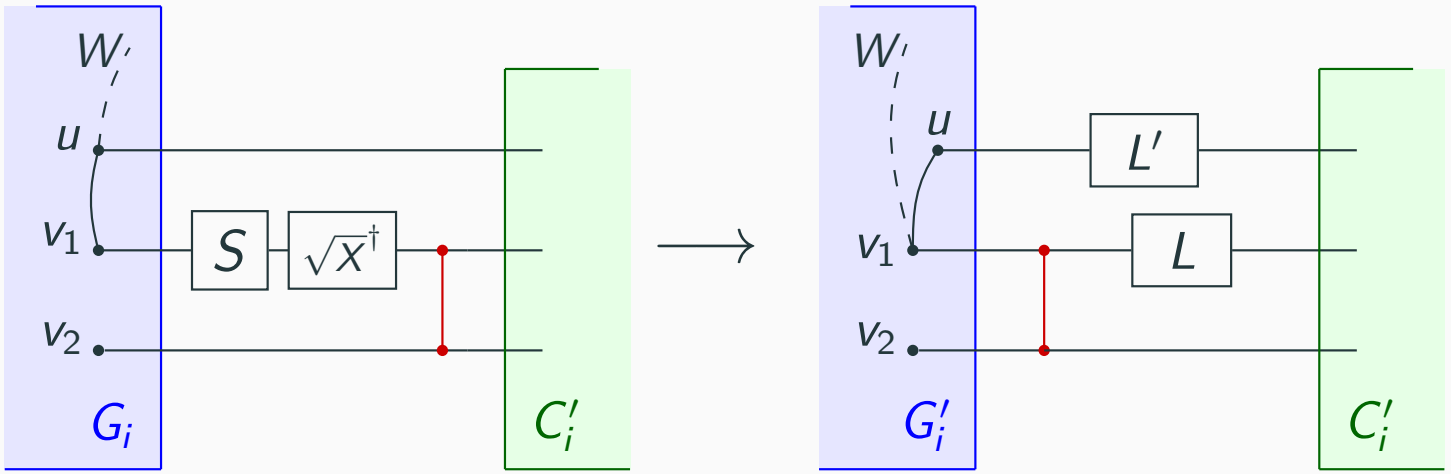


Lemma 1: These two circuits are equivalent

Proof of Lemma 1



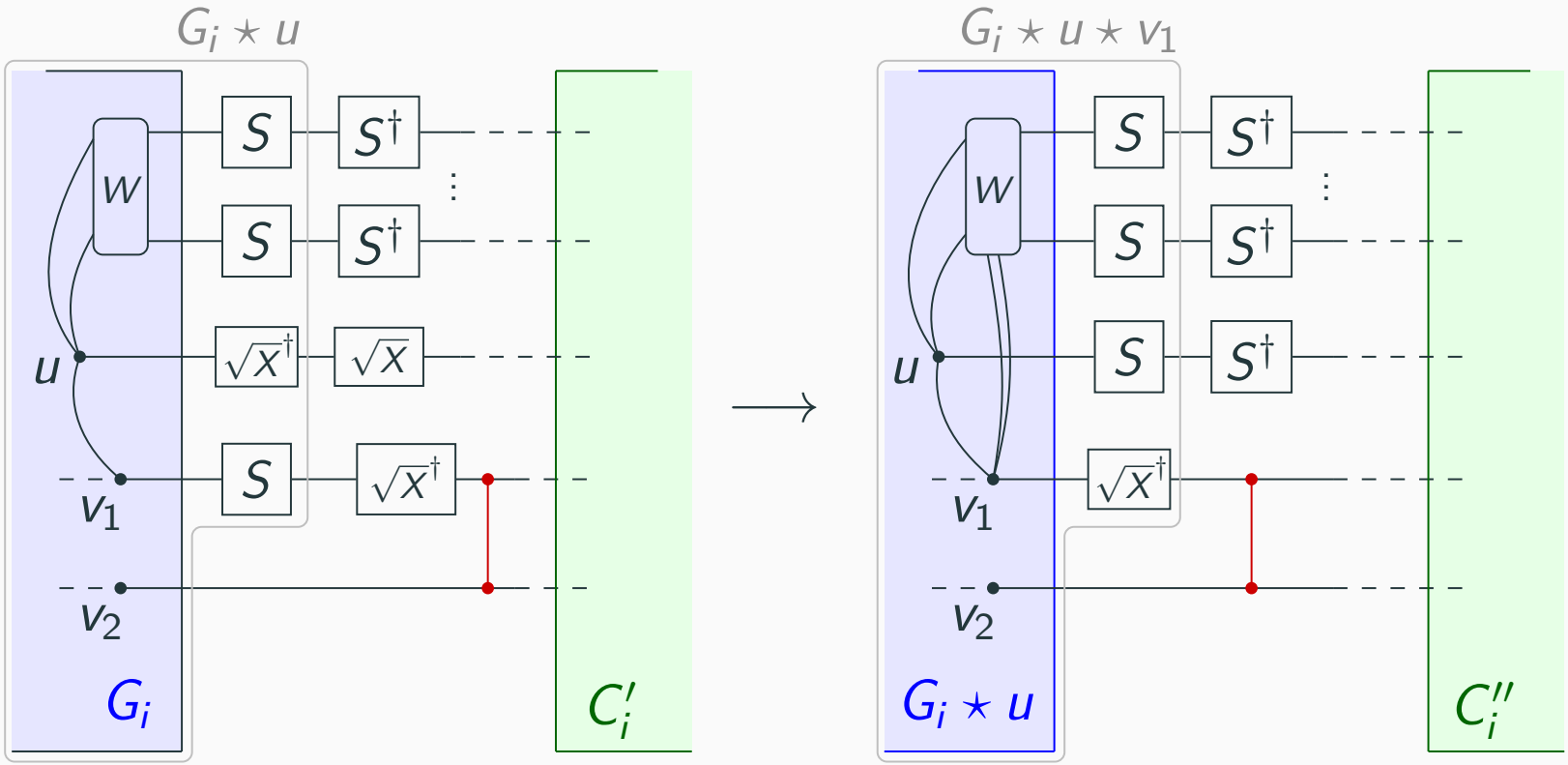
Preliminary Lemma 2: commuting and absorbing the CZ



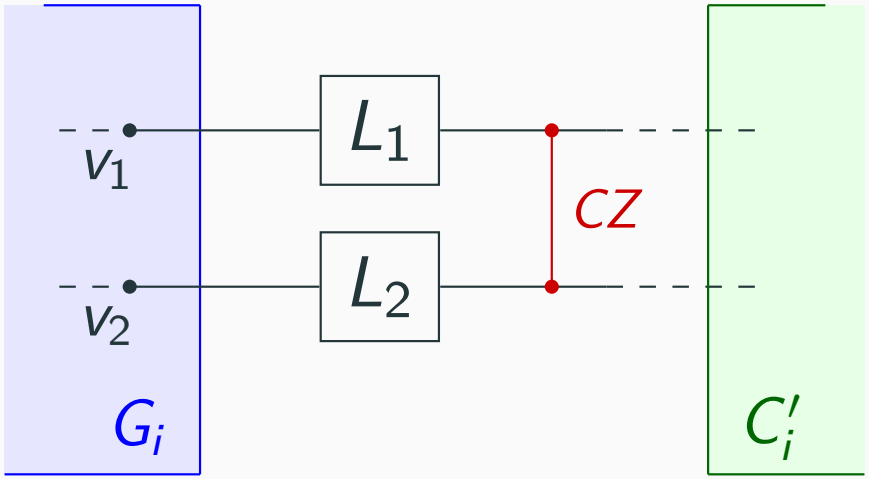
Lemma 2: These two circuits are equivalent

$W = N_u \setminus \{v_1\}$ and L, L' are generic local Clifford gates.

Proof of Lemma 2



Outline of the proof



Euler decomposition:

$$\exists a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{Z}/4\mathbb{Z} :$$
$$L_1 = S^{c_1}\sqrt{X}^{\dagger b_1}S^{a_1} \text{ and } L_2 = S^{c_2}\sqrt{X}^{\dagger b_2}S^{a_2}.$$

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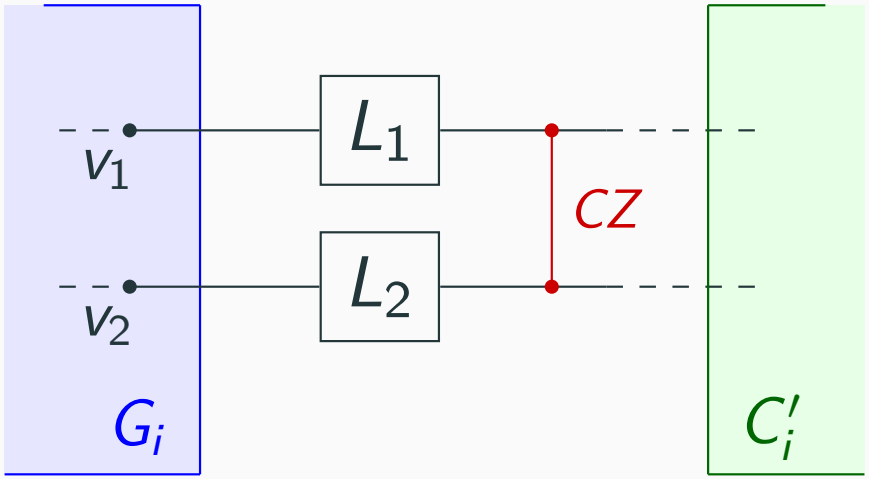

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$$\text{w.l.o.g. } L_1 := \sqrt{X}^{\dagger b_1} S^{a_1},\ L_2 := \sqrt{X}^{\dagger b_2} S^{a_2}.$$

Outline of the proof: three cases

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Three cases:

1. Both b_1 and b_2 are even.

2. Only one of b_1 and b_2 is odd.

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Three cases:

1. Both b_1 and b_2 are even.
2. Only one of b_1 and b_2 is odd.
3. Both b_1 and b_2 are odd.

Outline of the proof: Case 1

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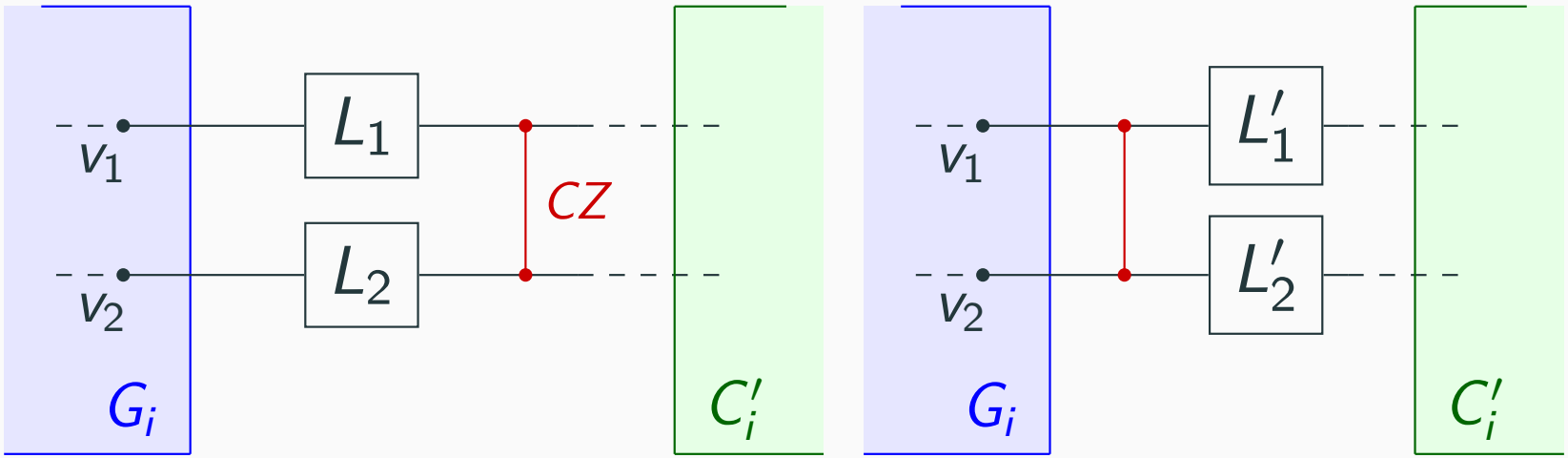
$$L_1 = X^{b_1/2} S^{a_1},\ L_2 = X^{b_2/2} S^{a_2}$$



Pauli and S gates: weakly commute with CZ

Outline of the proof: Case 2

2. Only one of b_1 and b_2 is odd, w.l.o.g. take b_1 odd.

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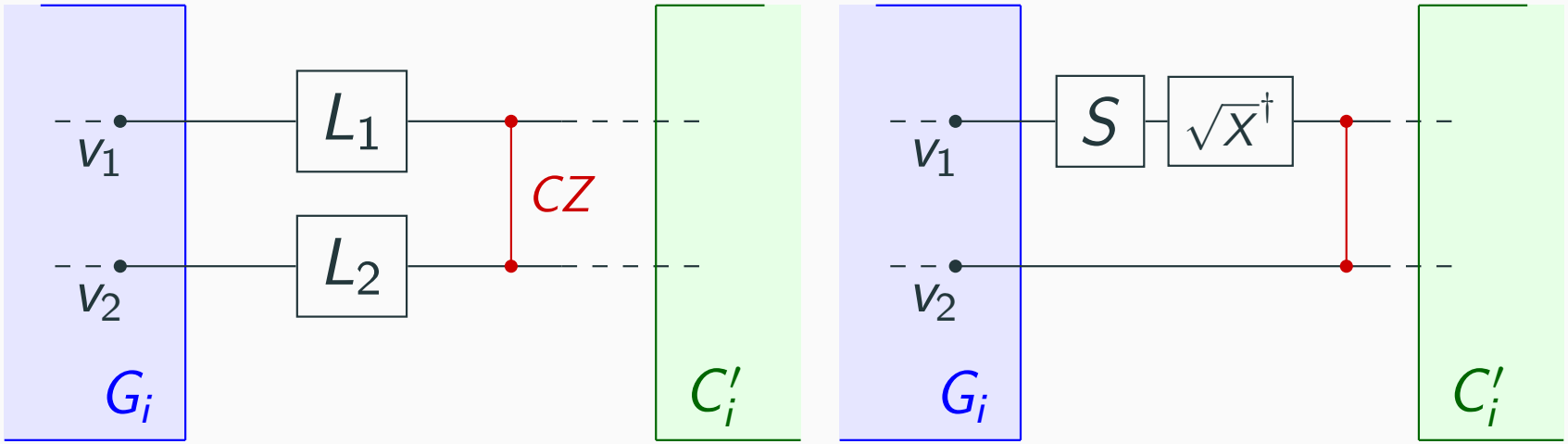

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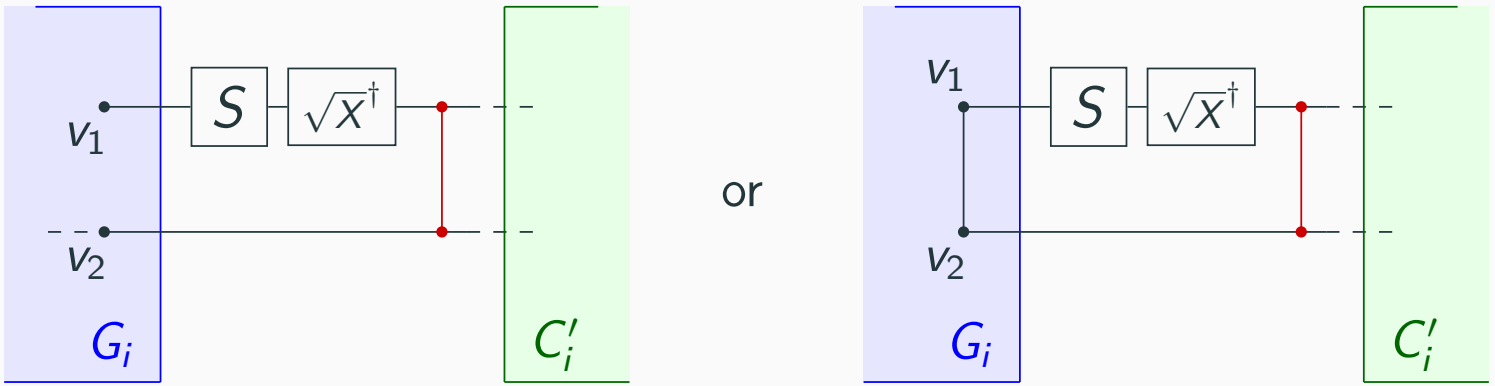


Either:

- v_1 has no neighbor in G_k, or v_1 and v_2 are isolated in G_k

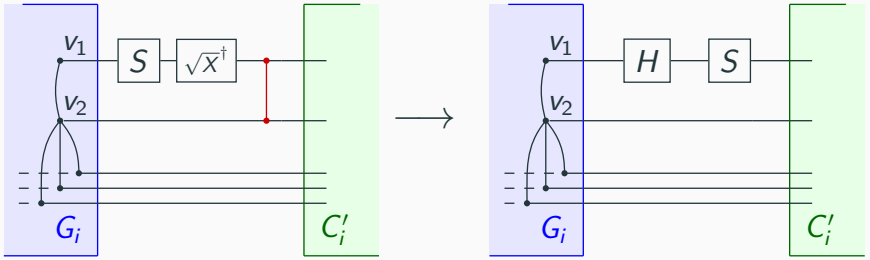
$\Rightarrow$ rewrite subcircuit as 0 or 1 CZ followed by local Cliffords;

Outline of the proof: Case 2

Either:

- v_1 has no neighbor in G_k / v_1, v_2 are each other's only neighbor: rewrite;

- the only neighbor of v_1 in G_k is v_2 and v_2 has at least one neighbor in G_k : Lemma 1;



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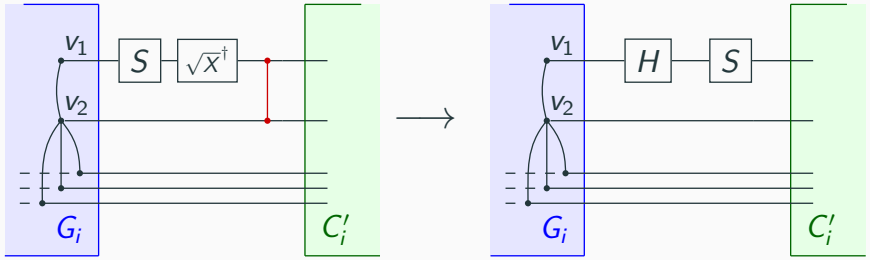
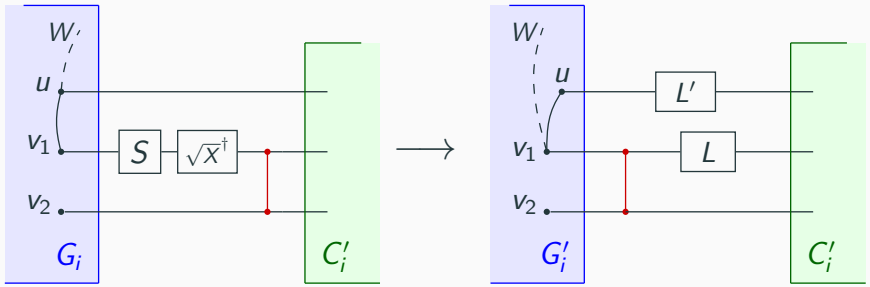


- v_1 has at least one neighbor u in G_k that is not v_2 : Lemma 2.



Outline of the proof: Case 3

3. Both b_1 and b_2 are odd.

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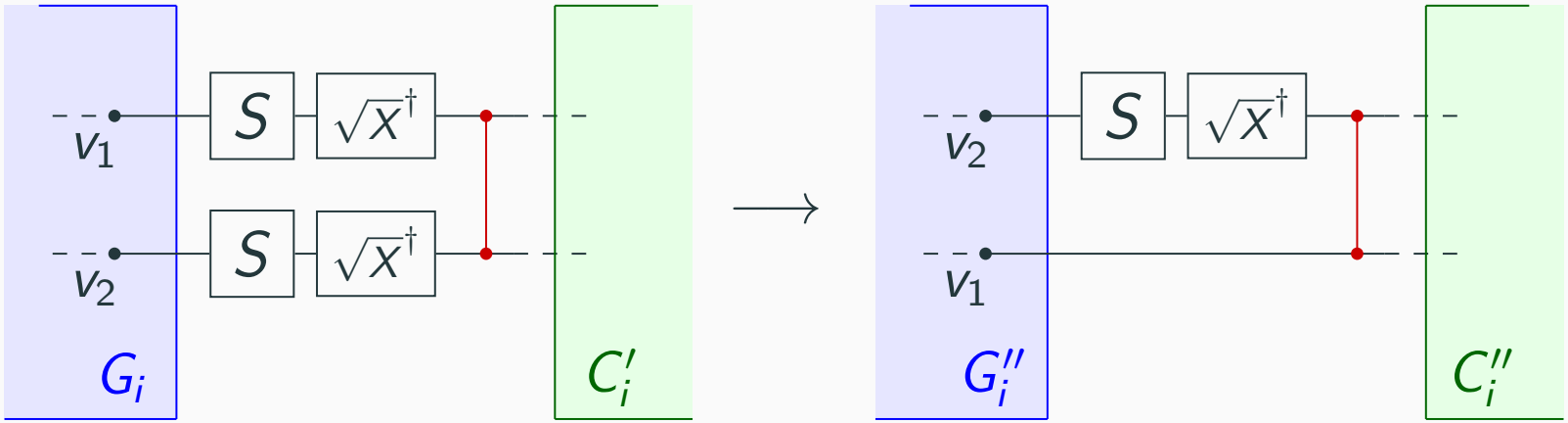
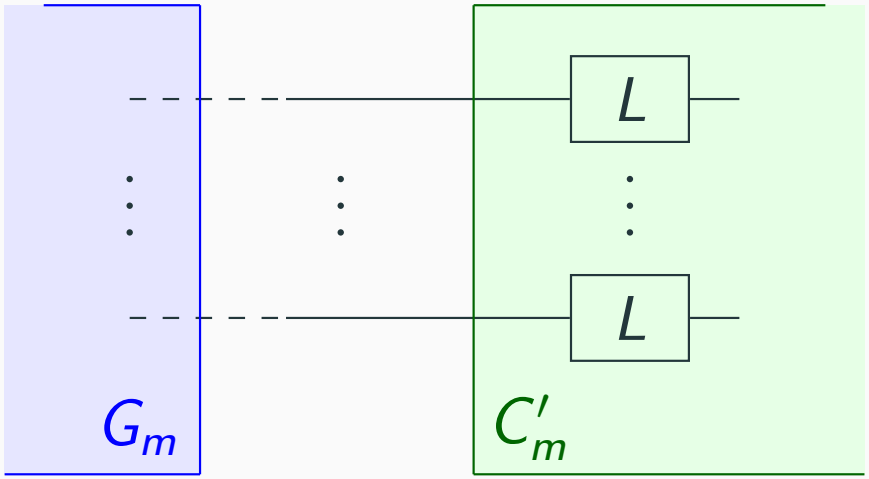


Pauli and S gates: weakly commute with CZ

$\Rightarrow$ Technicalities. Reduce **Case 3** to **Case 2**.

Outline of the proof: final touch

Last step: Van den Nest



Local Cliffords layer $\approx$ finite sequence of l. comp. $l_1, \ldots, l_s$

Outline of the proof: final touch

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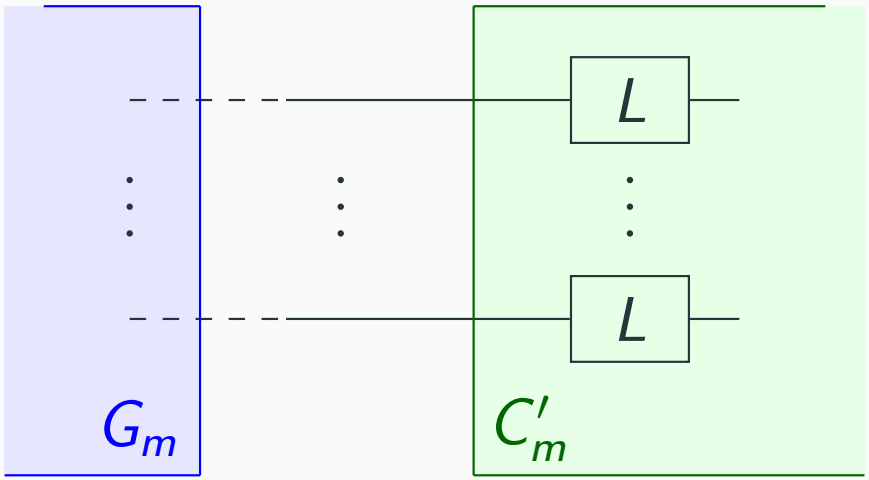

Local Cliffords layer $\approx$ finite sequence of l. comp. $l_1, \ldots, l_s$

Thus $G = G_m \star l_1 \star \cdots \star l_s$ is preparable on H. $\square$

Our results II

Theorem (Universality)

Any graph state can be prepared on any hardware using only CZ gates and local complementations.

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Remark

This method is implemented by a $\mathcal{O}(n^3)$ time constructive algorithm, that yields an output circuit with a CZ count of ***at most*** *the CZ count of the input circuit.*

Our results II

Theorem (Universality)

Any graph state can be prepared on any hardware using only CZ gates and local complementations.

Remark

This method is implemented by a $\mathcal{O}(n^3)$ time constructive algorithm, that yields an output circuit with a CZ count of ***at most*** *the CZ count of the input circuit.*

Theorem (Optimality)

The set of circuits on arbitrary constructed by our method always contains circuits with optimal CZ count.

Our results III

Pure graph theory restatement!

Corollary (Graph statement)

Let H be a connected graph on n vertices.
Then any graph on n vertices can be created from the empty graph on n vertices with only local complementations and edge toggling in H.

Our results III

Pure graph theory restatement!

Corollary (Graph statement)

Let H be a connected graph on n vertices.
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Proof by quantum circuit.
Can we find a purely combinatorial proof?

Thank you for your attention!

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