Recognizing Knödel and Fibonacci graphs

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Abstract

Knödel graphs and Fibonacci graphs are two classes of bipartite incident-graph of circulant digraphs. Both graphs have been extensively studied for the purpose of fast communications in networks, and they have deserved a lot of attention in this context. In this paper, we show that there exists an $O(n \log^5 n)$-time algorithm to recognize Knödel graphs of order $2n$, and that the same technique applies to Fibonacci graphs. The algorithm is based on a characterization of the cycles of length six in these graphs (bipartite incident-graphs of circulant digraphs always have cycles of length six). A consequence of our result is that the circulant digraphs whose chords are the power of two minus one, and the circulant digraphs whose chords are the Fibonacci numbers minus one, can be recognized in $O(n \log^5 n)$ time. An open problem that arises in this field is to derive an $O(m \log^{O(1)} n)$-time algorithm for any m-edge infinite family of bipartite incident-graphs of circulant digraphs.

Keywords: graph isomorphism, circulant graphs, chordal rings, broadcasting, gossiping.

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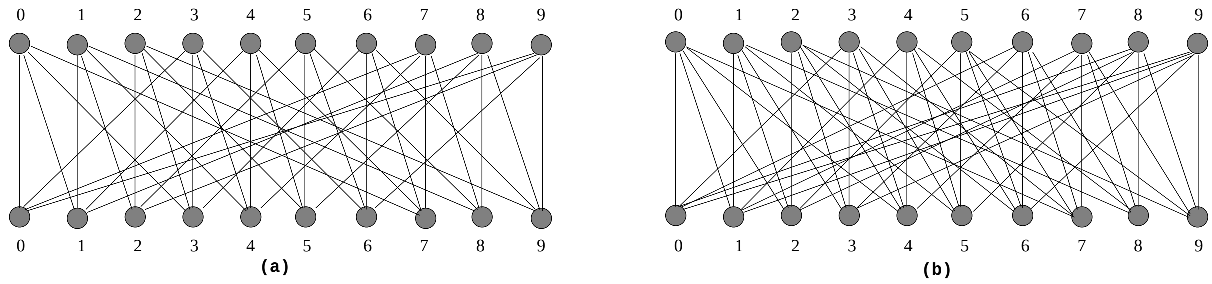


Figure 1: A Knödel graph of 20 vertices (a), and a Fibonacci graph of 20 vertices (b).

1 Introduction

So-called Knödel graphs and Fibonacci graphs have been used by Knödel [11], and Even and Monien [6] (see also [4, 12, 17]), respectively, for the purpose of performing efficient communications in networks. More precisely, consider a network of n nodes, and assume that communications among the nodes proceed by a sequence of synchronous calls between neighboring vertices. A *round* is defined as the set of calls performed at the same time. Knödel on one hand, and Even and Monien on the other hand, were interested in computing the minimum number of rounds necessary to perform an all-to-all broadcasting, also called *gossiping*, between the nodes (see [7, 8, 10] for surveys on gossiping and related problems). The communication constraints assume that a call involves exactly two neighboring nodes, and that a node can communicate to at most one neighbor at a time. Knödel considered the 2-way mode (full-duplex) in which the two nodes involved in the same call can exchange their information in one round, whereas Even and Monien considered the 1-way mode (half-duplex) in which the information can flow in one direction at a time, that is, during the call between x and y, only y can receive information from x, or x can receive information from y, not both. Under these hypotheses, it was shown in [11] that, for n even, one cannot perform gossiping in less that $\lceil \log_2 n \rceil$ rounds in the 2-way mode, and that there are graphs, called here Knödel graphs, that allow gossiping to be performed in $\lceil \log_2 n \rceil$ rounds. Even and Monien have shown in [6] that, for n even, one cannot perform gossiping in less than $2 + \lceil \log_\varrho \frac{n}{2} \rceil$ rounds in the 1-way mode, where $\varrho = \frac{1+\sqrt{5}}{2}$, and that there are graphs, called here Fibonacci graphs, that allow gossiping to be performed in that number of rounds (up to an additive factor of one).

Knödel graphs and Fibonacci graphs are bipartite graphs $G = (V_1, V_2, E)$ of $2n$ vertices. Each partition has n vertices labeled from 0 to $n-1$. In the Knödel graphs, there is an edge between $x \in V_1$ and $y \in V_2$ if and only if there exists $i \in \{0, 1, \ldots, k\}$ such that $y = x + 2^i - 1 \pmod{n}$, $k = \lfloor \log_2 n \rfloor$. In the Fibonacci graphs, there is an edge between $x \in V_1$ and $y \in V_2$ if and only if there exists $i \in \{0, 1, \ldots, k\}$ such that $y = x + F(i+1) - 1 \pmod{n}$, $k = F^{-1}(n) - 1$, where $F(i)$ denotes the ith Fibonacci number ($F(0) = F(1) = 1$, and $F(i) = F(i-1) + F(i-2)$ for $i \geq 2$) and $F^{-1}(n)$ denotes the integer i for which $F(i) \leq n < F(i+1)$. Both graphs are Cayley graphs on the dihedral group, and thus they are vertex-transitive. See Figure 1 for an example of a Knödel graph and a Fibonacci graph. Note that graphs on Figure 1 look pretty dense but Knödel graphs and Fibonacci graphs are of degree $O(\log n)$.

Knowing whether a graph G of n nodes allows gossiping to be performed optimally, that is in $\lceil \log_2 n \rceil$ rounds in the 2-way mode, and in (about) $\lceil \log_\varrho n \rceil$ rounds in the 1-way mode, is NP-complete [4]. In particular there are graphs that are not isomorphic to Knödel graphs (resp.

Fibonacci graphs), and that allow gossiping to be performed optimally in the 2-way mode (resp. 1-way mode). In this paper, we want to recognize Knödel graphs and Fibonacci graphs. In other words, given a graph G, we want to know whether G is isomorphic to a Knödel graph, or to know whether G is isomorphic to a Fibonacci graph.

A closely related topic deals with circulant digraphs. Recall that a digraph is circulant if nodes can be labeled so that the adjacency matrix is circulant, that is node x has $k+1$ out-neighbors $x + g_i \pmod n$ for $i = 0, \dots, k$ for some k and some given sequence of g_i's independent of x. Circulant digraphs are Cayley digraphs over $\mathbb{Z}_n$. Ponomarenko has given in [16] a polynomial-time algorithm to decide whether a given tournament is a circulant digraph (a tournament is a digraph obtained by giving an orientation to the edges of a complete graph). More recently, Muzychuk and Tinhofer [14] have shown that one can decide in polynomial-time whether a digraph of prime order is circulant. Deciding whether two circulant digraphs are isomorphic is also a difficult problem. Ádám [1] conjectured that two circulant digraphs are isomorphic if and only if the generators of one digraph can be obtained from the generators of the other digraph via a product by a constant. This conjecture is wrong [5] although it holds in many cases. For instance, Alspach and Parsons [2] have proved that Ádám's conjecture is true for values of n such as the product of two primes (see also [3, 13, 15]).

Nevertheless, even if they are closely related to circulant digraphs, Knödel graphs and Fibonacci graphs are not circulant graphs but bipartite incident-graphs of circulant digraphs, and they are thus sometimes called bi-circulant graphs. (The bipartite incident-graph of a digraph $H = (V, A)$ is a graph $G = (V_1, V_2, E)$ such that $V_1 = V_2 = V$, and for any $x_1 \in V_1$, and $x_2 \in V_2$, $\{x_1, x_2\} \in E \Leftrightarrow (x_1, x_2) \in A$. Note that two non isomorphic digraphs H and H' can yield isomorphic bipartite incident-graphs, e.g., $H = (\{u,v\}, \{(u,u),(v,v)\})$ and $H' = (\{u,v\}, \{(u,v),(v,u)\})$.) It is unknown whether there exists a polynomial-time algorithm to decide whether a given graph is isomorphic to a given circulant digraph or a given incident-graph of a circulant digraph. This is why we have studied specific algorithms for the case of Knödel graphs and Fibonacci graphs.

The paper is organized as follows. In Section 2, we study the form of solutions of equations involving powers of two. The characterization of these solutions allows us to recognize Knödel graphs in $O(n \log^5 n)$ time, as shown in Section 3. This algorithm is optimal up to a polylogarithmic factor since Knödel graphs have $\Theta(n \log n)$ edges. Section 4 concludes the paper with some remarks on bipartite incident-graphs of circulant digraphs defined by an arbitrary increasing sequence $(g_i)_{i \geq 0}$ of integers, including Fibonacci graphs.

2 Preliminary Results

Let $H = (V, A)$ be a circulant digraph of n vertices and of generators $g_0, \dots, g_k$, and let $G = (V_1, V_2, E)$ be the corresponding bipartite incident-graph. By 6-*cycle*, we mean an elementary cycle of length six.

Lemma 1 *There is a 6-cycle in G if and only if one can find a sequence of six generators*

$$(g_{i_0}, g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}, g_{i_5})$$

and $\alpha \in \{0,1,2\}$ such that

$$\begin{cases} g_{i_0} + g_{i_2} + g_{i_4} = g_{i_1} + g_{i_3} + g_{i_5} + \alpha\, n \\ g_{i_j} \neq g_{i_{j+1 \pmod 6}} \text{ for any } j \in \{0,1,2,3,4,5\} \end{cases} \tag{1}$$

Proof. Let $(u_0, u_1, u_2, u_3, u_4, u_5)$ be a 6-cycle in G (all the u_i's are pairwise distinct), and assume without loss of generality that $u_0 = 0 \in V_1$. We have:

$$\begin{cases} u_1 = u_0 + g_{i_0} \\ u_1 = u_2 + g_{i_1} + \alpha_1\, n \\ u_3 = u_2 + g_{i_2} + \alpha_2\, n \\ u_3 = u_4 + g_{i_3} + \alpha_3\, n \\ u_5 = u_4 + g_{i_4} + \alpha_4\, n \\ u_5 = u_0 + g_{i_5} \end{cases}$$

where $\alpha_i \in \{-1, 0\}$ and $g_{i_j} \neq g_{i_{j+1 \pmod 6}}$ for any $j \in \{0,1,2,3,4,5\}$ since the cycle uses six different edges. Therefore we have

$$g_{i_0} + g_{i_2} + g_{i_4} = g_{i_1} + g_{i_3} + g_{i_5} + \alpha\, n$$

where $\alpha = (\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4)$, that is $-2 \leq \alpha \leq 2$ by definition of the α_i's. By possibly swapping even and odd g's, we get the claimed result with $0 \leq \alpha \leq 2$.

Conversely, let $(g_{i_0}, g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}, g_{i_5})$ be such that

$$\begin{cases} g_{i_0} + g_{i_2} + g_{i_4} = g_{i_1} + g_{i_3} + g_{i_5} + \alpha\, n \\ g_{i_j} \neq g_{i_{j+1 \pmod 6}} \text{ for any } j \in \{0,1,2,3,4,5\} \end{cases}$$

with $\alpha \in \{0,1,2\}$. Then let $u_0 \in V_1$, and let

$$\begin{cases} u_1 = u_0 + g_{i_0} \bmod n \\ u_2 = u_1 - g_{i_1} \bmod n \\ u_3 = u_2 + g_{i_2} \bmod n \\ u_4 = u_3 - g_{i_3} \bmod n \\ u_5 = u_4 + g_{i_4} \bmod n \\ u_6 = u_5 - g_{i_5} \bmod n \end{cases}$$

We have

$$u_6 = u_0 + (g_{i_0} + g_{i_2} + g_{i_4}) - (g_{i_1} + g_{i_3} + g_{i_5}) \pmod n.$$

Since

$$g_{i_0} + g_{i_2} + g_{i_4} = g_{i_1} + g_{i_3} + g_{i_5} \pmod n,$$

we get $u_6 = u_0$, and therefore $(u_0, u_1, u_2, u_3, u_4, u_5)$ is a cycle of length six in G. This cycle is elementary because $g_{i_j} \neq g_{i_{j+1 \pmod 6}}$ for any $j \in \{0,1,2,3,4,5\}$. ■

From Lemma 1, any bipartite incident-graph of a circulant digraph has 6-cycles since $g_{i_0} = g_{i_3}$, $g_{i_1} = g_{i_4}$, and $g_{i_2} = g_{i_5}$ is a solution of Equation 1. We will solve Equation 1 to characterize 6-cycles of Knödel graphs, and to identify the possible generators of a candidate to be a Knödel graph. Let $x_0, x_1, x_2, x_3, x_4, x_5$ be six integers in $\{0, \ldots, k\}$, and let n be any integer such that $2^k \leq n < 2^{k+1}$. From Lemma 1, we are interested in computing the solutions of the equation

$$\begin{cases} 2^{x_0} + 2^{x_2} + 2^{x_4} = 2^{x_1} + 2^{x_3} + 2^{x_5} + \alpha\, n \\ x_i \neq x_{i+1 \pmod 6} \text{ for any } i \in \{0,1,2,3,4,5\} \end{cases} \tag{2}$$

where $\alpha \in \{0,1,2\}$.

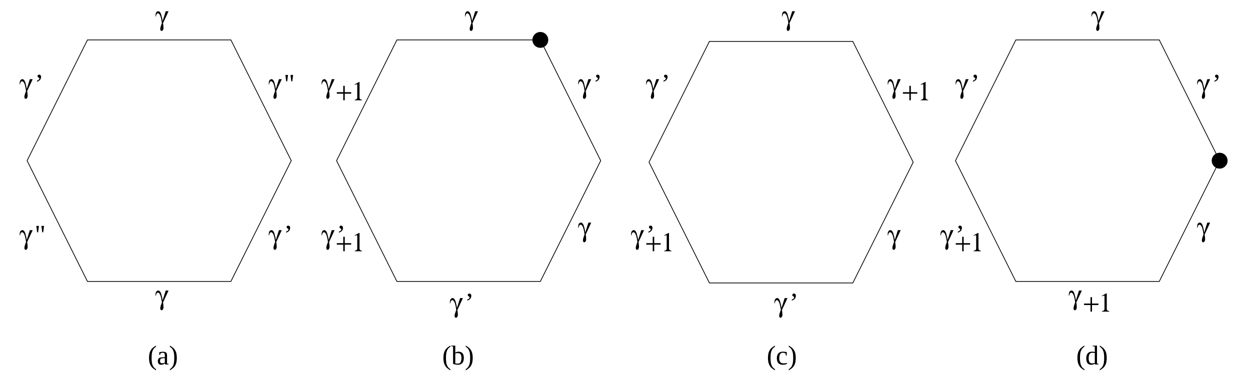


Figure 2: The four types of solutions of Equation 2 for $\alpha = 0$.

Lemma 2 *For $\alpha = 0$, Equation 2 has four types of solutions:*
$(x_0, x_1, x_2, x_3, x_4, x_5) =$

$$\begin{array}{llll}
\text{a)} & (\gamma, \gamma'', \gamma', \gamma, \gamma'', \gamma') & \gamma, \gamma', \gamma'' \in \{0, \ldots, k\} & \gamma \neq \gamma', \gamma' \neq \gamma'', \gamma'' \neq \gamma \\
\text{b)} & (\gamma, \gamma', \gamma, \gamma', \gamma'+1, \gamma+1) & \gamma, \gamma' \in \{0, \ldots, k-1\} & \gamma \neq \gamma' \\
\text{c)} & (\gamma, \gamma+1, \gamma, \gamma', \gamma'+1, \gamma') & \gamma, \gamma' \in \{0, \ldots, k-1\} & \gamma \neq \gamma' \\
\text{d)} & (\gamma, \gamma', \gamma, \gamma+1, \gamma'+1, \gamma') & \gamma, \gamma' \in \{0, \ldots, k-1\} & \gamma \neq \gamma'
\end{array}$$

up to cyclic permutations[1] *of the x_i's.*

Proof. The case x_0, x_2, x_4 pairwise distinct generates the first type of solutions. Assume $x_0 = x_2 = \gamma$ and $x_4 = \gamma' \neq \gamma$. There is an impossibility to solve Equation 2 if $\gamma = \gamma' - 1$ because it would imply either $x_0 = x_1$ or $x_0 = x_5$. If $\gamma \neq \gamma' - 1$, we get a solution if two x_{2i+1}'s are both equal to $\gamma' - 1$, and the third one is equal to $\gamma + 1$. This generates the three last types of solutions by changing $\gamma' - 1$ into γ'. ■

The solutions of Lemma 2 induces cycles of length 6. These 6-cycles have their edges labeled by *dimensions* as illustrated on Figure 2 (the generator $2^i - 1$ induces edges in dimension i). However, cycle (b) and cycle (d) are isomorphic (just travel (b) clockwise and (d) counterclockwise from the black node), that is the second and the fourth types of solutions induces the same labeled cycle. In the following, only cycles (a), (b), and (c) will be considered.

Notation. The number of blocks of consecutive 1's in the binary representation of n will be denoted by $B_1(n)$.

For instance $B_1((1101100111010)_2) = 4$, $B_1((100)_2) = 1$, $B_1((101)_2) = 2$, and $B_1((0)_2) = 0$. Integers of the form

$$n = (\overline{1}\underbrace{00\ldots00}\overline{1}\underbrace{00\ldots0\overline{0}}\overbrace{11\ldots11}\underline{0}\overbrace{11\ldots11}\underline{0}\overbrace{11\ldots1\underline{1}}\underbrace{00\ldots00})_2$$

satisfy $B_1(n) = 5$, and there is a solution to Equation 2 for $\alpha = 1$ with x_1, x_3, x_5 equal to the underlined bit-positions, and x_0, x_2, x_4 equal to the over-lined bit-positions. The reader can check that the function B_1 satisfies the simple following property:

[1] A permutation σ of p symbols is a cyclic permutation if $\sigma(x_1, x_2, \ldots, x_p) = (x_2, \ldots, x_p, x_1)$.

Lemma 3 *For any n and x, $B_1(n+2^x) \geq B_1(n)-1$. Moreover, if $B_1(n+2^x) = B_1(n)-1$, then one of the blocks of consecutive 1's of $n+2^x$ contains at least two 1-entries.*

We then have the following lemma:

Lemma 4 *If $B_1(n) \geq 6$, then Equation 2 has no solution for $\alpha \neq 0$.*

Proof. If $B_1(n) \geq 6$, then the sum of n and three powers of two cannot result in the sum of three powers of two. Indeed, the binary representation of $2^{x_0}+2^{x_2}+2^{x_4}$ has at most three 1-entries, that is $B_1(2^{x_0}+2^{x_2}+2^{x_4}) \leq 3$. On the other hand, from Lemma 3, for every n, $B_1(n+2^{x_1}+2^{x_3}+2^{x_5}) \geq B_1(n)-3$. Moreover, for n such that $B_1(n) \geq 6$, the binary representation of $n+2^{x_1}+2^{x_3}+2^{x_5}$ has at least four 1-entries. This completes the proof for $\alpha = 1$. The result holds for $\alpha = 2$ too because $B_1(2n) = B_1(n)$. ■

Lemma 4 has an important consequence that is, for most of the integers n, Knödel graphs of order $2n$ have 6-cycles only of the form given on Figure 2. There are orders however for which Equation 2 has solutions for $\alpha \neq 0$. This is typically the case for $n = 2^k$. Actually, this special case deserves a particular interest motivated by the simplicity of the solution.

Lemma 5 *If $n = 2^k$ then every 4-cycle of the Knödel graph of order $2n$ is a labeled cycle*

$$(k, \gamma-1, \gamma, \gamma-1) \text{ for } \gamma \in \{1, \ldots, k\}.$$

Proof. By similar arguments as in the proof of Lemma 1, one can check that 4-cycles exist if and only if there exist four generators $g_i = 2^{x_i}-1$, $0 \leq i \leq 3$, $x_i \neq x_{i+1} \pmod 4$, satisfying $2^{x_0}+2^{x_2} = 2^{x_1}+2^{x_3}+\alpha n$ for $\alpha \in \{0,1\}$. The equation $2^{x_0}+2^{x_2} = 2^{x_1}+2^{x_3}$ has no solution for $x_i \neq x_{i+1} \pmod 4$. Therefore, 4-cycles exist only for solutions of $2^{x_0}+2^{x_2} = 2^{x_1}+2^{x_3}+2^k$, that is for $x_0 = k$, and $x_1 = x_3 = x_2 - 1$, $x_2 \in \{1, \ldots, k\}$. Thus the solution is $(x_0, x_1, x_2, x_3) = (k, \gamma-1, \gamma, \gamma-1)$, up to a square-cyclic permutation[2] of the x_i's. ■

3 Recognizing Knödel Graphs

In order to recognize Knödel graphs, we use the following basic property:

Lemma 6 *Let G be a graph of order $2n$, and let $k = \lfloor \log_2 n \rfloor$. Assume that some edges of G are colored by 0 or 1. G is a Knödel graph whose edges in dimension 0 and 1 are colored 0 and 1 respectively if and only if:*

1. *Any path P using colors 0 and 1, alternatively, from an arbitrary node is Hamiltonian; and*
2. *Assuming the jth node of P is labeled $((j+1) \bmod 2, \lfloor j/2 \rfloor)$, $j \geq 0$, we have, for every $i \in \{0, \ldots, k\}$ and every $x \in \{0, \ldots, n-1\}$, there is an edge connecting node $(0,x)$ with node $(1, x+2^i-1 \bmod n)$, and no extra edges.*

[2]A permutation σ of p symbols is a square-cyclic permutation if $\sigma(x_1, x_2, x_3 \ldots, x_p) = (x_3, \ldots, x_p, x_1, x_2)$.

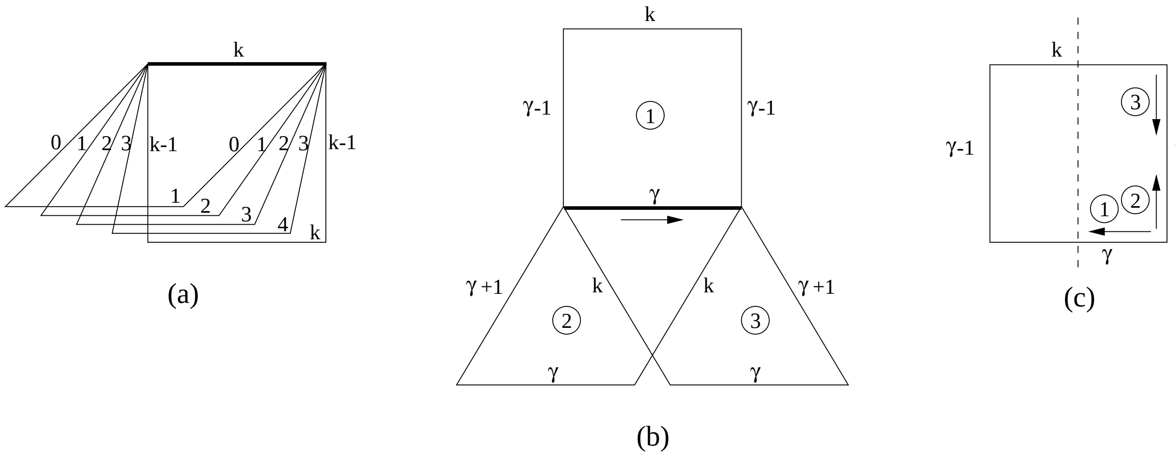


Figure 3: The 4-cycles in a Knödel graph of order $2 \cdot 2^k$.

Proof. Let $G = (V_1, V_2, E)$ be a Knödel graph whose nodes are labeled (i, x), $i = 0, 1$ and $x \in \{0, \ldots, n-1\}$ so that $V_1 = \{(0, x), x \in \{0, \ldots, n-1\}\}$, and $V_2 = \{(1, x), x \in \{0, \ldots, n-1\}\}$. Assume that the edges in dimension 0 and 1 of G are colored 0 and 1 respectively. Then any 0/1-path P is Hamiltonian. Let u be the starting point of P. Since a Knödel graph is vertex-transitive, one can assume w.l.g. that $u = (1, 0)$. Therefore, the labeling induced by the path corresponds to the connections of a Knödel graph.

Conversely, let G be a graph whose some edges are colored by 0 or 1. Let P be a path using colors 0 and 1, alternatively, from an arbitrary node of G. Assume P is Hamiltonian, label the vertices according to the rule of the second property, and assume the connection rule fulfills. Then G is a Knödel graph by definition. ■

Let us start with $n = 2^k$, $k \geq 2$. From Lemma 5, we know that there is only one type of labeled 4-cycle in a Knödel graph of order $2n$, namely $(k, \gamma - 1, \gamma, \gamma - 1)$, for $\gamma \in \{1, \ldots, k\}$. Therefore, for any edge of dimension k, there are k 4-cycles using that edge (see Figure 3(a)). For any edge of dimension $k-1$, there are two 4-cycles using that edge (see Figure 3(b) where cycle 2 and cycle 3 are the same). For any edge of dimension γ, $\gamma \in \{1, \ldots, k-2\}$, there are three 4-cycles using that edge (see Figure 3(b)). Finally, for any edge of dimension 0, there are two 4-cycles using that edge (the cycle 1 in Figure 3(b) does not exist for $\gamma = 0$). This counting yields the following corollary of Lemma 5:

Corollary 1 *There exists an $O(n \log^3 n)$-time algorithm to recognize Knödel graphs of order $2n = 2^{k+1}$, $k \geq 0$.*

Proof. Assume $k \geq 4$ (note that for any $k \leq k_0 = O(1)$, recognizing Knödel graphs of order 2^{k+1} can be done in constant time). Given an input graph $G = (V, E)$, we count the number $C_4(e)$ of 4-cycles passing through any edge $e \in E$. From the counting of the number of 4-cycles passing through an edge in a Knödel graph, if there is an edge e such that $C_4(e) \neq 2$, $C_4(e) \neq 3$, and $C_4(e) \neq k$ then G is not a Knödel graph. Otherwise, every edge e with $C_4(e) = 2$ is a candidate to be an edge of dimension 0 or dimension $k-1$, and every edge e with $C_4(e) = k$ (recall $k \neq 2$ and $k \neq 3$) is a candidate to be an edge of dimension k. From Figure 3(b), edges in dimension $\gamma = k-1$ are edges e such that $C_4(e) = 2$ and included in a 4-cycle containing an edge of dimension

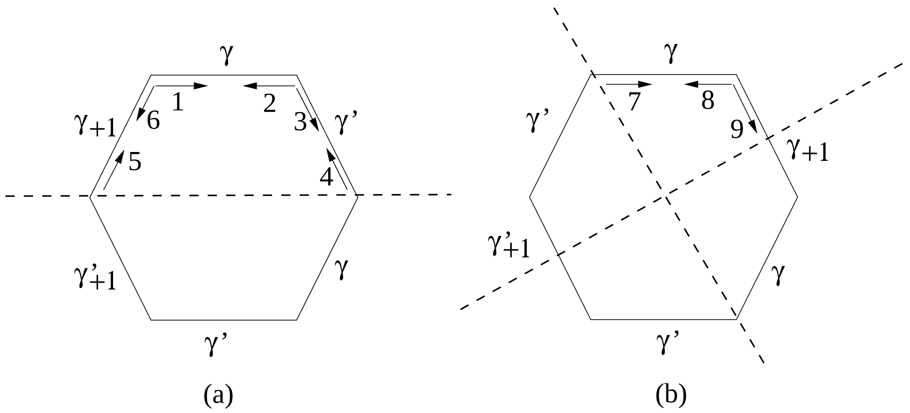


Figure 4: (a) The six possible travels of the cycle of type (b) in Figure 2. (b) The three possible travels of the cycle of type (c) in Figure 2.

k which is not adjacent to e in this cycle. This allows us to distinguish dimensions 0 and $k-1$. From dimensions 0 and k, one can identify dimension 1 by considering all paths of type $0, k, 0$. The end vertices of each such path are connected by an edge of dimension 1 (see Figure 3(a)). Color the edge of dimension 0 and 1 of G by colors 0 and 1, respectively, and check the conditions of Lemma 6.

This algorithm has a cost of $O(n \log^3 n)$ because, for every edge e, counting the number of 4-cycles using that edge takes at most a time of $O(\log^2 n)$ assuming node-adjacency testing in constant time. (The two end-vertices of every edge are both adjacent to $O(\log n)$ nodes. Thus, by testing all the possible edges between these nodes, one can determine the 4-cycles in $O(\log^2 n)$ time.) Checking the conditions of Lemma 6 takes $O(n \log n)$ time. ■

Let us carry on our study by considering integers n such that $B_1(n) \geq 6$. In this case, one can apply Lemmas 2 and 4, and we are dealing with the four types of cycles of Figure 2 (recall that cycles (b) and (d) are isomorphic). Figure 2(a) implies that, for any edge of dimension γ, $\gamma \in \{0, \ldots, k\}$, there are $k(k-1)$ 6-cycles of type (a) using that edge.

The contribution of cycles of type (b) in Figure 2 to each dimension is more difficult to calculate. We proceed as for the 4-cycles studied in the case $n = 2^k$. The counting for $n = 2^k$ can be formalized as follows. Consider again Figure 3. On Figure 3(c), there are four edges (whose one is of a fixed dimension, dimension k), and two possible senses of travel, clockwise and counterclockwise. For any γ such that $1 \leq \gamma \leq k-1$, there are potentially four positions for γ. However, only three of them are valid because $\gamma \neq k$. Moreover, once the position of γ has been fixed, we have only two ways to travel around the cycle. This fact gives six possible labeled cycles for any edge to belong to. However, only three of these 4-cycles are distinct because there is a symmetry along the axis perpendicular to dimension k. This symmetry reduces the number of travels by a factor of 2: for each travel of Figure 3(c), there is a corresponding travel in Figure 3(b) starting in the direction indicated by the arrow.

Coming back to the 6-cycle of Figure 2(b), there are six edges, and two possible directions (clockwise and counterclockwise). Thus we get twelve possibilities to travel along the edges of a labeled 6-cycle. However, we actually get only six travels for cycle (b) of Figure 2 because of a symmetry along the axis parallel to γ, γ' (see Figure 4(a)). These six travels are:

$$\left.\begin{array}{llllllll}
1. & (\gamma & \gamma' & \gamma & \gamma' & \gamma'+1 & \gamma+1) \\
2. & (\gamma & \gamma+1 & \gamma'+1 & \gamma' & \gamma & \gamma') \\
3. & (\gamma & \gamma' & \gamma & \gamma+1 & \gamma'+1 & \gamma') \\
4. & (\gamma & \gamma' & \gamma'+1 & \gamma+1 & \gamma & \gamma')
\end{array}\right\} \gamma, \gamma' \in \{0, \ldots, k-1\}, \gamma \neq \gamma';$$

and

$$\left.\begin{array}{llllllll}
5. & (\gamma & \gamma-1 & \gamma' & \gamma-1 & \gamma' & \gamma'+1) \\
6. & (\gamma & \gamma'+1 & \gamma' & \gamma-1 & \gamma' & \gamma-1)
\end{array}\right\} \gamma \in \{1, \ldots, k\}, \gamma' \in \{0, \ldots, k-1\}, \gamma' \neq \gamma-1.$$

We do the same analysis for 6-cycles of type (c) in Figure 2. The counting uses the fact that Figure 2(c) is symmetric along the axis perpendicular to the edges $\gamma+1$ and $\gamma'+1$, and along the axis parallel to $\gamma+1, \gamma'+1$ (see Figure 4(b)). Therefore, we get three new possibilities:

$$\left.\begin{array}{llllllll}
7. & (\gamma & \gamma+1 & \gamma & \gamma' & \gamma'+1 & \gamma') \\
8. & (\gamma & \gamma' & \gamma'+1 & \gamma' & \gamma & \gamma+1)
\end{array}\right\} \gamma, \gamma' \in \{0, \ldots, k-1\}, \gamma \neq \gamma';$$

and

$$9. \quad (\gamma \quad \gamma-1 \quad \gamma' \quad \gamma'+1 \quad \gamma' \quad \gamma-1) \ , \gamma \in \{1, \ldots, k\}, \gamma' \in \{0, \ldots, k-1\}, \gamma' \neq \gamma-1.$$

Note that:

- travels 1, 3, and 7 are the same if replacing γ' by $\gamma+1$ in all these travels,
- travels 2, 4, and 8 are the same if replacing γ' by $\gamma+1$ in all these travels,
- travels 3, 4, and 9 are the same if replacing γ' by $\gamma-1$ in travels 3 and 4, and γ' by γ in 9,
- travels 1, 5, and 8 are the same if replacing γ' by $\gamma-1$ in travels 1 and 8, and γ' by γ in 5,
- travels 2, 6, and 7 are the same if replacing γ' by $\gamma-1$ in travels 2 and 7, and γ' by γ in 6, and
- travels 5, 6, and 9 are the same if replacing γ' by $\gamma-2$ in all these travels.

Now we can count the number of 6-cycles using a given edge of a Knödel graph.

Lemma 7 *Assume $B_1(n) \geq 6$, and let e be an edge of a Knödel graph of order $2n$. Let $C_6(e)$ denote the number of 6-cycles using edge e. We have:*

$$C_6(e) = \begin{cases} k^2+5k-10 & \text{if } e \text{ is of dimension } 0 \\ k^2+8k-19 & \text{if } e \text{ is of dimension } 1 \\ k^2+8k-21 & \text{if } e \text{ is of dimension } \gamma,\ 2 \leq \gamma \leq k-2 \\ k^2+8k-17 & \text{if } e \text{ is of dimension } k-1 \\ k^2+2k-5 & \text{if } e \text{ is of dimension } k \end{cases}$$

Proof. Let e be an edge of dimension γ, $0 \leq \gamma \leq k$. Let us count the contribution to that edge of the nine travels identified in Figure 4. Let $\gamma = 0$. Travel 1 contributes for $k-1$ cycles, one for each $\gamma' \neq 0$. Similarly, travel 2 contributes for $k-1$ cycles, one for each $\gamma' \neq 0$. Travel 3 contributes for $k-2$ cycles only, because travels 1 and 3 are the same if $\gamma' = 1$ in both travels. Similarly, travel 4 contributes for $k-2$ cycles only, because travels 2 and 4 are the same if $\gamma' = 1$ in both travels. Note that travels 3 and 4 are always different if $\gamma = 0$ because they are the same only if $\gamma' = \gamma - 1 = -1$ which is out of the range 0 to $k-1$. Travels 5, 6, and 9 do not contribute because they assume $\gamma \geq 1$. Finally, travels 7 and 8 contribute for $k-2$ each. Applying the same counting for all the dimensions, we get the following table:

$\gamma = 0$		$\gamma = 1$		$2 \leq \gamma \leq k-2$		$\gamma = k-1$		$\gamma = k$	
travel	# cycle	travel	# cycle	travel	# cycle	travel	# cycle	travel	# cycle
1	$k-1$	1	$k-1$	1	$k-1$	1	$k-1$	1	0
2	$k-1$	2	$k-1$	2	$k-1$	2	$k-1$	2	0
3	$k-2$	3	$k-2$	3	$k-2$	3	$k-1$	3	0
4	$k-2$	4	$k-3$	4	$k-3$	4	$k-2$	4	0
5	0	5	$k-2$	5	$k-2$	5	$k-2$	5	$k-1$
6	0	6	$k-2$	6	$k-3$	6	$k-3$	6	$k-2$
7	$k-2$	7	$k-3$	7	$k-3$	7	$k-2$	7	0
8	$k-2$	8	$k-3$	8	$k-3$	8	$k-2$	8	0
9	0	9	$k-2$	9	$k-3$	9	$k-3$	9	$k-2$
Tot.	$6k-10$	*Tot.*	$9k-19$	*Tot.*	$9k-21$	*Tot.*	$9k-17$	*Tot.*	$3k-5$

We get $6k-10$ cycles for dimension 0, $9k-19$ cycles for dimension 1, $9k-21$ cycles for dimension γ, $2 \leq \gamma \leq k-2$, $9k-17$ cycles for dimension $k-1$, and $3k-5$ cycles for dimension k. Then add $k(k-1)$ cycles from the solutions of type (a) in Figure 2 to get the claimed result. ∎

Corollary 2 *There exists an $O(n \log^5 n)$-time algorithm to recognize Knödel graphs of order $2n$, for all n such that $B_1(n) \geq 6$.*

Proof. From Lemma 7, one can identify edges of dimensions 0 and 1 in $O(n \log^5 n)$-time. In time $O(n \log n)$, one can check the conditions of Lemma 6. ∎

Theorem 1 *There exists an $O(n \log^5 n)$-time algorithm to recognize Knödel graphs of any order.*

The rest of the section is dedicated to the proof of that theorem. Let us assume $k \geq k_0$ for $k_0 = O(1)$ large enough. From Corollaries 1 and 2, the theorem holds for n power of two, or n such that $B_1(n) \geq 6$. Thus, assume that n satisfies $B_1(n) < 6$, $n \neq 2^k$. The key argument is that, for almost all such n, if $C_6(e)$ denotes the number of 6-cycles passing through an edge e of a Knödel graph, then $C_6(e) \neq C_6(e')$ for any e and e' of dimensions 0 and k respectively, and $C_6(e) \neq C_6(e')$ for any e dimension 0 or k, and any e' of dimension i, $i \neq 0$ and $i \neq k$. Moreover, for the few n's not satisfying this condition, one can use additional arguments to identify dimension 0 and k.

The difficulty of the proof comes from the fact that, if $B_1(n) < 6$, then Equation 2 has solutions for $\alpha \neq 0$, and proceeding to a precise counting of the number of 6-cycles passing through the edges of a Knödel graph is tricky. From the knowledge of the edges of dimension 0 and k, we will show that one can determine the set of edges of dimension 1. Then it will remain only to check the conditions of Lemma 6.

To summarize, the algorithm is the following:

Algorithm *Recognize.*
`Input:` a regular graph $G = (V, E)$ of $2n$ vertices, and degree $k = \lfloor \log_2 n \rfloor$;
`Output:` tell whether or not G is isomorphic to the Knödel graph K of order $2n$.

Phase 1. For every $i \in \{0, \ldots, k\}$, compute the number of 6-cycles passing through any edge of dimension i of K;

Phase 2. For every $e \in E$, compute $C_6(e)$ = number of 6-cycles passing through the edge e of G;

Phase 3. Identify the two sets $S_0 \subset E$, and $S_1 \subset E$, of edges of G that are possibly edges of dimension 0 and 1 in K, respectively;

Phase 4. Check the conditions of Lemma 6.

The first phase has a cost of $O(\log^6 n)$ to derive all solutions of Equation 2. Assuming node-adjacency testing in constant time, the second phase has a cost at most $O(n \log^5 n)$ because the degree of every vertex is $O(\log n)$. We will see later that the third phase costs $O(n \log^5 n)$ time. Finally, the fourth phase cost $O(n \log n)$ time.

To prove the correctness of Algorithm *Recognize*, the difficult part is to show that Phase 3 is doable. For that purpose, let n be an integer that is different from a power of 2, and satisfying $B_1(n) < 6$. If Equation 2 has no solution for $\alpha \neq 0$, then, as far as the number of 6-cycles is concerned, we are let with a similar case as the ones studied in Lemma 7, and thus the theorem holds by the same arguments as in the proof of Corollary 2. The aim of the rest of the proof is to study the Knödel graphs of order $2n$ for which Equation 2 has solutions for $\alpha \neq 0$.

Lemma 8 *Determining the edges of dimensions* 0 *and* k *allows to determine the edges of dimensions* 0 *and* 1*.*

Proof. Given the list of all the edges of dimension 0, and the list of all the edges of dimension k, let us consider 6-cycles of type

$$(k, \gamma, 0, \gamma', 0, \gamma'')$$

where γ, γ' and γ'' are *a priori* unknown, apart the fact that $\gamma \neq k$, $\gamma \neq 0$, $\gamma' \neq 0$, $\gamma'' \neq 0$, and $\gamma'' \neq k$. From Equation 2, such a 6-cycle satisfies:

$$2^k + 2^0 + 2^0 = 2^\gamma + 2^{\gamma'} + 2^{\gamma''} + \alpha\, n$$

where $\alpha \in \{-2, -1, 0, 1, 2\}$. Now, $\alpha = 2$ is impossible because $2n > 2^{k+1}$. If $\alpha = 1$, then $2 = 2^\gamma + 2^{\gamma'} + 2^{\gamma''} + n'$ where $n = 2^k + n'$, $0 < n' < 2^k$. This is impossible because $2^\gamma + 2^{\gamma'} + 2^{\gamma''} + n' \geq 4$. If $\alpha \leq -1$, then $-\alpha n + 2^k + 2 > 2^\gamma + 2^{\gamma'} + 2^{\gamma''}$ because $2^{k+1} \geq 2^\gamma + 2^{\gamma'} + 2^{\gamma''}$ due to the fact that $\gamma \neq k$ and $\gamma'' \neq k$. It remains the case $\alpha = 0$ which implies that at least two of the γ's are the same. Thus we are let with the equation $2^k + 2 = 2^{p+1} + 2^q$, and hence either $p = 0$ and $q = k$, or $p = k - 1$ and $q = 1$. Since two γ's are distinct from k, the only 6-cycles of type

$$(k, \gamma, 0, \gamma', 0, \gamma'')$$

are those of type

$$(k, k-1, 0, k-1, 0, 1)$$

up to any permutation of the odd positions. This yields the two non isomorphic labeled cycles of Figure 5.

Therefore, by looking at all 6-cycles containing non-adjacent edges of dimensions $0, 0$, and k, we can construct a set S of edges such that $e \in S$ if and only if e is either of dimension 1, or of dimension $k - 1$. To distinguish dimension 1 from dimension $k - 1$, we construct, for every edge $\{u, v\}$ of dimension k, the three 6-cycles corresponding to Figure 5. We get a situation as depicted

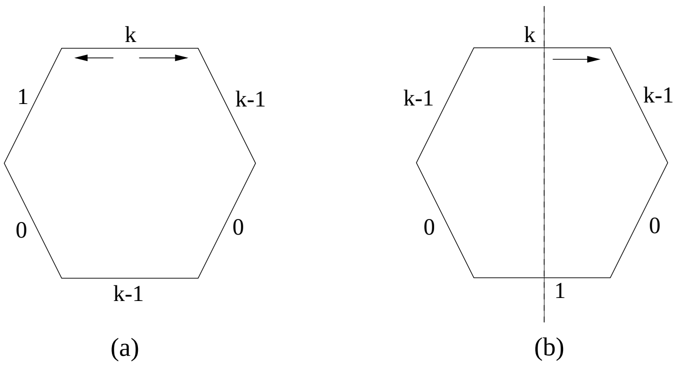


Figure 5: The 6-cycles containing non-adjacent edges of dimensions $0, 0$, and k.

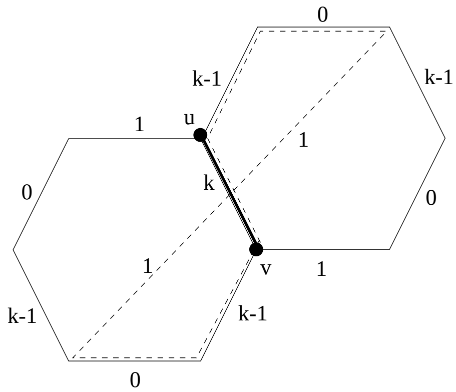


Figure 6: The three 6-cycles containing opposite edges of dimensions $0, 0$, and k, passing through an edge of dimension k.

on Figure 6. The edges of dimension 1 are those incident to nodes u or v, and through which passes only one of the three considered 6-cycles. ■

From that lemma, determining the edges of dimensions 0, and k allows to determine the edges of dimension 0 and 1 for the purpose of Phase 3 of the algorithm *Recognize*. It remains to show how to perform the identification of dimensions 0, and k.

First, from Lemmas 4 and 7, recall that, for $\alpha = 0$, the contributions to the number of 6-cycles passing through an edge e and the solutions of Equation 2 are of the form

$$k^2 + pk + O(1) \text{ where } p = \begin{cases} 5 & \text{if } e \text{ is in dimension } 0; \\ 2 & \text{if } e \text{ is in dimension } k; \\ 8 & \text{otherwise.} \end{cases}$$

Thence, if Equation 2 would not have other solutions than those for $\alpha = 0$, then the edges in dimension 0, and k could be easily identified for k large enough. Unfortunately, $\alpha \neq 0$ provides additional solutions which modify this counting. Fortunately, we will show that one can control this modification without too much effort.

Let us first study the solutions of Equation 2 when $\alpha = 2$. The equation

$$2^{x_0} + 2^{x_2} + 2^{x_4} = 2^{x_1} + 2^{x_3} + 2^{x_5} + 2^{k+1} + 2n'$$

where $n = 2^k + n'$, implies

$$x_2 = x_4 = k, \text{ and } 2^{x_0} = 2^{x_1} + 2^{x_3} + 2^{x_5} + 2n'$$

up to a permutation of the even indices. This equation has generally no solution apart for specific values of n' because, up to a permutation of the x_i's, the equation

$$2^{x_0} = m + 2^{x_1} + 2^{x_3} + 2^{x_5}$$

has at most four solutions if $B_1(m) \leq 3$, and no solution if $B_1(m) \geq 4$. Indeed, if $B_1(m) \geq 4$ then, from Lemma 3, either $B_1(m + 2^{x_1} + 2^{x_3} + 2^{x_5}) > 1$ or $B_1(m + 2^{x_1} + 2^{x_3} + 2^{x_5}) = 1$ with at least two 1-entries in the single block of $m + 2^{x_1} + 2^{x_3} + 2^{x_5}$. In both cases, it cannot result in a single power of 2. If $B_1(m) \leq 3$, then we get at most four solutions (yielded by the case $B_1(m) = 3$.

In conclusion, if $\alpha = 2$, then, for any n, Equation 2 has a number of solutions which is upper bounded by a constant.

The tricky case is actually $\alpha = 1$, yielding the equation

$$2^{x_0} + 2^{x_2} + 2^{x_4} = 2^{x_1} + 2^{x_3} + 2^{x_5} + 2^k + n'$$

where $0 < n' < 2^k$, $n = 2^k + n'$, and $x_i \neq x_{i+1} \pmod 6$. We distinguish two cases according to whether or not an x_i of the left hand side is equal to k. Indeed, if none of the x_i's of the left hand side is equal to k, then at least two of these x_i's must be equal to $k-1$ because $2^{x_1}+2^{x_3}+2^{x_5}+n' > 0$. In this case, we are let with the equation $2^{x_0} = 2^{x_1} + 2^{x_3} + 2^{x_5} + n'$ in which the sum of $n' > 0$ with three powers of two must result in a single power of two. As we have seen, he number of such type of solutions is bounded by a constant.

If one of the x_i's of the left hand side is equal to k, say $x_4 = k$, then we are let with

$$2^{x_0} + 2^{x_2} = 2^{x_1} + 2^{x_3} + 2^{x_5} + n'. \tag{3}$$

We will consider three cases according to the value of $B_1(n')$. Before that, let us prove the following lemmas.

Lemma 9 *Let I and J be two non empty sets of indices. If $B_1(m) \geq |I| + |J|$, then the equation $\sum_{i \in I} 2^{x_i} = m + \sum_{j \in J} 2^{y_j}$, has no solution.*

Proof. From Lemma 3, if this equation has a solution $(x_1, \ldots, x_{|I|}, y_1, \ldots, y_{|J|})$, then

$$|I| \geq B_1(\sum_{i \in I} 2^{x_i}) = B_1(m + \sum_{j \in J} 2^{y_j}) \geq B_1(m) - |J|.$$

Therefore, if $B_1(m + \sum_{j \in J} 2^{y_j}) > B_1(m) - |J|$, then the lemma holds. Otherwise, that is if if $B_1(m + \sum_{j \in J} 2^{y_j}) = B_1(m) - |J|$, then one block of consecutive 1's of $m + \sum_{j \in J} 2^{y_j}$ has at least two 1-entries. Thus there are at least $B_1(m) - |J| + 1$ 1-entries in $m + \sum_{j \in J} 2^{y_j}$. Now, there are at most $|I|$ 1-entries in $\sum_{i \in I} 2^{x_i}$. Therefore, $B_1(m) - |J| + 1 \leq |I|$. ■

Lemma 10 *Let $I = \{1, \ldots, i_0\}$ and $J = \{1, \ldots, j_0\}$ be two non empty sets of indices of cardinality $O(1)$. If the equation*

$$m + \sum_{J \setminus \{j_0\}} 2^{y_j} = \sum_{i \in I \setminus \{i_0\}} 2^{x_i}$$

has no solution, then the equation $m + \sum_{j \in J} 2^{y_j} = \sum_{i \in I} 2^{x_i}$ has at most $O(1)$ solutions.

Proof. The proof is by induction on $B_1(m)$. Assume $B_1(m) = 1$. If $|I| \geq 2$, and $|J| \geq 2$ then the equation $m + \sum_{J\setminus\{j_0\}} 2^{y_j} = \sum_{i \in I\setminus\{i_0\}} 2^{x_i}$ has at least one solution. If $|I| = |J| = 1$, then the equation $m + \sum_{j\in J} 2^{y_j} = \sum_{i\in I} 2^{x_i}$ has a unique solution. If $|I| = 1$ and $|J| = 2$, then the equation $m + \sum_{j\in J} 2^{y_j} = \sum_{i\in I} 2^{x_i}$ has at most two solutions (up to permutation of the indices). Similarly, if $|I| = 2$ and $|J| = 1$, then the equation $m + \sum_{j\in J} 2^{y_j} = \sum_{i\in I} 2^{x_i}$ has at most two solutions (up to permutation of the indices), apart if $m = 2^\ell$, in which case the equation has $\Omega(\log m)$ solutions. However, if $m = 2^\ell$, $|I| = 2$ and $|J| = 1$, then the equation $m + \sum_{J\setminus\{j_0\}} 2^{y_j} = \sum_{i\in I\setminus\{i_0\}} 2^{x_i}$ has a solution.

Assume now that $B_1(m) > 1$. Assume that the equation $m + \sum_{j\in J} 2^{y_j} = \sum_{i\in I} 2^{x_i}$ has a non constant number of solutions, and let us show that the equation $m + \sum_{J\setminus\{j_0\}} 2^{y_j} = \sum_{i\in I\setminus\{i_0\}} 2^{x_i}$ has at least one solution. Let S be the set of solutions of $m + \sum_{j\in J} 2^{y_j} = \sum_{i\in I} 2^{x_i}$. For every bit-position b, $0 \leq b \leq \lfloor \log_2 m \rfloor$, such that the b-th bit of m is 0, let S_b be the subset of solutions $(x, y) \in S$ such that all y_j's are distinct from b, and $m + \sum_{J_b} 2^{y_j} \leq 2^b$ where J_b is the set of y_j's smaller that b. We consider two cases:

Case 1. $\sum_b |S_b| \in O(1)$, that is $\sum_b |S_b|$ is finite. In that case, since $|S| \notin O(1)$, that is $|S|$ is not bounded by a constant, there is a non constant number of solutions satisfying that a carry jumps from block to block in $m + \sum_J 2^{y_j}$, this carry being relayed by y_j's. Hence, $|J| \geq B_1(m) + \sum_{i=1}^{B_1(m)-1}(z_i - 1)$ where z_i is the number of 0-entries of m between the i-th and the $(i + 1)$-th block of m. However, $|J| = B_1(m) + \sum_{i=1}^{B_1(m)-1}(z_i - 1)$ or $|I| = 1$ would yield a constant number of solutions. Therefore $|I| > 1$ and $|J| > B_1(m) + \sum_{i=1}^{B_1(m)-1}(z_i - 1)$. This implies that the equation $m + \sum_{J\setminus\{j_0\}} 2^{y_j} = \sum_{i\in I\setminus\{i_0\}} 2^{x_i}$ has at least a solution.

Case 2. At least one set $|S_b|$ contains a non constant number of solutions. Then there exist I^-, I^+, J^-, J^+, $I = I^- \cup I^+$, $J = J^- \cup J^+$, such that at least one of the two equations

$$2^{b+1}m^+ + \sum_{J^+} 2^{y_j} = \sum_{i\in I^+} 2^{x_i} \text{ and } m^- + \sum_{J^-} 2^{y_j} = \sum_{i\in I^-} 2^{x_i}$$

has a non constant number of solutions, while the other at has least one solution, where m^- (resp. m^+) is the integer whose binary representation consists of the b bits of m of lower order (resp. the $\lfloor \log_2 m \rfloor - b$ bits of m of higher order). Indeed, I and J are of cardinality $O(1)$, and thus there is a constant number of such partition of I and J. This concludes the proof of Lemma 10 by application of the induction hypothesis. ■

Now, we are ready to proceed to our case study (recall that $n = 2^k + n'$):

Case 1. $B_1(n') \geq 3$.

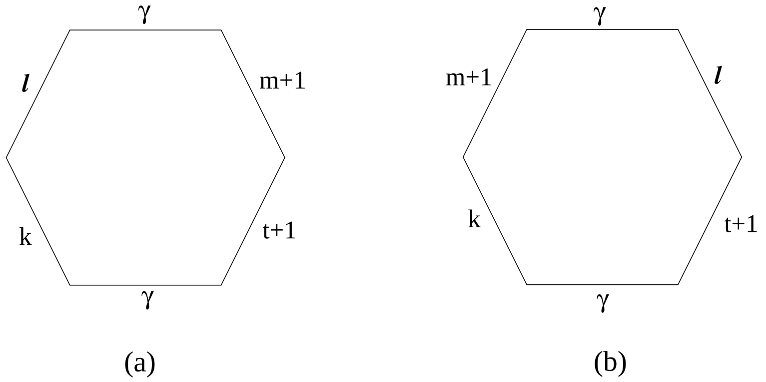
In this case, from Lemma 10, there is a constant number of solutions for Equation 3. Indeed, due to Lemma 9, the equation $2^{x_0} = 2^{x_1} + 2^{x_3} + n'$ has no solution.

Case 2. $B_1(n') = 2$.

In this case, Equation 3 has again a constant number of solutions if

$$n' \neq \sum_{i=m+2}^{t} 2^i + \sum_{i=\ell}^{m} 2^i.$$

Indeed, for such values of n', the equation $2^{x_0} = 2^{x_1} + 2^{x_3} + n'$ has no solution by Lemma 9, and thus the result holds by application of Lemma 10. For $n' = \sum_{i=m+2}^{t} 2^i + \sum_{i=\ell}^{m} 2^i$, the



$$\gamma \in \{0, \ldots, k-1\} \setminus \{\ell, m+1, t+1\}.$$

Figure 7: Additional solutions provided by Equation 3 if $B_1(n') = 2$.

equation $2^{x_0} = 2^{x_1} + 2^{x_3} + n'$ has a unique solution (up to permutation of the x_{2i+1}'s), that is: $x_0 = t+1$, $x_1 = \ell$, and $x_3 = m+1$. Thus, by adding the solution $x_2 = x_5 = \gamma$, Equation 3 yields the two non isomorphic labeled 6-cycles depicted on Figure 7. These cycles contribute to add $4k + O(1)$ 6-cycles for every edge in dimension k. Indeed, if $t+1 \neq k$, then Cycles (a) and (b) of Figure 7 contribute for $2k + O(1)$ each. If $t+1 = k$, then Cycles (a) and (b) are isomorphic, but Cycle (a) alone then contributes for $4k + O(1)$. Note that none of such cycles has been counted for $\alpha = 0$, as one can check on Figure 2. For the other dimensions, we will consider four cases. For that purpose, let us denote by $(p_0, \ldots, p_k)$ the vector such that $k^2 + p_i\, k + O(1)$ is the number of 6-cycles passing through an edge of dimension i, $0 \leq i \leq k$ (the contributions for $\alpha \neq 0$ do not modify the leading term k^2 set for $\alpha = 0$). We will show that $p_0 \neq p_k$, and, apart few cases, for every i, $1 \leq i \leq k-1$, $p_0 \neq p_i$, and $p_k \neq p_i$. These inequalities allow to distinguish dimensions 0 and k from the others.

Case 2.1. $\ell > 0$ and $t+1 < k$. Then $p_0 = 5$, $p_k = 2+4 = 6$, and $p_i \geq 8$ for any i, $0 < i < k$. For instance, $p_\ell = 8 + 4 = 12$.

Case 2.2. $\ell = 0$ and $t+1 < k$. Then $p_0 = 5+4 = 9$, $p_k = 2+4 = 6$, and for any i, $0 < i < k$, either $p_i = 8$ or $p_i \geq 10$. For instance, $p_{m+1} = 8 + 4 = 12$, whereas $p_m = 8$.

Case 2.3. $\ell > 0$ and $t+1 = k$. Then $p_0 = 5$, $p_k = 2+4 = 6$, and for any i, $0 < i < k$, $p_i \geq 8$.

Case 2.4. $\ell = 0$ and $t+1 = k$. Then $p_0 = 5+2 = 7$ (Cycles (a) and (b) are isomorphic), $p_k = 2+4 = 6$, and for any i, $0 < i < k$, $p_i \geq 8$.

Case 3. $B_1(n') = 1$. Then $n' = \sum_{i=\ell}^{m} 2^i$, $m \geq \ell$. Based on Lemma 10, let us study the solutions of the equations

$$(E_1)\ \ 2^{x_2} = 2^{x_1} + 2^{x_5} + n' \ \text{ and } \ (E_2)\ \ 2^{x_2} = 2^{x_1} + n'.$$

Equation (E_2) is motivated by the fact that, by repetively applying Lemma 10, we get that if the equation $m + \sum_{j \in J} 2^{y_j} = \sum_{i \in I} 2^{x_i}$ has a non constant number of solutions, then these solutions are obtained from the solutions of the set of equations $m + \sum_{J \setminus \{J'\}} 2^{y_j} = \sum_{i \in I \setminus \{I'\}} 2^{x_i}$, $I' \subset I$, $I' \neq \emptyset$, $J' \subset J$, $J' \neq \emptyset$.

There are two solutions for (E_1) up to permutation of x_1 and x_5:

$$\begin{cases} x_1 = \ell \\ x_5 = m+1 \\ x_2 = m+2 \end{cases} \quad \text{and} \quad \begin{cases} x_1 = x_5 = \ell - 1 \\ x_2 = m+1 \end{cases}$$

Equation (E_2) has a unique solution:

$$\begin{cases} x_1 = \ell \\ x_2 = m+1. \end{cases}$$

Therefore, we get three types of solutions for Equation 3:

$$1) \begin{cases} x_1 = \ell \\ x_5 = m+1 \\ x_2 = m+2 \\ x_0 = x_3 = \gamma \end{cases} \quad \text{or} \quad 2) \begin{cases} x_1 = x_5 = \ell - 1 \\ x_2 = m+1 \\ x_0 = x_3 = \gamma \end{cases} \quad \text{or} \quad 3) \begin{cases} x_1 = \ell \\ x_2 = m+1 \\ x_0 = \gamma \\ x_3 = x_5 = \gamma - 1 \end{cases}$$

up to valid permutations of the indices. There are other solutions (for instance, if $n' = 2^\ell$, $x_1 = x_3 = x_5 = \ell$ and $x_0 = x_2 = \ell + 1$) but, from Lemma 10, they contribute for a constant number of cycles. The non isomorphic labeled 6-cycles corresponding to the three previous sets of solutions are depicted on Figure 8. Solution 1 yields Cycles 1 and 2; Solution 2 yields Cycle 3; And Solution 3 yields Cycles 4, 5 and 6. These 6-cycles are non isomorphic to the 6-cycles of Figure 2. The remaining part of the proof consist to count the contribution of these solutions as a function of the values of ℓ and m. This cases study is summarized below.

		0	1	$\ell-1$	ℓ	$m+1$	$m+2$	γ	$k-1$	k
1	$m < k-2$ and $\ell > 1$	5	–	12	18	20	12	8	–	14
2	$m < k-2$ and $\ell = 1$	9	18	–	–	20	12	8	–	14
3	$m < k-2$ and $\ell = 0$	15	–	–	–	18	12	8	–	12
4	$m = k-2$ and $\ell > 1$	5	–	12	16	–	–	8	18	14
5	$m = k-2$ and $\ell = 1$	9	16	–	–	–	–	8	18	14
6	$m = k-2$ and $\ell = 0$	13	–	–	–	–	–	8	16	12
7	$m = k-1$ and $\ell > 1$	5	–	10	11	–	–	8	–	10
8	$m = k-1$ and $\ell = 1$	7	11	–	–	–	–	8	–	10
9	$m = k-1$ and $\ell = 0$	8	–	–	–	–	–	8	–	8

In this table, each cell contains the coefficient p_i in front of k in the expression of the number of 6-cycles passing through an edge of dimension i. For instance, if $m < k-2$ and $\ell > 1$, then row 1 says that $p_0 = 5$, $p_{\ell-1} = 12$, $p_\ell = 18$, $p_{m+1} = 20$, $p_{m+2} = 12$, $p_k = 14$, and, for all the other dimensions γ, $p_\gamma = 8$. The sign "–" stands either to indicate that a cell is meaningless (for instance the column $\ell - 1$ is meaningless if $\ell = 0$), or to indicate that the value of p_i is given elsewhere (for instance, if $\ell = 1$, then the value of $p_{\ell-1}$ is given in column 0).

For the values of ℓ and m corresponding to row 1, 2, 4, 5, 6, or 8, the dimensions 0 and k can be distinguished. Three rows require specific attention: rows 3, 7, and 9.

On row 3, the counting show that $p_k = p_{m+2}$. To distinguish dimension k from dimension $m+2$, we consider 6-cycles of type $(0, \gamma, 0, m+1, d, m+1)$ where $d \in \{m+2, k\}$ and $\gamma \in \{0, \ldots, k\}$ (note that dimension $m+1$ can be distinguished). We get:

$$2^d + 2 = 2^\gamma + 2^{m+2} + \alpha n, \quad \alpha \in \{-2, -1, 0, 1, 2\}.$$

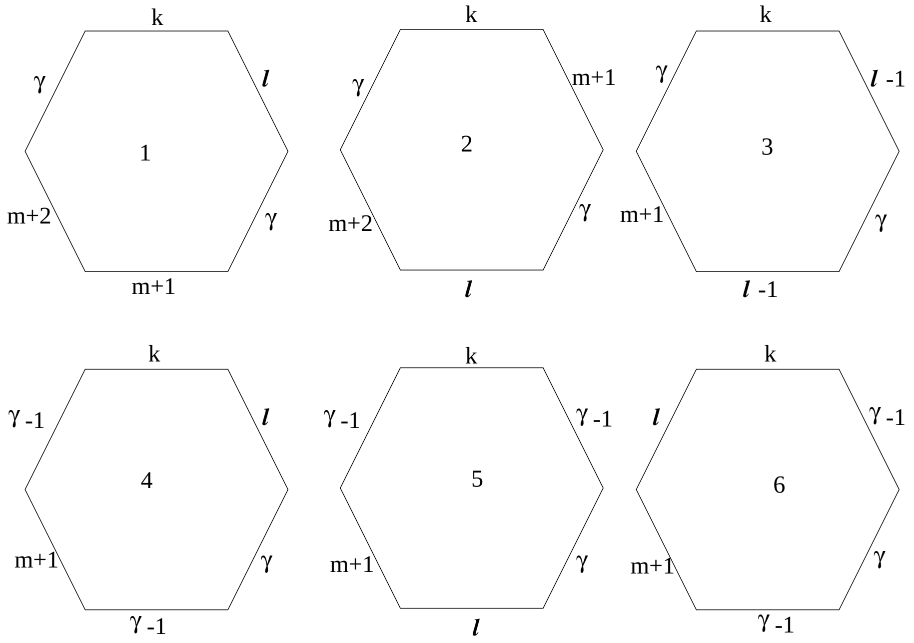


Figure 8: Additional solutions provided by Equation 3 with $B_1(n') = 1$.

If $\alpha = 1$ or $\alpha = 2$, then $2^\gamma + 2^{m+2} + \alpha n > 2^d + 2$ for $d \in \{m+2, k\}$. For $\alpha = 0$, $d = k$ in the equation yields $2^k + 2 = 2^\gamma + 2^{m+2}$ which has no solution since $m + 2 < k$. For $\alpha = -1$ or $\alpha = -2$, $-\alpha n + 2^d + 2 > 2^\gamma + 2^{m+2}$ for $d \in \{m+2, k\}$. Therefore, this equation forces $d = m + 2$, and, in that case, the last edge is in dimension 1. Therefore, dimension $m + 2$ can be distinguished from dimension k.

The equality between p_k and $p_{\ell-1}$ on row 7 can be solved similarly by considering the 6-cycles of the form $(0, d, 0, d', \ell, \gamma)$ where d and d' are chosen in $\{\ell - 1, k\}$. The equation $2 + 2^\ell = 2^d + 2^{d'} + 2^\gamma + \alpha n$, $\alpha \in \{-2, -1, 0, 1, 2\}$, forces $\gamma = 1$. Therefore, dimension 0 and 1 can be distinguished, which is enough to conclude (Lemma 6).

Row 9 corresponds to Knödel graphs of order $2^{k+2} - 2$. It is the last but not the least case considered in this proof. Indeed, $n = 2^{k+1} - 1$ corresponds to an edge-transitive graph, and it is therefore not surprising to find the same value for all the p_i's in the table. To solve that case, let us look at the number of 6-cycles using a path of length 3. Solutions for $\alpha = 0$ contribute for a constant number of cycles (see Figure 2). Equation 2 has no solution for $\alpha = 2$. Indeed, on one hand

$$2^{x_1} + 2^{x_3} + 2^{x_5} + 2n = 2^{x_1} + 2^{x_3} + 2^{x_5} + 2^{k+2} - 2 \geq 2^{k+2} + 1,$$

and on the other hand

$$2^{x_0} + 2^{x_2} + 2^{x_4} \leq 2^{k+1} + 2^k.$$

The case $\alpha = 1$ yields the two cycles of Figure 9 where $\gamma \neq 0$. These solutions contribute for a constant number of 6-cycles if the path is distinct from $(k, 0, k)$, and contribute for k to a path $(k, 0, k)$. Therefore, edges of dimension 0 and dimension k can be identified.

This completes the cases study. In every case, we have proved that, for $k \geq k_0 = O(1)$, dimensions 0 and k can be distinguished from the other dimensions by counting the number of

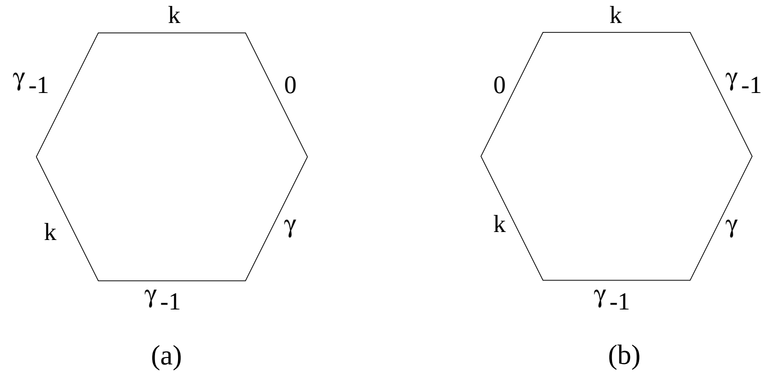


Figure 9: Specific solutions for $n = 2^{k+1} - 1$.

6-cycles passing through every edge (or path of length 3 for $n = 2^{k+1} - 1$). This identification can be done in $O(n \log^5 n)$ time for every n, even for those requiring specific treatments. This completes the proof of the theorem by application of Lemma 8, and of Lemma 6.

Remark. For n large enough, if $n \neq 2^{k+1} - 1$, then Knödel graphs of order $2n$ are non edge-transitive. (From [9], Knödel graphs of order $2^{k+2} - 2$ are edge-transitive.) Indeed, the proof of correctness of Algorithm *Recognize*, given before, uses the fact that, for $k \geq k_0 = O(1)$, if $n \neq 2^{k+1} - 1$, then the number of 6-cycles using edges of some dimensions is not the same.

Here is a simple corollary of Theorem 1.

Corollary 3 *There exists an $O(n \log^5 n)$-time algorithm to recognize the circulant digraphs of chords the $\lfloor \log_2 n \rfloor$ first powers of two minus one, i.e., $\langle 1, 3, \ldots, 2^k - 1 \rangle_n$ with $k = \lfloor \log_2 n \rfloor$.*

Proof. Let G be the input digraph. Add a loop to every vertex of G. For every arc, compute the number of *non necessarilly elementary* 6-cycles, passing through that arc. The result of this counting is the same as the counting for the Knödel graph. The chord 1 can therefore be identified. The cycle obtained by following that chord produces an hamiltonian cycle which can be used to label the vertices from 0 to $n - 1$. It just remains to check the correctness of the adjacency. ■

4 Conclusion and Further Research

In this paper, we have shown that Knödel graphs can be recognized in $O(n \log^5 n)$ time by counting the number of 6-cycles using any edge. One can use the same technique for Fibonacci graphs (see Appendix A), although the analysis looks more difficult. Therefore, the natural question arising in this field is to ask for which sequences g_i the "6-cycle technique" applies. Let us formalize this question. Given an infinite and increasing sequence of integers $\Gamma = (g_i)_{i \geq 0}$, consider the sequence $(G_n^\Gamma)_{n \geq 0}$ of circulant digraphs of order n such that, for any $n \geq 0$, G_n^Γ has generators $g_0, g_1, \ldots, g_k$ where k is the largest integer such that $g_k \leq n - 1$ (in other words, $g_k \leq n - 1 < g_{k+1}$). Then let $(H_n^\Gamma)_{n \geq 0}$ be the corresponding sequence of bipartite incident-graphs of order $2n$. The problem is the following:

Problem 1
Instance: *An integer n, and a graph G of $2n$ vertices;*
Question: *Is G isomorphic to H_n^Γ?*

Note that the sequence Γ is fixed in Problem 1, and thus that this problem is different from the problem of deciding whether a graph is isomorphic to the bipartite incident-graph of a circulant digraph, or to decide whether two bipartite incident-graphs of circulant digraphs are isomorphic. These two latter problems are known to be difficult, even if they might be simpler than the problem of deciding whether a graph is a Cayley graph.

We have seen that Problem 1 can be solved in $O(m \log^4 n)$-time if $\Gamma = (2^i - 1)_{i \geq 0}$, where m denotes the number of edges. We conjecture that the same result holds if $\Gamma = (F(i+1) - 1)_{i \geq 0}$. The question is: for which family the techniques used to solve the problem for Knödel graphs can be extended, and how? Actually, as soon as we know how to solve Equation 1, then we are able to enumerate the 6-cycles, and, probably, to use the same techniques as the techniques used for Knödel graphs.

Note that if one cannot directly identify the dimensions g_i's, that is if we can just color the edges such that there is a one-to-one correspondence between the colors and the dimensions but without knowing the values of the dimensions, then one can conclude anyway if $k = O(\frac{\log n}{\log \log n})$ by testing the $k!$ possibilities of labeling the colors. However, if k is too large, $k = \Omega(\log n)$ for instance, then one needs to identify precisely every dimensions, and this might be difficult by using Equation 1 only. For instance, if $g_i = 4^i$, then Equation 1 has only one type of solutions, namely (g, g', g'', g, g', g'') for $g \neq g'$, $g' \neq g''$, and $g \neq g''$. That is all edges have the same number of 6-cycles using them. There is a way though to break the symmetry in the specific case $g_i = 4^i$, by counting 10-cycles. This should yield a polynomial algorithm for this sequence. More generally, any sequence of the form p^i, p integral ≥ 2, might be solvable by looking at $(2p+2)$-cycles, and thus in time $O(m \log^{2p+1} n)$. Sequences giving rise to sparse graphs, e.g., $g_i = 2^{2^i}$, or sequences giving rise to dense graphs, e.g., $g_i = i^p$, p integral ≥ 2, seem to require different techniques.

We let as an open problem the characterization of the sequences Γ for which Problem 1 can be solved in $O(m \log^{O(1)} n)$-time.

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A Recognizing Fibonacci Graphs

We just sketch how techniques given for Knödel graphs can be extended to Fibonacci graphs. Equation 1 can be rewritten for Fibonacci graphs as:

$$\left\{\begin{array}{l} F(x_0)+F(x_2)+F(x_4)=F(x_1)+F(x_3)+F(x_5)+\alpha\, n \\ x_i \neq x_{i+1} \pmod 6 \text{ for any } i\in\{0,1,2,3,4,5\} \end{array}\right. \tag{4}$$

where $\alpha \in \{0,1,2\}$. In order to recognize Fibonacci graphs, we use the following basic property:

Lemma 11 *Let G be a graph of order $2n$, and let $k=F^{-1}(n)-1$. Assume that some edges of G are colored by 0 or 1. G is a Fibonacci graph whose edges in dimension 0 and 1 are colored 0 and 1 respectively if and only if:*

1. *Any path P using colors 0 and 1, alternatively, from an arbitrary node is Hamiltonian; and*
2. *Assuming the jth node of P is labeled $((j+1) \bmod 2, \lfloor j/2 \rfloor)$, $j \geq 0$, we have, for every $i \in \{0,\dots,k\}$ and every $x\in\{0,\dots,n-1\}$, there is an edge connecting node $(0,x)$ with node $(1, x+F(i)-1 \bmod n)$, and no extra edges.*

Proof. Similar to the proof of Lemma 6. ■

We can use 4-cycles rather than 6-cycles when n is a Fibonacci number:

Lemma 12 *If $n=F(k+1)$ then the 4-cycles of the Fibonacci graph of order $2n$ are all of type $(k,\gamma-1,\gamma,\gamma-2)$, for $\gamma\in\{2,\dots,k\}$.*

Proof. Similar to the proof of Lemma 5. ■

Corollary 4 *There exists an $O(n\log^3 n)$-time algorithm to recognize Fibonacci graphs of order $2n=2F(k)$, $k\geq 0$.*

Proof. Similar to the proof of Corollary 1. ■

When n is not a Fibonacci number, the method derived for Fibonacci graphs of order $2n$ requires to solve Equation 4. For $\alpha=0$, this is easy:

Lemma 13 *For $\alpha=0$, Equation 4 has seven types of solutions. These solutions are expressed on Figure 10.*

Proof. Similar to the proof of Lemma 2. ■

For $\alpha\neq 0$, we use the following definition:

Definition. Let $n>0$ be an integer. $F^{-1}(n)$ is defined as the largest index i such that $F(i)\leq n$. The *Fibonacci decomposition* of n is a binary sequence $x=(x_{F^{-1}(n)},\dots,x_1)_F$ where x_i, $i=1,\dots,F^{-1}(n)$, is defined by $x_{F^{-1}(n)}=1$, and, for $i<F^{-1}(n)$,

$$x_i=\left\{\begin{array}{ll} 1 & \text{if } i=F^{-1}\left(n-\sum_{j=i+1}^{F^{-1}(n)} x_j F(j)\right) \\ 0 & \text{otherwise.} \end{array}\right.$$

The Fibonacci decomposition satisfies the following:

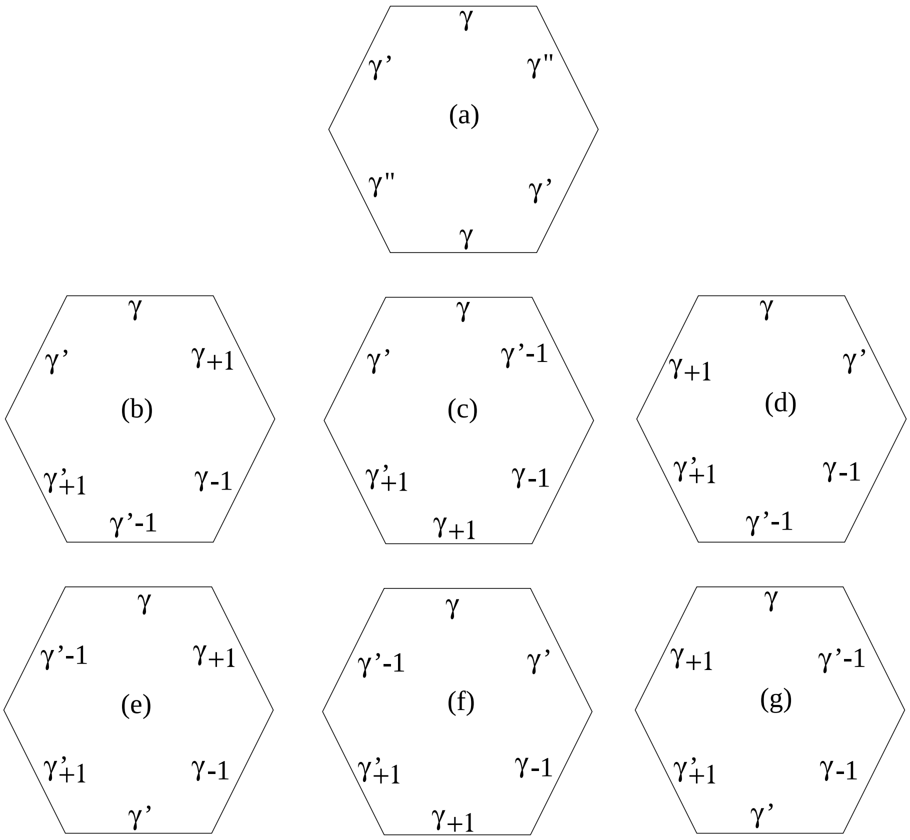

Figure 10: The seven types of solution of Equation 4 for $\alpha = 0$. In (a), $\gamma \neq \gamma'$, $\gamma' \neq \gamma''$, and $\gamma'' \neq \gamma$. In (b)-(g), $\gamma \neq \gamma'$, and $\gamma \neq \gamma' \pm 1$.

Lemma 14 *There is no consecutive* 1*'s in the Fibonacci decomposition of any positive integer.*

Proof. Assume that there are two consecutive 1's in the Fibonacci decomposition x of an integer n, and let $k = F^{-1}(n)$. Then, let i be the largest index for which $x_i = x_{i-1} = 1$. We have $n - \sum_{j=i+2}^{k} x_j F(j) \geq F(i) + F(i-1)$ because $x_{i+1} = 0$. Thus $n - \sum_{j=i+2}^{k} x_j F(j) \geq F(i+1)$, and thus, by definition of x, $x_{i+1} = 1$, a contradiction. ■

In the Fibonacci decomposition of an integer, two consecutive bits "10" plays the same role as a single bit in the binary decomposition of an integer since $(10\ldots1010)_F + (1)_F = (100\ldots0000)_F$ in the Fibonacci decomposition, in the same way $(1\ldots11)_2 + (1)_2 = (10\ldots00)_2$ in the binary decomposition.

Notation. $B_{10}(n)$ denotes the number of blocks of consecutive 2-bit strings "10" in the Fibonacci decomposition of n (the rightmost 1-entry may be not followed by a 0-entry).

For instance, $B_{10}((1010010)_F) = 2$, $B_{10}((1000101001)_F) = 3$, and $B_{10}((100)_F) = 1$. Note that

$$n = (\bar{1}0\underbrace{0\ldots0}\bar{1}0\underbrace{0\ldots\bar{0}}\overbrace{1010\ldots1010}\underline{0}0\overbrace{1010\ldots1010}\underline{0}0\overbrace{1010\ldots101\underline{0}}\underbrace{0\ldots0})_F$$

satisfies $B_{10}(n) = 5$, and there is a solution of Equation 2 for $\alpha = 1$ with x_1, x_3, x_5 equal to the underlined bit-positions, and x_0, x_2, x_4 equal to the over-lined bit-positions. However, we have:

Lemma 15 *If $B_{10}(n) \geq 6$, then Equation 4 has no solution for $\alpha \neq 0$.*

Proof. Similar to the proof of Lemma 4. ■

To prove a result similar to Corollary 2, we could proceed by performing the exact computation of $C_6(e)$ defined as the number of 6-cycles using a given edge e of a Fibonacci graph of $2n$ vertices with $B_{10}(n) \geq 6$. More generally, to prove a result similar to Theorem 1, we could study the solutions of Equation 4 for $\alpha \neq 0$. We could distinguish two cases depending whether one of the x_i's of the left hand side is equal to k or not. For that purpose, an integer $n \neq F(k)$ is said to be *block-special* if it is one of the following forms:

$$n = \begin{cases} 10^a(10)^b0^c & a \geq 0, b > 0, c \geq 0; \text{ or} \\ 10^a(10)^b0^c(10)^d0^e & a \geq 0, b > 0, 1 \leq c \leq 4, d > 0, e \geq 0, \text{ or} \\ 10^a(10)^b0^c(10)^d0^e(10)^f0^g & a \geq 0, b > 0, 1 \leq c \leq 2, d > 0, 1 \leq e \leq 2, f > 0, g \geq 0; \end{cases}$$

One can show that if n is not block-special then any solution of Equation 4 for $\alpha \neq 0$ implies that one of the x_i's of the left hand side is equal to k. Moreover, if n is block-special then solutions of Equation 4 exist in which none of the x_i's of the left hand side is equal to k. However, for any of such solution, one of the x_i's of the left hand side is equal to $k-1$, another is equal to $k-2$, and the third one is determined by the value of n. This means that such solutions contribute for a constant number of 6-cycles passing through a given edge. Since we do not concentrate on the constant terms, we rather concentrate our attention to the equation:

$$F(x_0) + F(x_2) = F(x_1) + F(x_3) + F(x_5) + \alpha\, n' \tag{5}$$

where $\alpha \in \{0, 1, 2\}$, and $n' = n - F(k)$. To get the approximate number of solutions of Equation 5 up to a constant additive factor, we can proceed by a case study depending on $B_{10}(n)$, as we did in the proof of Theorem 1. We conjecture that this analysis would succeed, and thus that dimension 0 and 1 can be distinguished from the other dimensions. We did not proceed through this analysis as it would consist in a somewhat fastidious repetition of arguments similar to those used in the proof of Theorem 1.