Shorter Labeling Schemes for Planar Graphs

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Representation of a Graph

adjacency list

matrix

1 node = 1 pointer in the data-structure
(it does not carry any specific information)
Implicit Representation

To associate with the nodes more information, typically adjacency, and to remove the data-structure.

Interval graphs: \( u \mapsto I(u) \subseteq [1, 2n] \)

Edges: \( u - v \iff I(u) \cap I(v) \neq \emptyset \)

Compact representation: \( O(\log n) \) bits/node
Possibly time \( O(n) \) algorithms vs. \( O(n+m) \)
Labeling Schemes

\( P = \) a graph property defined on pairs of nodes
\( F = \) a graph family

A \( P \)-labeling scheme for \( F \) is a pair \((\lambda, f)\) such that \( \forall G \in F, \forall u, v \in V(G) \):

- [labeling] \( \lambda(u, G) \) is a binary string
- [decoder] \( f(\lambda(u, G), \lambda(v, G)) = P(u, v, G) \)

**Goal:** to minimize the maximum label size

**In this talk:** \( P(u, v, G) \) is true \( \iff uv \in E(G) \)
Basic Example: Trees
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Basic Example: Trees

\[ \lambda(u,T) = (u, \text{parent}(u)) \text{ or } (u, u) \]

\[ f(uv, xy) = (v = x \text{ or } u = y) \]
Basic Example: Trees

For trees with $n$ nodes: $\sim 2\log n$ bits/node

(\text{the constant does matter \cite{Abiteboul et al. - SICOMP '06}})
Induced-Universal Graphs

[Babai, Chung, Erdös, Graham, Spencer ’82]

A graph $U$ is an **induced-universal** graph for the family $F$ if every graph of $F$ is isomorphic to an induced subgraph of $U$. 
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Induced-Universal Graphs

induced-universal graph $U$  
graphs of $F$

c$\log n$-bit labeling $\Leftrightarrow$ induced-universal graph of $n^c$ nodes
Universal Graphs for Trees
(for $n=6$ nodes)

From the $(u,\text{parent}(u))$ labeling
⇒ induced-universal graph of $n^2=36$ nodes

Using DFS for T: $(u,v)$
⇒ $u>v$ or $u=v=1$
Universal Graphs for Trees (for $n=6$ nodes)

From the $(u, \text{parent}(u))$ labeling
$\Rightarrow$ induced-universal graph of $n^2=36$ nodes

Using DFS for T: $(u,v)$
$\Rightarrow u>v$ or $u=v=1$
$\Rightarrow n(n-1)/2+1=16$ nodes
Universal Graphs for Trees
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Universal Graphs for Trees (universal trees)

\[
\begin{align*}
n = 1: & \quad \bullet \\
n = 2: & \quad \bullet - \bullet \\
n = 3: & \quad \bullet - \bullet - \bullet \\
n = 4: & \quad \bullet - \bullet - \bullet - \bullet \\
n = 5: & \quad \bullet - \bullet - \bullet - \bullet - \bullet \\
n = 6: & \quad \text{14 nodes} \\
n = 7: &
\end{align*}
\]

On graphs which contain all small trees, II.

F.R.K. Chung — R.L. Graham — N. Pippenger

Colloquia Mathematica Societatis János Bolyai

Labeling Schemes for Planar Graphs

Edge partition: combining schemes

Arboricity-\( k \) graphs: \((k+1)\log n\) bits

\[ \Rightarrow \text{Planar (}k=3\text{): }4\log n \text{ bits} \quad [\text{KNR – STOC’88}] \]
Better Labeling Schemes

For trees: $\log n + O(\log^* n), \log n + O(1)$
[Alstrup,Rauhe – FOCS’02]
[Alstrup,Dahlgaard,B.T.Knudsen – FOCS’15 & JACM’17]
⇒ Arboricity-$k$: $k \log n + O(1)$
⇒ Planar: $3 \log n + O(1)$

For treewidth-$k$: $\log n + O(k \log \log n)$
[G.,Labourel – ESA’07]
⇒ Planar & Minor-free: $2 \log n + O(\log \log n)$
A New Bound

For planar & bounded genus: $\frac{4}{3} \log n + O(\log \log n)$

⇒

Induced-universal graph of $n^{4/3 + o(1)}$ nodes for $n$-node planar graphs (and bounded genus graphs)

Labeling the nodes is polynomial
Decoding adjacency takes constant time
Sketch of Proof (1/2)

Edge partition: $G = S \cup B$ (Strips & Border)

$S$: components have $d$ layers $\sim n^{1/3}$
$B$: has treewidth $\leq 5$ and $n/d \sim n^{2/3}$ nodes

BFS & shift $\in [1..d]$

$d \geq 3$

$B_{i-1}$ $S_i$ $B_i$ $S_{i+1}$ $B_{i+1}$
Sketch of Proof (1/2)

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\[ \text{BFS & shift in } [1..d] \]

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Sketch of Proof (2/2)

Labeling for B: \( \log(n/d) \) ✔

Labeling for S: \( \log n + \log d \) new!

[up to +\( O(\log \log n) \) terms]

Problem: nodes in \( V(B) \) pay both labels
\[ \Rightarrow \log(n/d) + \log n + \log d = 2\log n \] ❌

Improved labeling for S: nodes in \( V(B) \) pay only \( \log|B| = \log(n/d) \) bits!

\[ \Rightarrow \text{nodes in } S\backslash V(B): \log n + \log d = \frac{4}{3} \log n \] ☺

\[ \Rightarrow \text{nodes in } V(B): \log(n/d) + \log(n/d) = \frac{4}{3} \log n \] ☹
**Improved Scheme for Treewidth-\(k\)**

\[ G = \text{treewidth-}k, \ V(G) = V_1 \cup V_2, \ K_u = N[u] = \text{simplicial complex of } u, \ |K_u| \leq k+1. \]

**Lemma.** \( G \) has a scheme providing, for each \( u \), \( \text{id}(u) \) and \( \lambda(u) \) st. \( \forall v \in K_u \) \( \text{id}(v) \) can be extracted from \( \lambda(u) \). Moreover, for \( u \in V_i \)

\[
|\lambda(u)| = \log|V_i| + O(k \log \log |V(G)|).
\]

\( u - v \Leftrightarrow \text{id}(v) \in \{\text{id}(K_u)\} \) or \( \text{id}(u) \in \{\text{id}(K_v)\} \)
Labeling Scheme for $S$

**Key lemma.** [2018,2019] If $G$ is planar, there is a node partition into monotone paths taken from any given BFS such that contracting each one leads to a treewidth-8 graph.

Label $\lambda(u)$ consists of:
- the treewidth-8 scheme
- the depth of $u$ in $S$ (unless $u \in B$)
- 3 bits/path in the treewidth-8 scheme, i.e., $3 \times 8 = 24$ extra bits

[Extend to genus-$g$ graphs]
Open Problems

1. Improve to $c \log n$ with $c < \frac{4}{3}$ for planar
2. Extend to minor-free graphs
3. Improve to $\log n + \theta(k)$ for treewidth-$k$
4. Prove lower bounds for planar or minor-free

Best lower bound for planar: $\log n + \Omega(1)$ ☹️

No hereditary family with $n! \cdot 2^{O(n)}$ labeled graphs (trees, planar, bounded genus, bounded treewidth, minor-free ...) is known to require labels of $\log n + \omega(1)$ bits.
THANK YOU!