Representation of a Graph

adjacency list

1 node = 1 pointer in the data-structure
(it does not carry any specific information)
Implicit Representation

To associate with the nodes more information, typically adjacency, and to remove the data-structure.

Interval graphs: $u \rightarrow I(u) \subseteq [1,2n]$

Edges: $u-v \Leftrightarrow I(u) \cap I(v) \neq \emptyset$

Compact representation: $O(\log n)$ bits/node

Possibly time $O(n)$ algorithms vs. $O(n+m)$
Labelling Schemes

$P$ = a graph property defined on pairs of nodes
$F$ = a graph family

A $P$-labelling scheme for $F$ is a pair $(\lambda, f)$ such that $\forall G \in F, \forall u, v \in V(G)$:

- [labelling] $\lambda(u, G)$ is a binary string
- [decoder] $f(\lambda(u, G), \lambda(v, G)) = P(u, v, G)$

**Goal:** to minimize the maximum label size

**In this talk:** $P(u, v, G)$ is true $\iff uv \in E(G)$
Basic Example: Trees
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Basic Example: Trees

\[ \lambda(u, T) = (u, \text{parent}(u)) \text{ or } (u, u) \]

\[ f(uv, xy) = (v = x \text{ or } u = y) \]
Basic Example: Trees

For trees with $n$ nodes: $\sim 2\log n$ bits/node

($\lambda(u,T) = (u,\text{parent}(u))$ or $(u,u)$)

$f(uv,xy) = (v=x \text{ or } u=y)$

For trees with $n$ nodes: $\sim 2\log n$ bits/node

(the constant does matter [Abiteboul et al. - SICOMP '06])
Induced-Universal Graphs

[Babai, Chung, Erdös, Graham, Spencer ’82]

A graph $U$ is an **induced-universal** graph for the family $F$ if every graph of $F$ is isomorphic to an *induced* subgraph of $U$. 
Induced-Universal Graphs

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Induced-Universal Graphs

\begin{align*}
\text{induced-universal graph } U & \quad \text{graphs of } F \\
\text{c} \log n \text{-bit labelling } & \iff \text{induced-universal graph of } n^c \text{ nodes}
\end{align*}
Universal Graphs for Trees (for $n=6$ nodes)

From the $(u, \text{parent}(u))$ labelling
$\Rightarrow$ induced-universal graph of $n^2=36$ nodes

Using DFS for T: $(u,v)$
$\Rightarrow u>v$ or $u=v=1$
Universal Graphs for Trees
(for $n=6$ nodes)

From the $(u, \text{parent}(u))$ labelling
⇒ induced-universal graph of $n^2 = 36$ nodes

Using DFS for T: $(u,v)$
⇒ $u>v$ or $u=v=1$
⇒ $n(n-1)/2 + 1 = 16$ nodes
Universal Graphs for Trees (for $n=6$ nodes)

From the $(u, \text{parent}(u))$ labelling
⇒ induced-universal graph of $n^2=36$ nodes

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⇒ $u>v$ or $u=v=1$
⇒ $n(n-1)/2+1=16$ nodes
Universal Graphs for Trees (universal trees)

\[
\begin{align*}
n = 1: & \quad - \\
n = 2: & \quad - - \\
n = 3: & \quad - \quad - \quad - \\
n = 4: & \quad - - - - - \\
n = 5: & \quad - - - - - - - \\
n = 6: & \quad - - - - - - - - - - \\
n = 7: & \quad - - - - - - - - - - - - - - - - - \\
\end{align*}
\]
Labelling Schemes for Planar Graphs

Edge partition: combining schemes

Arboricity-\(k\) graphs: \((k+1)\log n\) bits
\[ \Rightarrow \text{Planar (}\ k=3\text{): } 4\log n \text{ bits} \quad [\text{KNR – STOC’88}] \]
Better Labelling Schemes

For trees: $\log n + O(\log^* n)$
[Alstrup, Rauhe – FOCS’02]
⇒ Arboricity-$k$: $k\log n + O(\log^* n)$
⇒ Planar: $3\log n + O(\log^* n)$

For treewidth-$k$: $\log n + O(k \log \log n)$
[G., Labourel – ESA’07]
⇒ Planar & Minor-free: $2\log n + O(\log \log n)$

For trees: $\log n + O(1)$
[Alstrup, Dahlgaard, B.T. Knudsen – FOCS’15 & JACM’17]
A New Bound

For planar & bounded genus: \( \frac{4}{3} \log n + O(\log \log n) \)

\[ \Rightarrow \]

**Induced-universal graph of** \( n^{4/3+o(1)} \) **nodes for** \( n \)-node planar graphs (and bounded genus graphs)

Labelling the nodes is polynomial
Decoding adjacency takes constant time
Sketch of Proof (1/2)

Edge partition: $G = S \cup B$ (Strips & Border)

- **S**: components have depth $d \sim n^{1/3}$
- **B**: has treewidth $\leq 5$ and $n/d \sim n^{2/3}$ nodes

BFS & shift $\in [1..d]$
Sketch of Proof (1/2)

Edge partition: $G = S \cup B$ (Strips & Border)

$S$: components have depth $d \sim n^{1/3}$

$B$: has treewidth $\leq 5$ and $n/d \sim n^{2/3}$ nodes

BFS & shift $\in [1..d]$
Sketch of Proof (2/2)

Labelling for $B$: $\log(n/d)$ ✔

Labelling for $S$: $\log n + \log d$ new!

[up to $+O(\log\log n)$ terms]

Problem: nodes in $V(B)$ pay both labels

$\Rightarrow \log(n/d) + \log n + \log d = 2\log n$ 😞

Improved labelling for $S$: nodes in $V(B)$ pay only $\log|B| = \log(n/d)$ bits!

$\Rightarrow$ nodes in $S \setminus V(B)$: $\log n + \log d = \frac{4}{3} \log n$ 😊

$\Rightarrow$ nodes in $V(B)$: $\log(n/d) + \log(n/d) = \frac{4}{3} \log n$ 😞
Improved Scheme for Treewidth-$k$

$G = \text{treewidth-}k$, $V(G) = V_1 \cup V_2$, $K_u = \text{simplicial complex of } u$, $|K_u| \leq k+1$.

**Lemma.** $G$ has a scheme providing, for each $u$, $\text{id}(u)$ and $\lambda(u)$ st. $\text{id}(v)$ can be extracted $\forall v \in K_u$ from $\lambda(u)$. Moreover, for $u \in V_i$

$$|\lambda(u)| = \log|V_i| + O(k \log \log |V(G)|).$$

$u-v \Leftrightarrow \text{id}(v) \in \{\text{id}(K_u)\}$ or $\text{id}(u) \in \{\text{id}(K_v)\}$
**Key lemma.** [2018,2019] If $G$ is planar, there is a node partition into monotone paths taken from any given BFS such that contracting each one leads to a treewidth-8 graph.

Label $\lambda(u)$ consists of:

- the treewidth-8 scheme
- the depth of $u$ in $S$ (unless if $u \in B$)
- 3 bits/path in the treewidth-8 scheme, i.e., $3 \times 8 = 24$ extra bits

[Extend to genus-$g$ graphs]
Open Problems

1. Improve to \( c \log n \) with \( c < \frac{4}{3} \) for planar
2. Extend to minor-free graphs
3. Improve to \( \log n + \theta(k) \) for treewidth-\( k \)
4. Prove lower bounds for planar or minor-free

Best lower bound for planar: \( \log n + \Omega(1) \) 😞

No hereditary family with \( n! 2^{O(n)} \) labelled graphs (trees, planar, bounded genus, bounded treewidth, minor-free ...) is known to require labels of \( \log n + \omega(1) \) bits.
THANK YOU!