Edge Partition of Toroidal Graphs into Forests in Linear Time
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PLAN

1. Background
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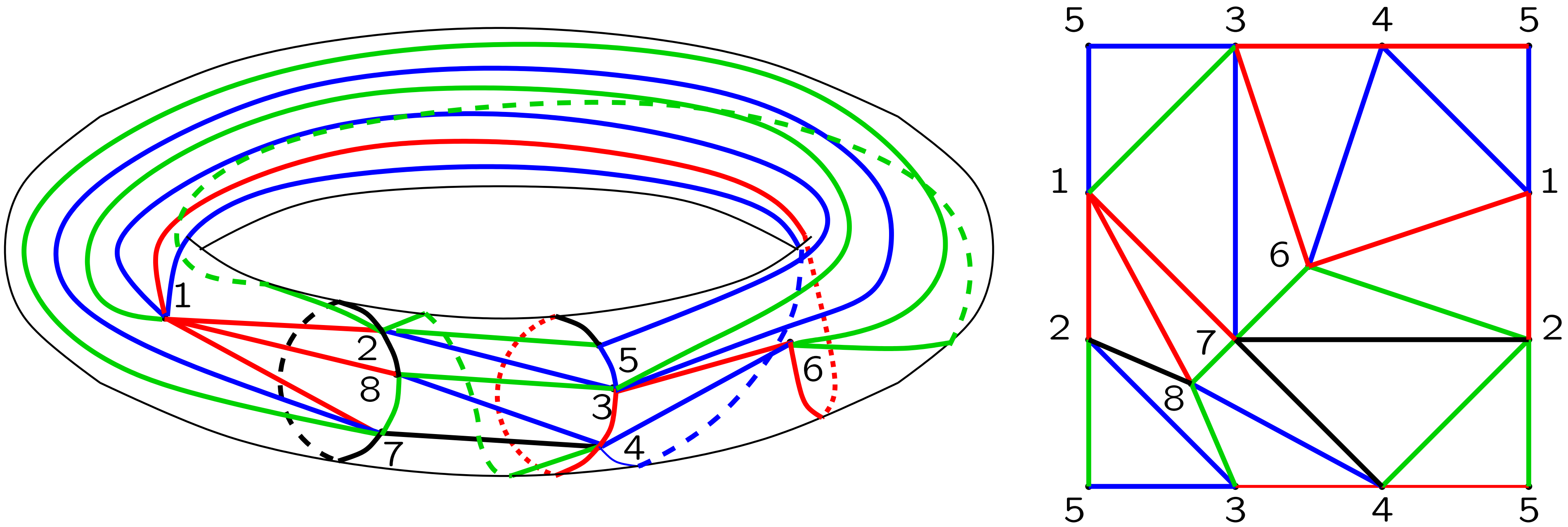
3. Possible Improvements and Open Problems

1. Background:toroidal maps



Example of toroidal map and its representation on the plane

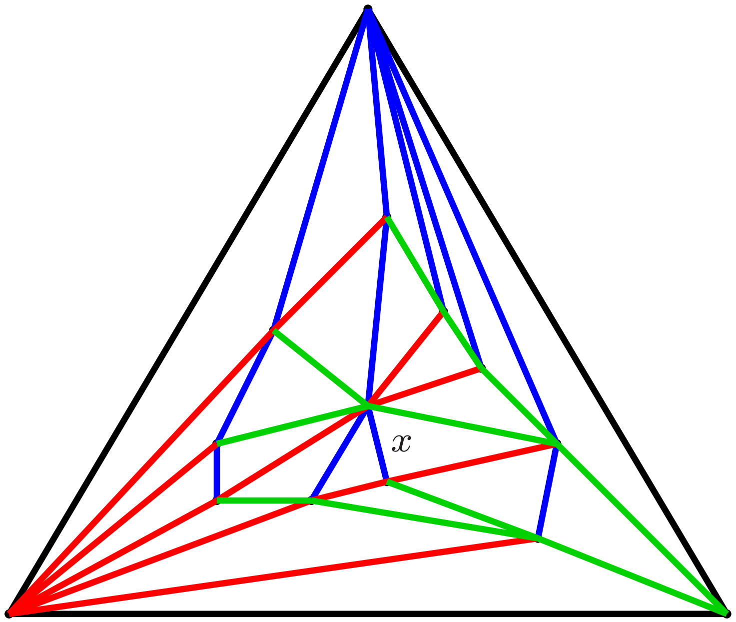
1. Background:known results for edge partition into trees and toroidal graphs

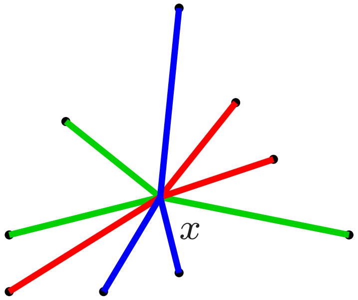
[Kundu 1974]:Triangulated toroidal graphs have 3 edge-disjoint spanning trees.

[Roskind and Tarjan 1985]:$O(m^2)$ time algorithm for finding maximal number of edge-disjoint spanning trees in a graph of m edges.

1. Background:Schnyder's realizer

Definition 1 [Schnyder 1989]
Schnyder's realizer: edge-partition of a plane triangulation into 3 trees $\{T_1, T_2, T_3\}$ each rooted on a vertex of the outer face and such that edges around inner vertices form blocks.
[Schnyder 1990]:This partition can be computed in $O(n)$ time.





2. Main result

Theorem 1

In linear time, we can partition the edges of a toroidal graph into three forests plus at most three edges.

Some remarks on the result

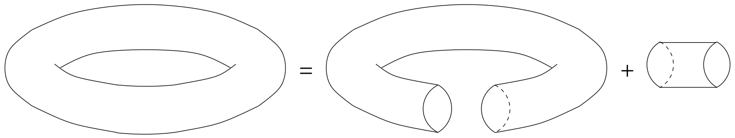
- This partition is tight for toroidal triangulations, because in this case we must remove at least 3 edges to obtain a graph of arboricity 3.

- In the case of a triangulated map, the algorithm gives us 3 edge-disjoint spanning trees.

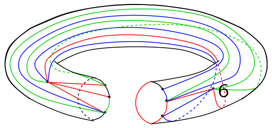
- The set A of three edges that are not in one of the 3 trees must fullfils the following properties for triangulated map: $\forall A' \subset A$, the edges of A' are not contained in a contractible region bounded by a cycle of length less than $|A'| + 3$.

Sketch of the algorithm

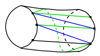
- **Step 1:** Graph partition into a planar subgraph H and a tambourine T.



- **Step 2:** Edge partition of H into 3 forests A_1, A_2 and A_3 plus at most 3 edges.



- **Step 3:** Edge partition into 3 forests of G from the edge partition of H.



Preliminary: Triangulation of the map G

First, we need a triangulated simple graph with no loops for the three steps of the algorithm.

Lemma 1

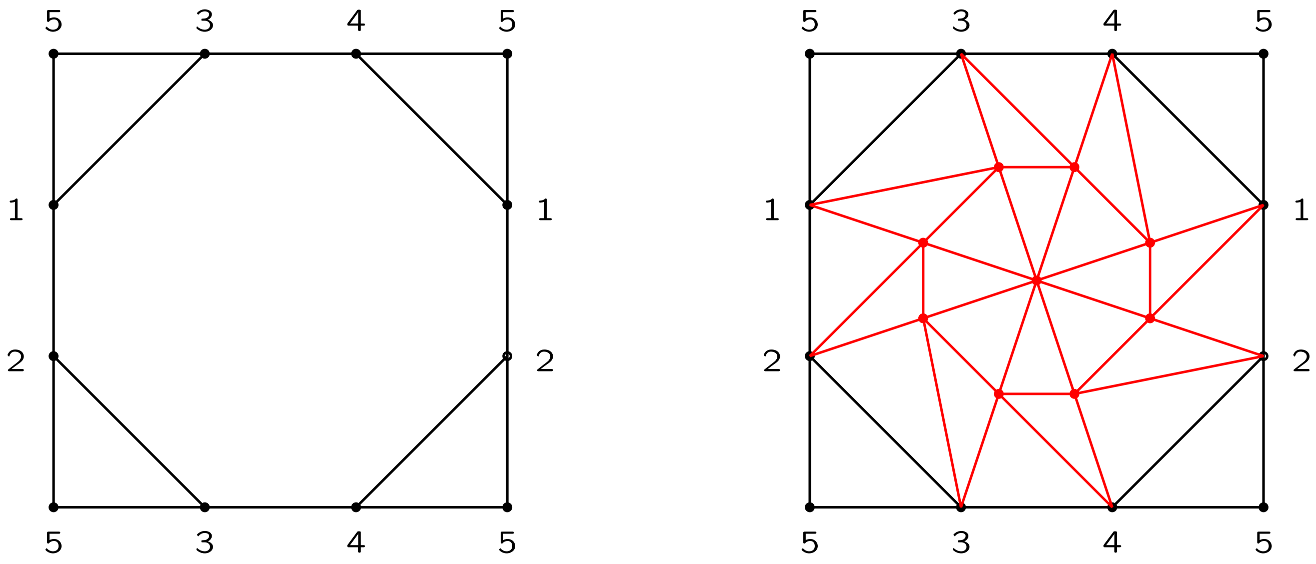
In linear time, we can triangulate a toroidal graph G without creating loops nor multiple edges by adding at most $O(n)$ vertices and edges.

Proof

To triangulate the face $F = \{x_0, \cdots, x_{r-1}\}$, we add a set $S = \{u_0, \cdots, u_r\}$ of vertices and the following set of edges:

$\{(x_i, u_i), (x_i, u_{i+1[\bmod r]}), (u_i, u_r) \mid \forall i, 0 \leqslant i \leqslant r\}$

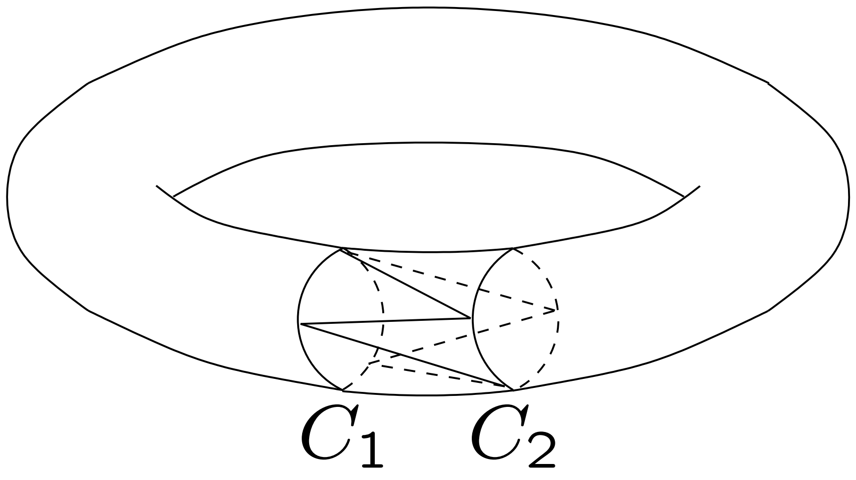
Example of triangulation of a toroidal map



Step 1: Graph partition into a planar subgraph H and a tambourine T

To use Schnyder's realizer, we need to planarize the map G. To do this, we remove from G a submap T called "tambourine" such that we obtain a planar graph H.

Map G



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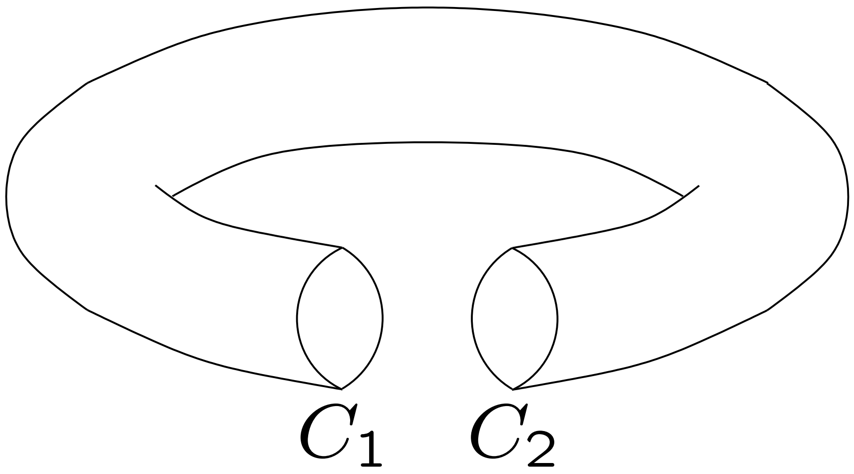
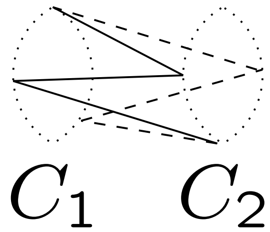
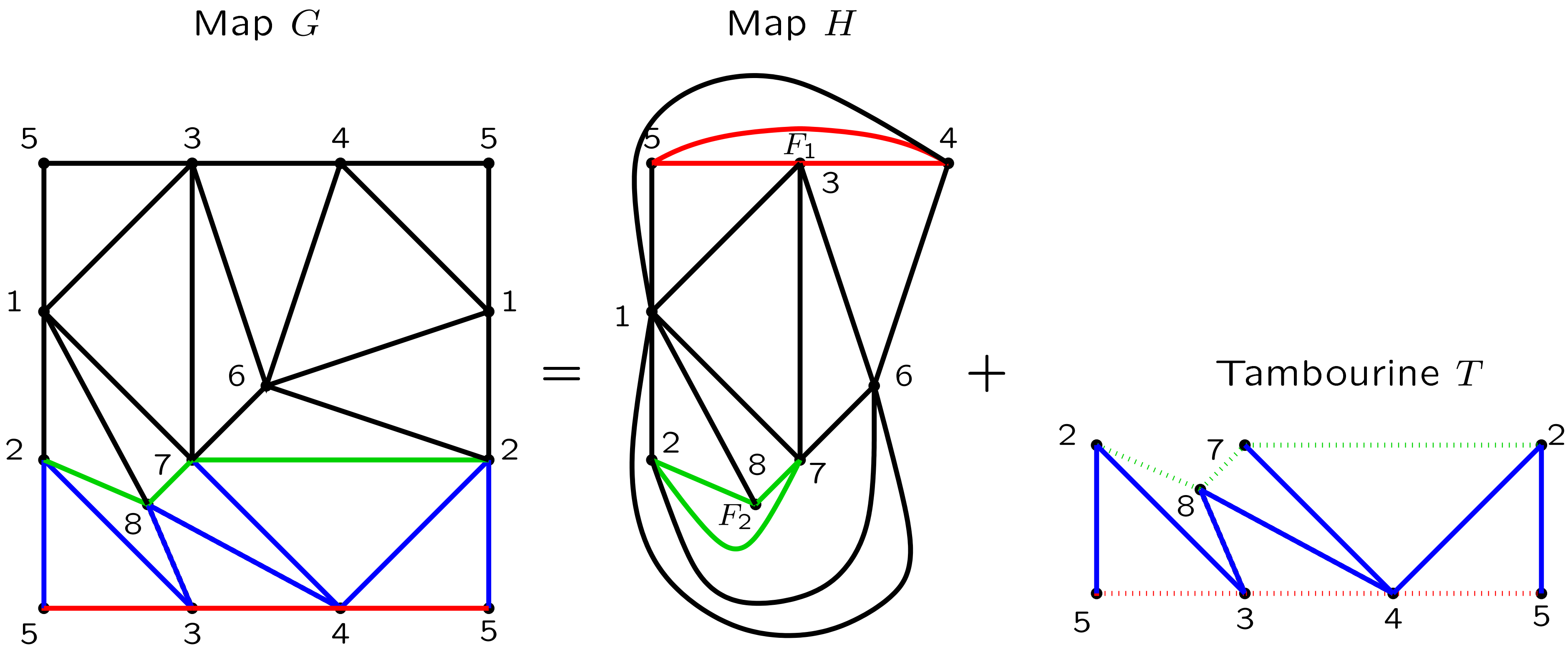
Submap H



Tambourine T



Example of such a planarization



: cycle C_1

: cycle C_2

: edges of the tambourine (C_1, C_2)

Step 2: Edge partition of H into 3 forests A_1, A_2 and A_3 plus at most 3 edges.

We need an edge partition of H such that $E(T)$ can be partitioned into 3 forests and added to H without creating any cycle.

Lemma 2
We can partition the edges of H such that every vertex x of $C_1 \cup C_2$, except at most 3 vertices has a missing color in the graph H, i.e., there is a color $i \in \{1, 2, 3\}$ such that the connected component of the forest A_i which contains the vertex x does not contain other vertices of $C_1 \cup C_2$.

Step 2: Edge partition of H into 3 forests A_1, A_2 and A_3 plus at most 3 edges.

To have this property, we contract each of the two faces C_1 and C_2 of H obtaining a new graph H'. We then compute a realizer of H' and from this realizer, we deduce the partition wanted for H. This step consists in four phases.

Step 2: Edge partition of H into 3 forests A_1, A_2 and A_3 plus at most 3 edges.

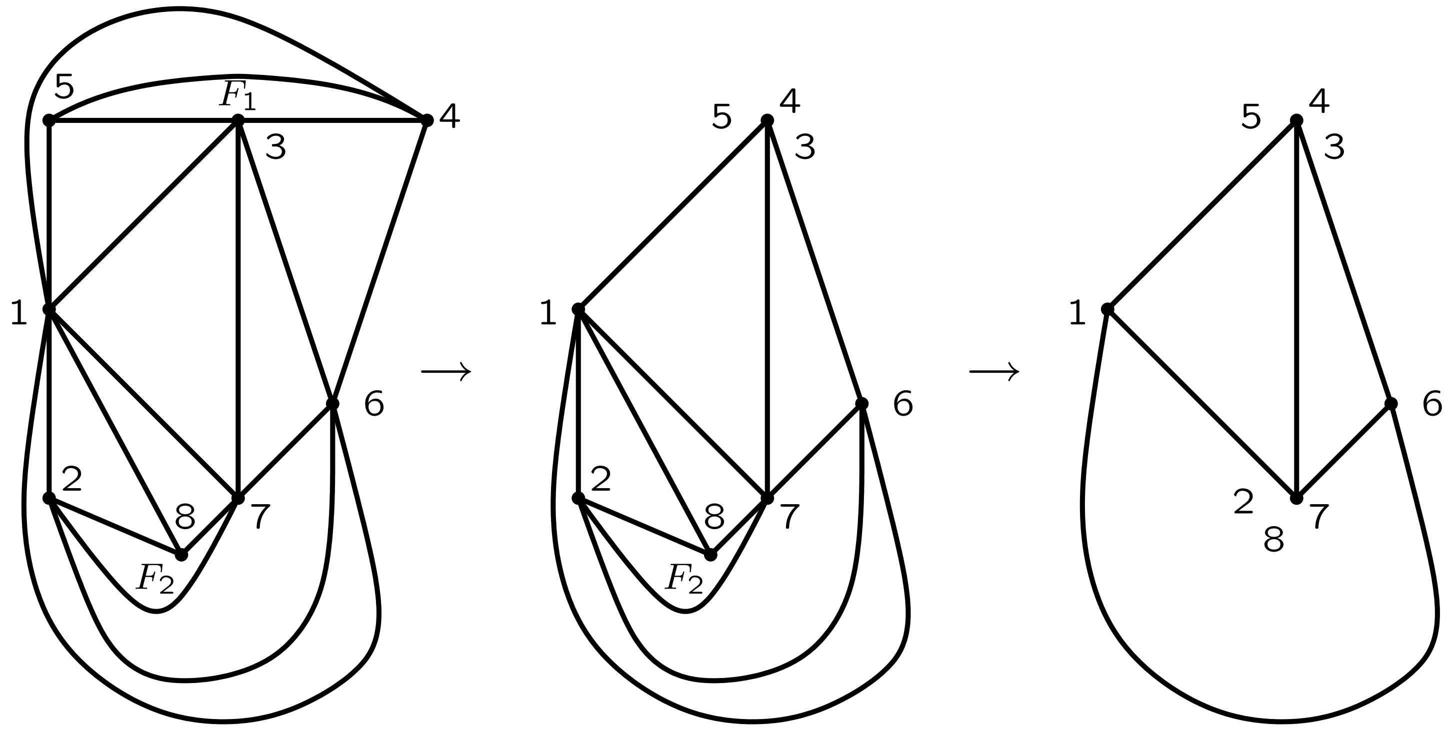
- **Phase 1:** contraction of the two faces to obtain a triangulated map H'.

- **Phase 2:** computation of Schnyder's realizer of map H'.

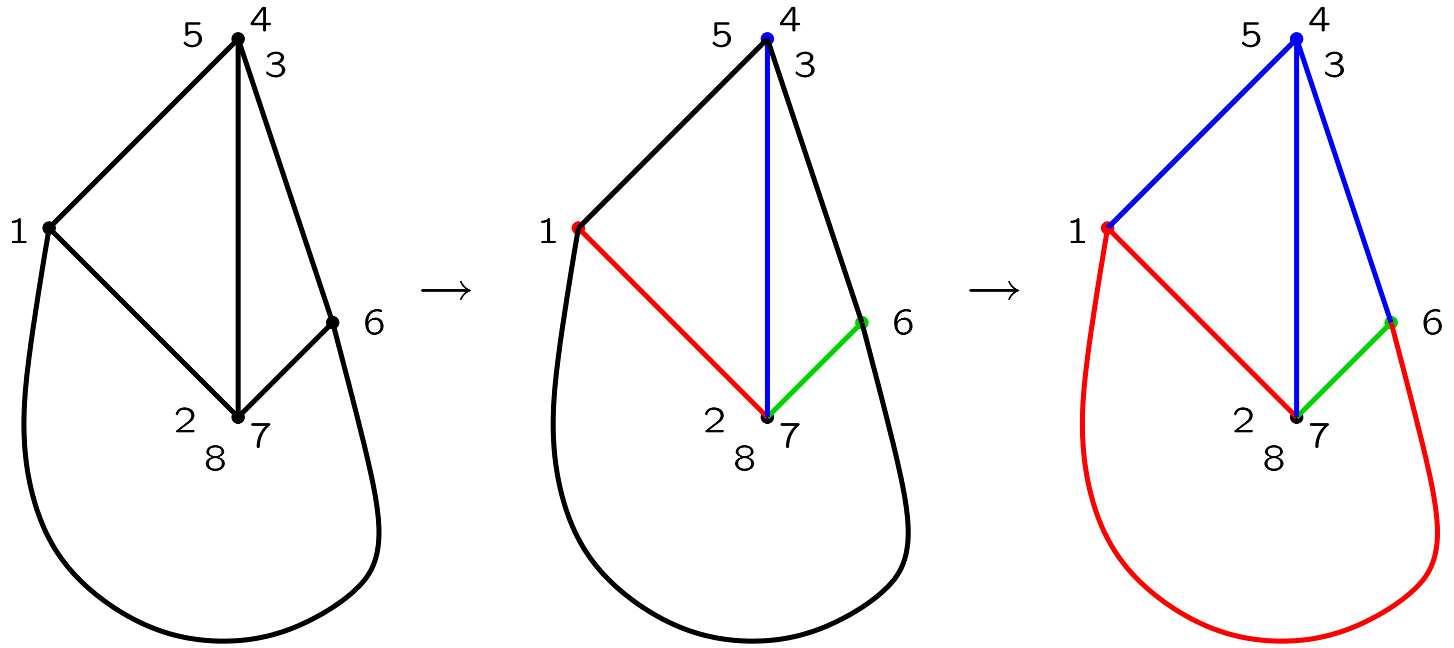
- **Phase 3:** edge-partition of map H except edges of F_2 from realizer of map H'.

- **Phase 4:** whole edge-partition of map H

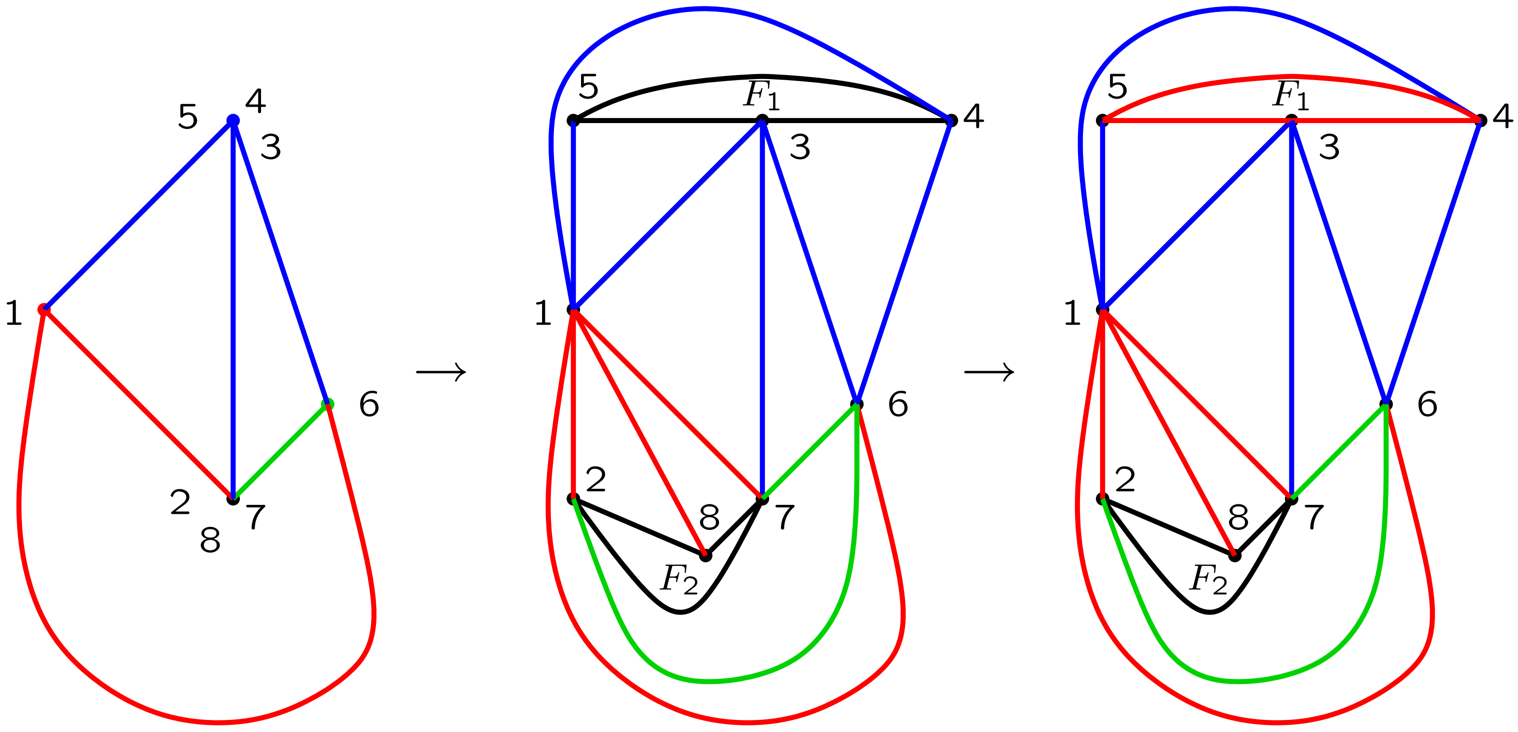
Phase 1:Contraction of the two faces to obtain a triangulated map H'



Phase 2:Computation of Schnyder's realizer of map H'



Phase 3:Edge-partition of map H except edges of F_2 from realizer of map H'



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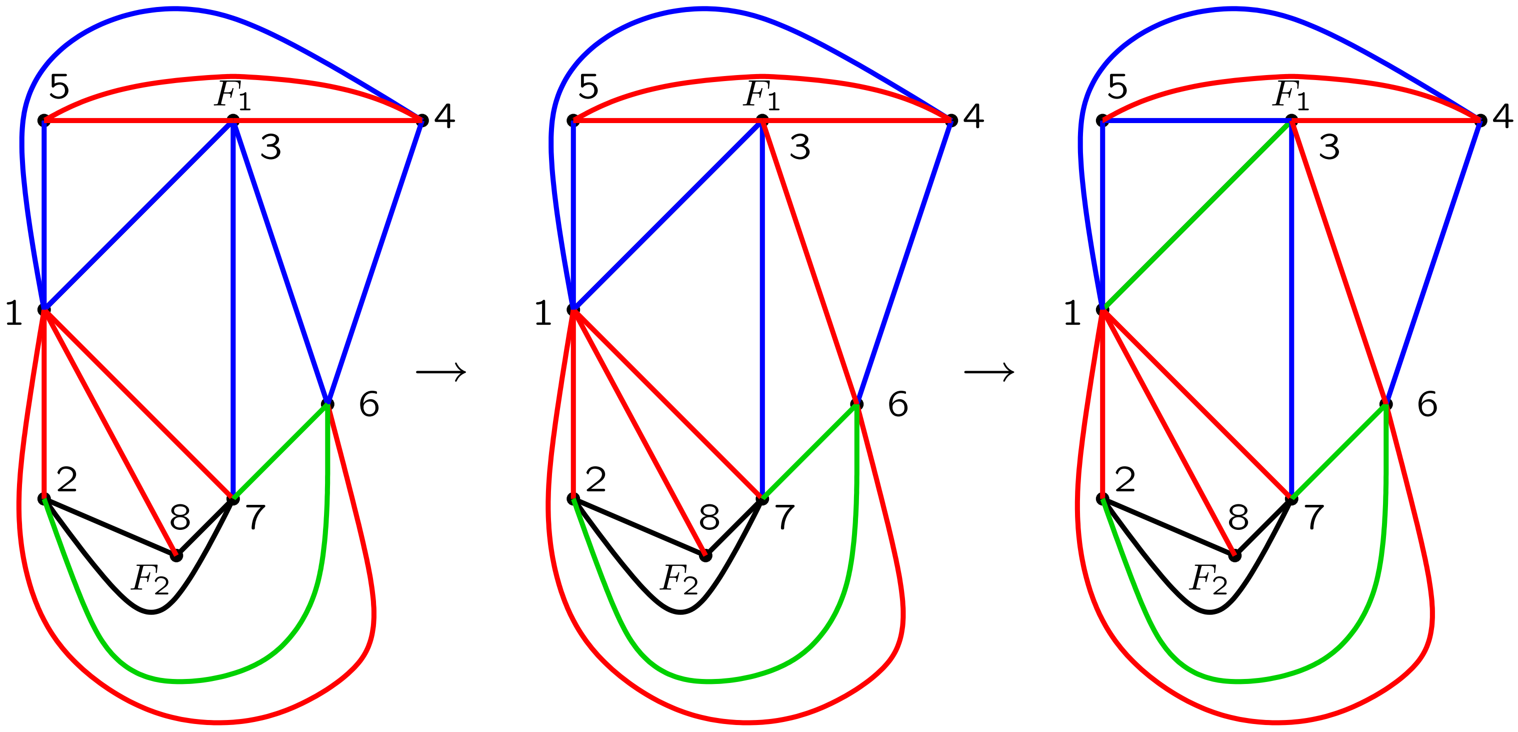
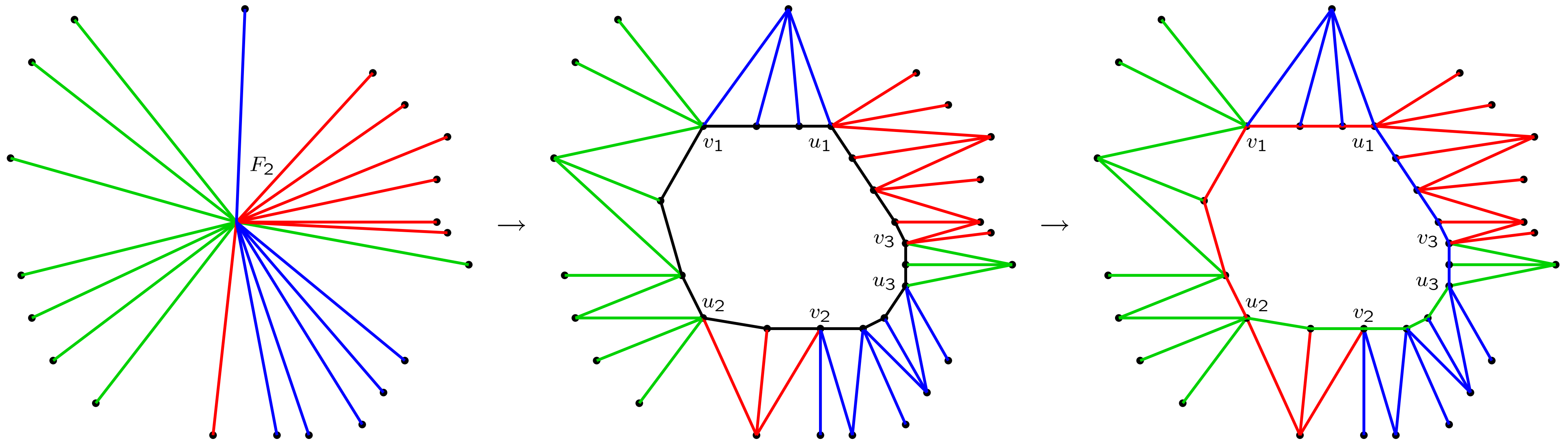


Phase 4: Whole edge-partition of map H



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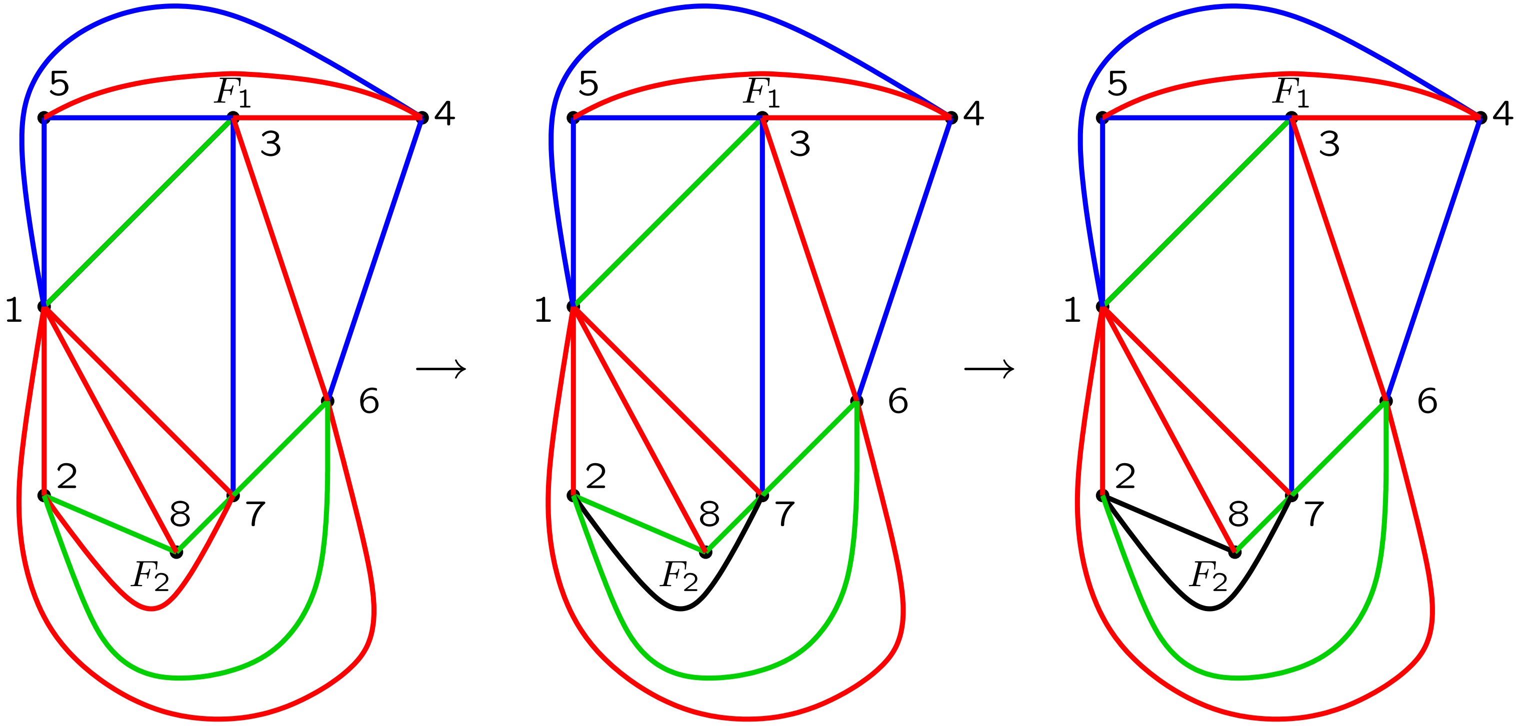
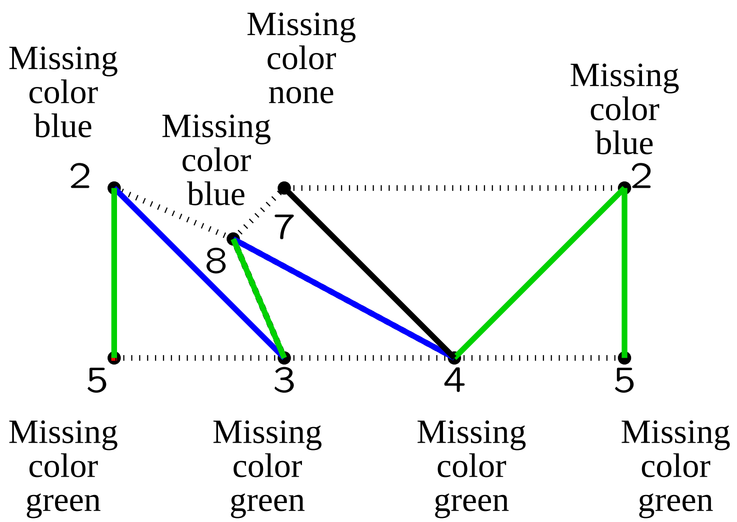


Step 3 : Edge partition into 3 forests of G from the edge partition of H.

It remains us to color edges of T according to the missing color of the vertices of the 2 cycles.



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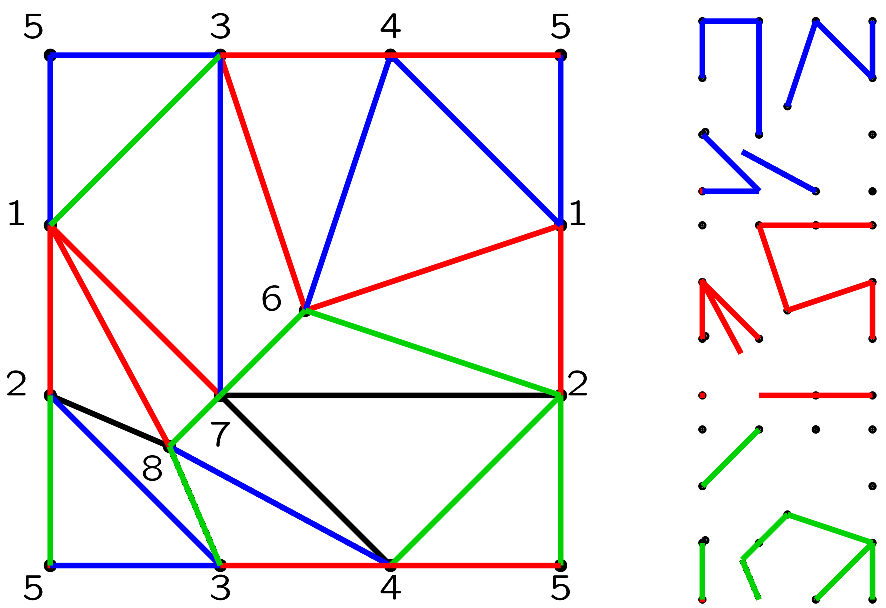

3. Possible Improvements and Open Problems

- Partition with extra properties about the edge ordering around each vertex in spite of the three Schnyder's trees for plane triangulation. In our partition, edges around vertex consists in 6 blocks except vertices of C_1 and C_2.

- Partition with bound on diameter of trees

- Generalisation of our result for graph of genus $g > 1$ especially for locally planar graph.