

Tight stretch factors for L_1 - and L_∞ -Delaunay triangulations



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ABSTRACT

In this paper we determine the exact stretch factor of L_∞ -Delaunay triangulations of points in the plane. We do this not only when the distance between the points is defined by the usual L_2 -metric but also when it is defined by the L_p -metric, for any $p \in [1, \infty]$. We then apply this result to compute the exact stretch factor of L_1 -Delaunay triangulations when the distance between the points is defined by the L_1 -, L_∞ -, or L_2 -metric.

In the important case of the L_2 -metric, we obtain that the stretch factor of L_1 -Delaunay and L_∞ -Delaunay triangulations is exactly $\sqrt{4 + 2\sqrt{2}} \approx 2.61$. This is the first time that the stretch factor of an L_p -Delaunay triangulation, for any $p \in [1, \infty]$, is determined exactly. We show, in particular, how to construct between any two points a and b of an L_1 -Delaunay or L_∞ -Delaunay triangulation a path whose length is no more than $\sqrt{4 + 2\sqrt{2}}$ times the Euclidean distance between a and b . This improves the bound of $\sqrt{10}$ by Chew (SoCG '86) [5]. We also describe families of point sets whose L_1 -Delaunay or L_∞ -Delaunay triangulation has a stretch factor that can be made arbitrarily close to $\sqrt{4 + 2\sqrt{2}}$.

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1. Introduction

Given a set of points P in the plane, the Voronoi diagram of P is the subdivision of the plane into Voronoi regions, one region $V(u)$ for each point u of P , such that region $V(u)$ contains those points in the plane that are closer to u than to any other point of P (see for instance [1, Chapter 2]). The Voronoi diagram can be viewed as a plane graph whose faces are the Voronoi regions and whose vertices are the points in the plane that lie on the boundary of three (or more) regions. The Delaunay graph on P is the dual graph, with vertex set P , of the Voronoi diagram. The Delaunay graph can naturally be viewed as a weighted graph in which the weight of an edge is the distance, typically Euclidean, between its endpoints. When the points of P are in *general position* (defined precisely in Section 2), the Voronoi diagram has maximum degree three and all internal faces of the Delaunay graph are triangles; the Delaunay graph is then referred to as the *Delaunay triangulation* on P .

Delaunay triangulations have broad applications in Computer Science including surface construction, meshing, and visualization (see [2], for example, for an overview of applications). We are particularly motivated by applications (such as wireless communication network construction [3] and online routing in such networks [4]) that use the triangulation as

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Table 1
Key stretch factor upper bounds (tight bounds are bold).

Paper	Graph	Metric	Stretch factor
[7]	L_2 -Delaunay	L_2	$\pi(1 + \sqrt{5})/2 \approx 5.08$
[8]	L_2 -Delaunay	L_2	$4\pi/(3\sqrt{3}) \approx 2.41$
[9]	L_2 -Delaunay	L_2	1.998
[6]	TD-Delaunay	L_2	2
[5]	L_1 -, L_∞ -Delaunay	L_2	$\sqrt{10} \approx 3.16$
[this paper]	L_1 -, L_∞ -Delaunay	L_2	$\sqrt{4 + 2\sqrt{2}} \approx$ 2.61
[this paper]	L_1 -, L_∞ -Delaunay	L_1, L_∞	3
[this paper]	L_∞ -Delaunay	L_p ($p \in [1, \infty]$)	see Fig. 1

a *spanner* of the Euclidean graph on P , defined as a spanning subgraph in which the distance in the subgraph between any pair of points is no more than a constant multiplicative ratio of the Euclidean distance between the points. The constant ratio is typically referred to as the *stretch factor* of the spanner. While the Delaunay triangulation has been studied extensively, obtaining a tight bound on its stretch factor has been elusive even after decades of attempts.

In the mid-1980s, it was not known whether a Delaunay triangulation is, in general, a spanner. In order to gain an understanding of the spanning properties of Delaunay triangulations, Chew considered related structures. In his seminal 1986 paper [5], he proved that an L_1 -Delaunay triangulation – the dual of the Voronoï diagram of P based on the L_1 -metric rather than the L_2 -metric – has a stretch factor bounded by $\sqrt{10}$. Chew then continued on and showed that a TD-Delaunay triangulation – the dual of a Voronoï diagram defined using a *Triangular Distance*, a distance function not based on a circle (L_2 -metric) or a square (L_1 -metric) but on an equilateral triangle – has a stretch factor of 2 [6]. This bound is, in fact, tight: one can construct TD-Delaunay triangulations with stretch factor arbitrarily close to 2. Finally, Dobkin et al. [7] showed that the (L_2 -) Delaunay triangulation of P is a spanner as well. The bound on the stretch factor they obtained was subsequently improved by Keil and Gutwin [8] as shown in Table 1. The bound by Keil and Gutwin stood unchallenged for many years until Xia recently improved the bound to below 2 [9].

There has also been some progress recently on understanding the lower bound on the stretch factor of an (L_2 -) Delaunay triangulation. The trivial lower bound of $\pi/2 \approx 1.5707$ has been improved to 1.5846 [10] and then to 1.5932 [11]. A large gap between the lower and upper bound still remains, however, and it is not clear at all that the techniques currently being used are appropriate for obtaining a tight bound on the stretch factor of a Delaunay triangulation.

While much effort has been made on studying the stretch factor of L_2 -Delaunay triangulations, since Chew's original work little has been done on L_p -Delaunay triangulations for $p \neq 2$. It is known that L_p -Delaunay triangulations are spanners: Bose et al. [12] have shown that Delaunay triangulations that are based on a convex distance function are spanners whose stretch factor depends only on the shape of the associated convex body. However, due to the general approach, the bounds on the stretch factor that they obtain are loose: the bound for L_2 -Delaunay triangulations, for example, is greater than 24.

The overall picture is that, in spite of much effort and with the exception of the triangular distance, the exact value of the stretch factor of Delaunay triangulations based on any convex function is unknown. In particular, the stretch factor of L_p -Delaunay triangulations is unknown for any $p \in [1, \infty]$.

Our contributions We show that the exact stretch factor of L_1 - and L_∞ -Delaunay triangulations is $\sqrt{4 + 2\sqrt{2}} \approx 2.61$, ultimately improving the upper bound of $\sqrt{10} \approx 3.16$ obtained by Chew [5]. This bound applies to the (traditional) case when the edge weight function is the Euclidean distance between its endpoints.

In addition to obtaining a tight bound on the stretch factor when the edge weight function is the Euclidean distance (i.e., L_2 -metric), we also obtain a tight bound on the stretch factor of L_∞ -Delaunay triangulations when the edge weight function is the L_p -metric, for any $p \in [1, \infty]$ (see Fig. 1). In this case, the weight of each edge in the triangulation is the distance between its endpoints according to the L_p -metric; the stretch factor is computed by comparing for every pair of points their distance in the triangulation with their distance in the plane according to the L_p -metric. From this general result we deduce that the stretch factor of L_1 - and L_∞ -Delaunay triangulations is exactly 3 when the edge weight function is the L_1 - or L_∞ -metric.

2. Preliminaries

We consider a finite set P of points in the two-dimensional plane with an orthogonal coordinate system consisting of a horizontal x -axis and a vertical y -axis. The coordinates of a point u will be denoted (x_u, y_u) . The L_p -norm of u is $\|(x_u, y_u)\|_p = (|x_u|^p + |y_u|^p)^{1/p}$ for $p \in [1, \infty)$; for $p = \infty$, $\|(x_u, y_u)\|_p = \lim_{p' \rightarrow \infty} \|(x_u, y_u)\|_{p'} = \max(|x_u|, |y_u|)$. Given points u and v , let $\Delta_x(u, v) = |x_v - x_u|$ and $\Delta_y(u, v) = |y_v - y_u|$. The L_p -metric is the metric defined by

$$d_p(u, v) = \left\| (\Delta_x(u, v), \Delta_y(u, v)) \right\|_p.$$

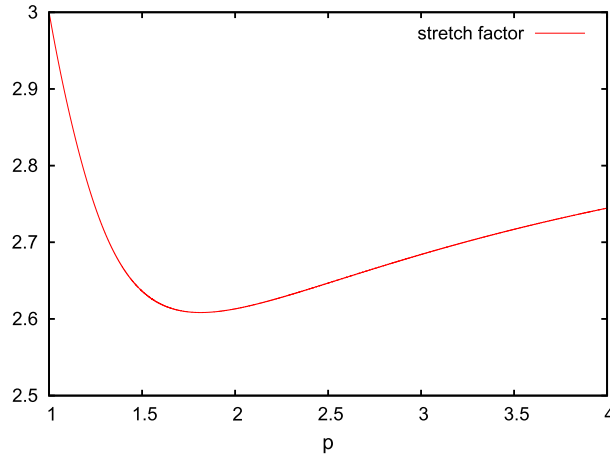


Fig. 1. The exact stretch factor of L_∞ -Delaunay triangulations with respect to the L_p -metric. When p is 1 or ∞ the stretch factor is 3. The stretch factor for $p = 2$ is $\sqrt{4 + 2\sqrt{2}} \approx 2.613125929$. The stretch factor is minimal for $p = 1.814669163$ when its value is 2.608358089.

We denote by \mathcal{E}_p^P (or simply \mathcal{E}_p when the set of points P is clear from the context) the complete weighted graph whose vertex set is P and whose edge weight function, for any edge $(u, v) \in \mathcal{E}_p$, is $d_p(u, v)$. The graph \mathcal{E}_2 , in particular, is known as the *complete Euclidean graph*.

The length of a path in a weighted graph is the sum of the weights of the edges of the path and the distance in the graph between two vertices is the length of the shortest path between them. We say that a subgraph H of a weighted graph G is a t -spanner of G if for any pair of vertices u, v of G , the distance between u and v in H is at most t times the distance between u and v in G ; the constant t is referred to as the *stretch factor* of H (with respect to G).

In this paper we study spanners of the graph \mathcal{E}_p , for $p \in [1, \infty]$, defined on a finite set of points P . Given a subgraph H of \mathcal{E}_p and points $u, v \in P$, we denote by $d_p^H(u, v)$ the distance between u and v in H ; note that $d_p^H(u, v) = d_p(u, v)$ if (u, v) is an edge of H . Therefore H is a t -spanner of \mathcal{E}_p if

$$d_p^H(u, v) \leq t \cdot d_p(u, v)$$

for all pairs of points u and v in P . We say that H is a t -spanner of P (or simply a t -spanner) with respect to the L_p metric if it is a t -spanner of \mathcal{E}_p .

We define a *family of spanners* to be a set of graphs H^P , one for every finite set P of points in the plane, such that for some constant $t > 0$, H^P is a t -spanner of \mathcal{E}_p^P for every set of points P . We say that the stretch factor of the family is *exactly* t with respect to the L_p metric if: 1) H^P is a t -spanner of \mathcal{E}_p^P for every set of points P and 2) for every $\epsilon > 0$ there exists a set of points P such that H^P is *not* a $(t - \epsilon)$ -spanner of \mathcal{E}_p^P .

Defining L_1 -Delaunay triangulations The families of spanners we consider are the L_1 - and L_∞ -Delaunay triangulations on a set P of points in general position (which we define precisely at the end of this section). As we saw in the introduction, the L_p -Delaunay triangulation on P , for $p \in [1, \infty]$, can be defined as the dual graph of the Voronoï diagram of P based on the L_p -metric. We will use an alternative, more direct definition though. A property of L_p -Delaunay triangulations — shared by all Delaunay triangulations based on a convex distance function — is that for each interior face, or triangle, of the triangulation there is an associated convex body (e.g., square for L_1 - or L_∞ - and circle for L_2 -Delaunay triangulations) that contains the vertices of the triangle on its boundary and that contains no point of P in its interior (or, said more succinctly, that is *empty*). More specifically, let a *square* in the plane be a square whose sides are parallel to the x - and y -axis and let a *tipped square* be a square rotated by 45° . For every triple of points $u, v, w \in P$, u, v, w form an interior face (triangle) in the L_1 -Delaunay triangulation on P if and only if there is an empty tipped square that contains u, v, w on its boundary [5].

This property does not completely define L_1 -Delaunay triangulations however, because some edges of an L_1 -Delaunay triangulation may not border an interior face (when the points in P are collinear, in fact, no edge borders an interior face). The following, however, does completely define the L_1 -Delaunay triangulation on a set of points P : for every pair of points $u, v \in P$, (u, v) is an edge in the L_1 -Delaunay triangulation on P if and only if there is an empty tipped square that has u and v on its boundary [5]. We note that this definition assumes that no four points of P lie on the boundary of an empty tipped square.

From L_1 - to L_∞ -Delaunay triangulations If a *square*, rather than a tipped square, is used in this last definition then a different triangulation is defined; it corresponds to the dual of the Voronoï diagram based on the L_∞ -metric. We refer to this triangulation as the L_∞ -Delaunay triangulation. This triangulation is nothing more than the L_1 -Delaunay triangulation of the set of points P after rotating all the points by 45° around the origin. Chew’s bound of $\sqrt{10}$ on the stretch factor, with

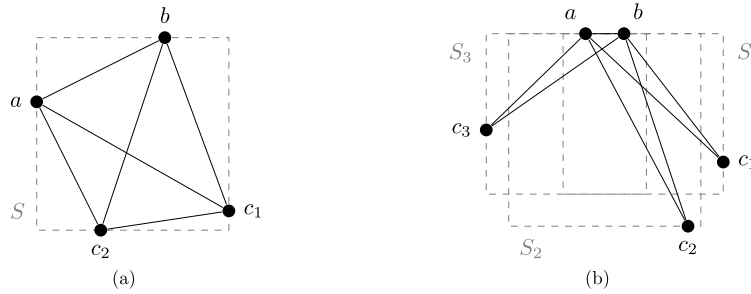


Fig. 2. (a) The L_∞ -Delaunay triangulation is not well defined when four points lie on the boundary of an empty square. (b) Another set of points for which the L_∞ -Delaunay triangulation is not well defined. In this example, points a and b lie on one side (N) of square S_1 and c_1 lies on an adjacent side (E). The restriction that no four points of P lie on the boundary of an empty square implies that if $a, b, c_1 \in P$ and S_1 is empty then $c_2, c_3 \notin P$ and, in fact, no point of P could lie in the plane quadrant defined by sides N and E of square S_1 .

respect to the Euclidean distance, of the L_1 -Delaunay triangulation [5] applies to L_∞ -Delaunay triangulations as well. In this paper, most of our arguments focus on L_∞ -Delaunay triangulations and use squares (rather than tipped squares). When we refer to the *side* of a square, we assume it includes the square vertices; in other words, sides intersect at the square vertices. To differentiate the four sides of a square we use the notation N (for the north side), E (east), S (south), and W (west). We also use this notation to describe the relative position of an edge connecting two points lying on the boundary of a square: for example, a WN edge connects a point on the W side and a point on the N side.

General position requirement for L_∞ -Delaunay triangulations One issue with L_∞ -Delaunay triangulations (in fact, all Delaunay triangulations) is that they are not well-defined for all sets of points P . If four points of P happen to lie on the boundary of an empty square then the L_∞ -Delaunay triangulation on P , using our working definition, would contain the two ways to triangulate the four points – as illustrated in Fig. 2(a) – and is thus not plane (with interior faces that are triangles) at all. We therefore restrict P to be a set of points such that no four points lie on the boundary of an empty square. In fact, in order to avoid technical complications in our proofs and pathological examples such as the one illustrated by Fig. 2(b), we choose to restrict P a bit further and insist that no two points of P lie on the same side of an empty square. Note that this implies that endpoints of an edge in the L_∞ -Delaunay triangulation must have different x - and y -coordinates.

In summary, the assumption in this paper will be that the points of P are in *general position*, which for us means that no four points lie on the boundary of an empty square and that no two points lie on the same side of an empty square.

3. The lower bound

In this section we prove lower bounds on the stretch factor of L_1 - and L_∞ -Delaunay triangulations. To do this we require three technical lemmas. The first gives a general lower bound for L_∞ -Delaunay triangulations:

Lemma 1. *Let $p \in [1, \infty]$. For every $\varepsilon > 0$, there exists a set P of points in the plane such that the L_∞ -Delaunay triangulation on P with respect to the L_p -metric has stretch factor at least ³*

$$\max_{y \in [0,1]} s_p(y) - \varepsilon, \quad \text{where } s_p(y) = \frac{1 + 2^{1/p} + y}{\|(1, y)\|_p}.$$

Proof. Given $p \in [1, \infty]$, $y \in [0, 1]$ and a positive integer k , we define the set of points P as follows. Let point a be the origin and let points b, c_1 , and c_2 have coordinates $(1, y)$, $(\delta, 1 + y - 2\delta)$, and $(1 - \delta, 2\delta - 1)$, respectively, where $\delta = \frac{1+y}{k+3}$ (see Fig. 3(a)). Additional k points are placed on line segment $[ac_1]$ and another k on line segment $[c_2b]$ in such a way that the difference in y -coordinates between successive points on a segment is δ . Let $a = p_0, p_1, p_2, p_3, \dots, p_k, p_{k+1} = c_1$ be the labels, in order as they appear when moving from a to c_1 , of the points on segment $[ac_1]$ and let $c_2 = q_0, q_1, q_2, q_3, \dots, q_{k+1} = b$ be the labels, in order as they appear when moving from c_2 to b , of the points on segment $[c_2b]$.

Consider the square S_1 of side length $1 - \delta$ and having a and p_1 on its west (left) and north (top) sides, respectively (see Fig. 3(b)). Since $\Delta_x(a, c_2) = 1 - \delta$ and $\Delta_y(p_1, c_2) = 1 - \delta$, point c_2 is exactly the southeast vertex of square S_1 . By symmetry, it follows that for every $i = 1, 2, \dots, k + 1$, if S_i is the square of side length $1 - \delta$ with p_{i-1} and p_i on its west and north sides, then point q_{i-1} is exactly the southeast vertex of S_i . This means that all points p_j with $j \neq i - 1$ as well as all points p_j with $j \neq i - 1, i$ must lie outside S_i . Therefore, for every $i = 1, 2, \dots, k + 1$, points p_{i-1}, p_i , and q_{i-1} define a triangle in the L_∞ -Delaunay triangulation T on P . A similar argument shows that for every $i = 1, 2, \dots, k + 1$, points q_{i-1}, q_i , and p_i define a triangle in the L_∞ -Delaunay triangulation T – illustrated in Fig. 3(a) – as well.

³ Throughout the paper, to simplify the exposition we will abuse notation and, for any function $f()$, write $f(\infty)$ instead of $\lim_{p \rightarrow \infty} f(p)$ when this limit is well defined.

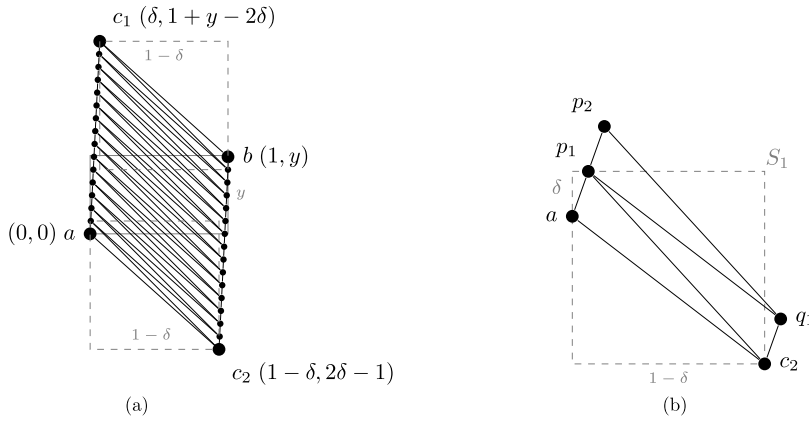


Fig. 3. (a) An L_∞ -Delaunay triangulation with points in general position. The coordinates of points a , b , c_1 , and c_2 are $(0, 0)$, $(1, y)$, $(\delta, 1 + y - 2\delta)$, and $(1 - \delta, 2\delta - 1)$, respectively. (b) A closer look at the bottom faces of this triangulation.

Having defined the set of points P and described the L_∞ -Delaunay triangulation T on P , we now analyze the stretch factor of T . A shortest path from a to b in T with respect to the L_p -metric is, for example, $a, p_1, p_2, \dots, p_k, c_1, b$. The length of this path is $d_p(a, c_1) + d_p(c_1, b)$ which tends to $1 + y + 2^{1/p}$ as k tends to ∞ (and thus δ tends to 0). The distance between a and b is $\|(1, y)\|_p$. Therefore, it is possible to construct an L_∞ -Delaunay triangulation whose stretch factor is arbitrarily close to $s_p(y)$. \square

We evaluate the maximum value of function $s_p(y)$ in the interval $[0, 1]$:

Lemma 2. When $1 < p < \infty$, the maximum of function $s_p(y)$ in the interval $[0, 1]$ is

$$\left((1 + 2^{1/p})^{p/(p-1)} + 1 \right)^{(p-1)/p}$$

and is reached at $y = y_p = (1 + 2^{1/p})^{1/(1-p)}$. For $p = 1$ and $p = \infty$, the maximum is 3 and is reached at $y = y_1 = 0$ for $p = 1$ and at $y = y_\infty = 1$ for $p = \infty$.

Proof. When $1 < p < \infty$, the derivative of $s_p(y)$ is

$$s'_p(y) = \frac{1 - y^{p-1}(1 + 2^{1/p})}{(1 + y^p)^{1+1/p}}.$$

Let $y_p = (1 + 2^{1/p})^{1/(1-p)}$. Since $0 < y_p < 1$, $s'_p(y_p) = 0$, $s'_p(y) > 0$ for $y \in (0, y_p)$, and $s'_p(y) < 0$ for $y \in (y_p, 1)$, the maximum of $s_p(y)$ for $y \in [0, 1]$ is reached at $y = y_p$ and is thus equal to

$$\begin{aligned} s_p(y_p) &= \frac{1 + 2^{1/p} + (1 + 2^{1/p})^{1/(1-p)}}{(1 + (1 + 2^{1/p})^{p/(1-p)})^{1/p}} \\ &= \frac{(1 + 2^{1/p})(1 + (1 + 2^{1/p})^{p/(1-p)})}{(1 + (1 + 2^{1/p})^{p/(1-p)})^{1/p}} \\ &= (1 + 2^{1/p})(1 + (1 + 2^{1/p})^{p/(1-p)})^{(p-1)/p} \\ &= \left((1 + 2^{1/p})^{p/(p-1)} (1 + (1 + 2^{1/p})^{p/(1-p)}) \right)^{(p-1)/p} \\ &= \left((1 + 2^{1/p})^{p/(p-1)} + 1 \right)^{(p-1)/p}. \end{aligned}$$

When $p = 1$, $s_1(y) = \frac{3+y}{1+y}$. Because it is a decreasing function of y in the interval $[0, 1]$, the maximum of $s_1(y)$ is reached at $y = y_1 = 0$ and thus $s_1(y_1) = 3$. When $p = \infty$, $s_\infty(y) = 2 + y$ and, within the interval $[0, 1]$, reaches its maximum of 3 at $y = y_\infty = 1$. \square

The following lemma shows that a (lower or upper) bound on the stretch factor, with respect to the L_1 -, L_2 -, and L_∞ -metrics, of L_∞ -Delaunay triangulations can be extended to L_1 -Delaunay triangulations (and vice-versa):

Lemma 3. There is an L_∞ -Delaunay triangulation with stretch factor t with respect to the L_2 -metric (resp. L_∞ -metric, L_1 -metric) if and only if there is an L_1 -Delaunay triangulation with stretch factor t with respect to the L_2 -metric (resp. L_1 -metric, L_∞ -metric).

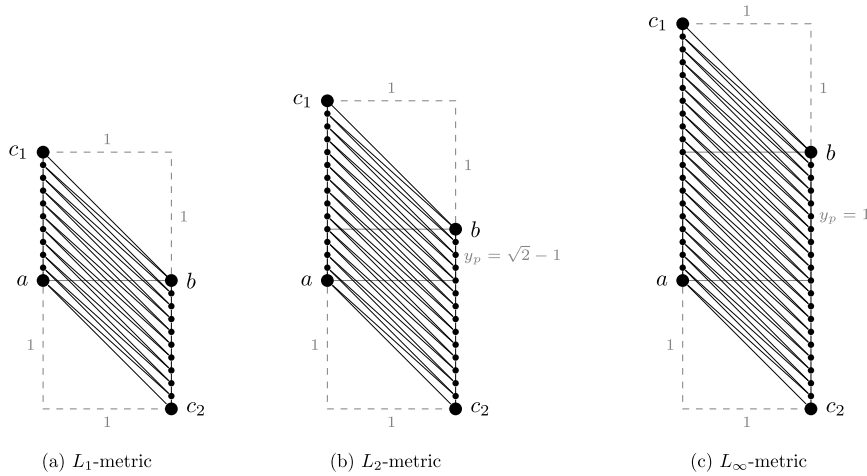


Fig. 4. Degenerate L_∞ -Delaunay triangulations that illustrate the lower bound on the stretch factor with respect to the (a) L_1 -metric, (b) L_2 -metric, and (c) L_∞ -metric.

Proof. Consider a set P of points in the plane and let P' be the set of points that is the image of P under a 45° rotation r of the plane. Suppose that the L_∞ -Delaunay triangulation T on P has stretch factor t with respect to the L_p -metric for $p \in \{1, 2, \infty\}$ and let T' be the L_1 -Delaunay triangulation on P' .

Observe that there is an edge $(r(u), r(v))$ in T' if and only there is an edge (u, v) in T since the image, under rotation r , of a square that contains no point of P in its interior and that has u and v on its boundary is a tipped square that contains no point of P' in its interior and that has $r(u)$ and $r(v)$ on its boundary. Observe also that for all $u, v \in P$,

$$d_2(u, v) = d_2(r(u), r(v)); \quad d_1(u, v) = \sqrt{2}d_\infty(r(u), r(v)); \quad d_\infty(u, v) = \frac{1}{\sqrt{2}}d_1(r(u), r(v)).$$

Hence, we have:

$$\frac{d_2^T(u, v)}{d_2(u, v)} = \frac{d_2^{T'}(r(u), r(v))}{d_2(r(u), r(v))}; \quad \frac{d_1^T(u, v)}{d_1(u, v)} = \frac{d_\infty^{T'}(r(u), r(v))}{d_\infty(r(u), r(v))}; \quad \frac{d_\infty^T(u, v)}{d_\infty(u, v)} = \frac{d_1^{T'}(r(u), r(v))}{d_1(r(u), r(v))}.$$

Therefore, if T has stretch factor t with respect to the L_2 -metric (resp. L_∞ -metric, L_1 -metric) then T' has stretch factor t with respect to the L_2 -metric (resp. L_1 -metric, L_∞ -metric). The proof in the opposite direction, from P' and T' to P and T , is symmetric. \square

We can now show lower bounds on the stretch factors of L_∞ - and L_1 -Delaunay triangulations with respect to the important Euclidean, L_1 -, and L_∞ -metrics:

Theorem 4. *There exist:*

- L_1 - and L_∞ -Delaunay triangulations whose stretch factor, with respect to the Euclidean distance, is arbitrarily close to $\sqrt{4 + 2\sqrt{2}} \approx 2.613 \dots$
- L_1 - and L_∞ -Delaunay triangulations whose stretch factor, with respect to the L_1 -metric, is arbitrarily close to 3.
- L_1 - and L_∞ -Delaunay triangulations whose stretch factor, with respect to the L_∞ -metric, is arbitrarily close to 3.

Furthermore, with respect to the L_p -metric, for $1 < p < \infty$, there exist L_∞ -Delaunay triangulations whose stretch factor is arbitrarily close to

$$\left((1 + 2^{1/p})^{p/(p-1)} + 1 \right)^{(p-1)/p}.$$

Proof. The statements for L_∞ -Delaunay triangulations follow from Lemma 1 and Lemma 2. Lemma 3 extends the lower bounds to L_1 -Delaunay triangulations for the Euclidean, L_1 -, and L_∞ -metrics. \square

In Fig. 4, we show several examples of L_∞ -Delaunay triangulations that illustrate Theorem 4. While the points in these examples are not in general position, they demonstrate more clearly the lower bounds. Fig. 4(a) represents a class of L_∞ -Delaunay triangulations whose stretch factor, with respect to the L_1 -metric, can be made arbitrarily close to

$$\frac{d_1(a, c_1) + d_1(c_1, b)}{d_1(a, b)} = \frac{1 + 2}{1} = 3$$

by adding enough regularly spaced points on segments $[ac_1]$ and $[c_1b]$ and triangulating as shown. Fig. 4(b) represents a class of triangulations whose stretch factor, with respect to the L_2 -metric can also be made arbitrarily close to:

$$\frac{d_2(a, c_1) + d_2(c_1, b)}{d_2(a, b)} = \frac{2\sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} = \sqrt{4 + 2\sqrt{2}} \approx 2.613\dots$$

Finally, Fig. 4(c) represents a class of triangulations whose stretch factor, with respect to the L_∞ -metric, can be made arbitrarily close to:

$$\frac{d_\infty(a, c_1) + d_\infty(c_1, b)}{d_\infty(a, b)} = \frac{2 + 1}{1} = 3.$$

4. Main result

In this section we obtain tight upper bounds on the stretch factor of L_1 - and L_∞ -Delaunay triangulations. They will follow directly from the following key theorem that bounds the distance between any two points a and b of an L_∞ -triangulation in terms of the length and width of a rectangle having a and b as vertices:

Theorem 5. *Let $p \in [1, \infty]$, let P be a set of points in general position in the plane, and let T be the L_∞ -Delaunay triangulation on P . For any pair of points a and b of P , let $R(a, b)$ be the rectangle with sides parallel to the x - or y -axis and with diagonal $[ab]$. If the size of $R(a, b)$ is $w \times h$ (with $0 \leq h \leq w$ and $0 < w$) then*

$$d_p^T(a, b) \leq (1 + 2^{1/p})w + h.$$

This section is devoted to the proof of this theorem. Before we start, we show how this theorem implies the main result of this paper:

Theorem 6. *The stretch factor of L_1 - and L_∞ -Delaunay triangulations is exactly*

- $\sqrt{4 + 2\sqrt{2}} \approx 2.6131259\dots$ with respect to the Euclidean distance.
- 3 with respect to the L_1 -metric or the L_∞ metric.

With respect to the L_p -metric, for $1 < p < \infty$, the stretch factor of L_∞ -Delaunay triangulations is exactly

$$((1 + 2^{1/p})^{p/(p-1)} + 1)^{(p-1)/p}.$$

Proof. By Theorem 5, an upper-bound on the stretch factor of an L_∞ -Delaunay triangulation with respect to the L_p -metric ($p \in [1, \infty]$) is the maximum of the function

$$\frac{(1 + 2^{1/p})w + h}{\|(w, h)\|_p} = \frac{1 + 2^{1/p} + h/w}{\|(1, h/w)\|_p} = s_p(h/w)$$

over values w and h such that $0 \leq h \leq w$ and $0 < w$ and where function s_p is as defined in Lemma 1. As observed in Lemma 2,

$$\max_{(h/w) \in [0, 1]} s_p(h/w) = s_p(y_p).$$

Therefore the upper bounds on the stretch factor of an L_∞ -Delaunay triangulation with respect to the L_p -metric ($p \in [1, \infty]$) match the lower bounds shown in Theorem 4. By Lemma 3, the same holds for L_1 -Delaunay triangulations for the L_1 -, L_2 -, and L_∞ -metrics. □

4.1. Overview of the argument and a structural lemma

We will prove Theorem 5 by showing that between any two points a and b of P there is a short enough path in T . The proof will be by induction on the (rank of the) distance between the points in P with respect to the L_∞ -metric. If edge $(a, b) \in T$ the path is simply edge (a, b) . Otherwise, if rectangle $R(a, b)$ contains a point c of P other than a and b , we will easily apply induction to show the existence of short enough paths from a to c and from c to b which together form a short enough path from a to b . The case when $R(a, b)$ does not contain any point of P (other than a and b) is more difficult and we need to develop tools to handle it.

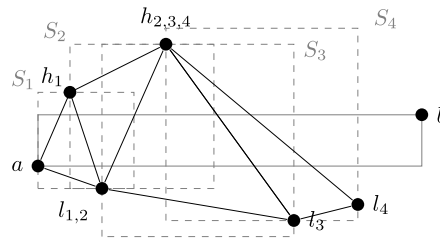


Fig. 5. Triangles T_1 (with points a, h_1, l_1), T_2 (with points h_1, h_2, l_2), and T_3 (with points l_2, h_3, l_3) and associated squares S_1, S_2 , and S_3 . When traveling from a to b along segment $[a, b]$, the edge that is hit when leaving triangle T_i is (h_i, l_i) .

To simplify the notation and the discussion, we assume that point a has coordinates $(0, 0)$ and point b has coordinates (w, h) with $0 \leq h \leq w$ and $0 < w$. The line containing segment $[ab]$ divides the plane into two half-planes; a point in the same half-plane as point $(0, 1)$ is said to be *above* segment $[ab]$, otherwise it is *below*. Let $T_1, T_2, T_3, \dots, T_k$ be the sequence of triangles of triangulation T that line segment $[ab]$ intersects when moving from a to b ; if $R(a, b)$ contains no point of P other than a and b , segment $[ab]$ does not contain the vertices of these triangles, other than a and b of course. Let h_1 and l_1 be the vertices of T_1 other than a , with h_1 lying above segment $[ab]$ and l_1 lying below. Every triangle T_i , for $1 < i < k$, intersects line segment $[ab]$ twice; let h_i and l_i be the endpoints of the edge of T_i that intersects segment $[ab]$ last, when moving on segment $[ab]$ from a to b , with h_i being above and l_i being below segment $[ab]$. Note that either $h_i = h_{i-1}$ and $T_i = \Delta(h_i, l_i, l_{i-1})$ or $l_i = l_{i-1}$ and $T_i = \Delta(h_{i-1}, h_i, l_i)$, for $1 < i < k$. We also set $h_0 = l_0 = a$ and $h_k = l_k = b$. For $1 \leq i \leq k$, we define S_i to be the empty square having the vertices of T_i on its boundary (see illustration in Fig. 5). Finally, we will say that a point u with coordinates (x_u, y_u) is *high* (resp. *low*) with respect to $R(a, b)$ if $0 \leq x_u \leq w$ and $y_u > h$ (resp. $y_u < 0$).

In order to prove Theorem 5 in the case when $R(a, b)$ does not contain any point of P , we will show (in Lemma 8) that either a) one of the paths $a, h_1, h_2, \dots, h_{k-1}, b$ or $a, l_1, l_2, \dots, l_{k-1}, b$ is short enough or b) there is a point $c = l_j$ or h_j and there is a path from a to c that is short enough and that can be combined with the path from c to b that exists by induction into a short enough path from a to b .

We start the formal argument with a lemma that describes the structure of triangles T_1, \dots, T_k and associated squares S_1, \dots, S_k :

Lemma 7. *If (a, b) is not an edge of T and if rectangle $R(a, b)$ contains no point of P other than a and b , then a lies on the W side of square S_1 , b lies on the E side of square S_k , points h_1, \dots, h_{k-1} are high and points l_1, \dots, l_{k-1} are low with respect to $R(a, b)$. Furthermore, for any i such that $1 < i < k$:*

- a) *Either $T_i = \Delta(h_{i-1}, h_i, l_{i-1} = l_i)$, points h_{i-1}, h_i , and $l_{i-1} = l_i$ lie on sides of S_i in clockwise order, with no two on the same side, and (h_{i-1}, h_i) is a WN, WE, or NE edge in S_i ,*
- b) *Or $T_i = \Delta(h_{i-1} = h_i, l_{i-1}, l_i)$, points $h_{i-1} = h_i, l_i$, and l_{i-1} lie on the sides of S_i in clockwise order, with no two on the same side, and (l_{i-1}, l_i) is a WS, WE, or SE edge in S_i .*

These properties are illustrated in Fig. 5.

Proof. Because (a, b) is not an edge of T , $h_1, l_1 \neq b$. Since h_1 and l_1 lie on the boundary of square S_1 and are, respectively, above and below segment $[ab]$, it follows that segment $[ab]$ intersects the interior of square S_1 . With a being the origin and b lying in the first quadrant, a therefore must lie on the W or S side of S_1 . If a was to lie on the S side then, since l_1 is below $[ab]$ and lies outside $R(a, b)$, the side length of S_1 would have to be larger than w , the width of $R(a, b)$, and S_1 would contain b in its interior which is a contradiction. Therefore a lies on the W side of S_1 and, by symmetry, b lies on the E side of S_k .

Every square S_i ($0 < i < k$) has points h_i and l_i on its boundary. Since h_i is above segment $[ab]$ and l_i is below, since both are outside rectangle $R(a, b)$, and since S_i cannot contain points a or b in its interior, S_i must have its N side above the north side of rectangle $R(a, b)$, its S side below the south side of rectangle $R(a, b)$, and its E and W sides intersecting the north and the south sides of rectangle $R(a, b)$. This implies that points h_i and l_i must be high and low, respectively, with respect to $R(a, b)$.

The three vertices of T_i can be either $h_i = h_{i-1}, l_{i-1}$, and l_i or h_{i-1}, h_i , and $l_{i-1} = l_i$. Because points of P are in general position, every edge of T_i hits two different sides of square S_i . Also, because h_{i-1} and h_i are high, they cannot lie on the S side of S_i , and because l_{i-1} and l_i are low, they cannot lie on the N side of S_i . If $T_i = \Delta(h_{i-1}, h_i, l_{i-1} = l_i)$, points h_{i-1}, h_i, l_i must lie on the sides of S_i in clockwise order. The only placements of h_{i-1} and h_i on the sides of S_i that satisfy all these constraints are such that they form a WN, WE, or NE edge in S_i . If $T_i = \Delta(h_{i-1} = h_i, l_{i-1}, l_i)$, points h_i, l_i, l_{i-1} must lie on the sides of S_i in clockwise order, and the placements of points l_{i-1} and l_i on the sides of S_i that satisfy the constraints are such that they form a WS, WE, or SE edge in S_i . \square

4.2. Inductive square, the Crossing Lemma, and the proof of Theorem 5

We define next the square S_j and the point of P lying on the boundary of S_j on which induction can be applied in the proof of Theorem 5 when rectangle $R(a, b)$ contains no point of P other than a and b . We introduce some additional terminology first. A vertex c of T_i is said to be *eastern* in S_i if it lies on the E side of S_i . An edge is said to be *gentle* if the line segment corresponding to it in the graph embedding has a slope within $[-1, 1]$; otherwise we say that it is *steep*. Note that by Lemma 7 and the assumption that points of P are in general position, if edge (l_j, h_j) is gentle then l_j or h_j must be eastern in S_j .

Definition 1. Assume that (a, b) is not an edge in T and that rectangle $R(a, b)$ contains no point of P other than a and b . Square S_j , for $1 \leq j < k$, is *inductive* if edge (l_j, h_j) is gentle. The eastern point $c = h_j$ or $c = l_j$ is the *inductive point* of inductive square S_j .

The following lemma will be the key ingredient of our inductive proof of Theorem 5. We will prove it later in this section.

Lemma 8 (Crossing Lemma). Assume that (a, b) is not an edge in T and that rectangle $R(a, b)$ contains no point of P other than a and b . If no square S_1, \dots, S_{k-1} is inductive then

$$d_p^T(a, b) \leq (1 + 2^{\frac{1}{p}})w + h.$$

Otherwise let S_j be the first inductive square in the sequence S_1, S_2, \dots, S_{k-1} . If h_j is the inductive point of S_j then

$$d_p^T(a, h_j) + (y_{h_j} - h) \leq (1 + 2^{\frac{1}{p}})x_{h_j}.$$

If l_j is the inductive point of S_j then

$$d_p^T(a, l_j) - y_{l_j} \leq (1 + 2^{\frac{1}{p}})x_{l_j}.$$

When an inductive point c exists, we can use Lemma 8 to bound $d_p^T(a, c)$ and then apply induction to bound $d_p^T(c, b)$. We can use these bounds to prove Theorem 5 if the position of point c relative to the position of point b is such that $w - x_c \geq |h - y_c|$. If that is not the case, we will use the following lemma:

Lemma 9. Assume that (a, b) is not an edge in T and that rectangle $R(a, b)$ contains no point of P other than a and b and let the coordinates of point $c = h_i$ or $c = l_i$ satisfy $w - x_c < |h - y_c|$.

- a) If $c = h_i$, and thus $w - x_{h_i} < y_{h_i} - h$, then there exists j , with $i < j \leq k$ such that all edges in path $h_i, h_{i+1}, h_{i+2}, \dots, h_j$ are NE edges in their respective squares and $w - x_{h_j} \geq y_{h_j} - h \geq 0$.
- b) If $c = l_i$, and thus $w - x_{l_i} < h - y_{l_i}$, then there exists j , with $i < j \leq k$ such that all edges in path $l_i, l_{i+1}, l_{i+2}, \dots, l_j$ are SE edges in their respective squares and $w - x_{l_j} \geq h - y_{l_j} \geq 0$.

Proof. We only prove the case $c = h_j$ as the case $c = l_i$ follows using a symmetric argument.

We construct the path $h_i, h_{i+1}, h_{i+2}, \dots, h_j$ iteratively. If $h_i = h_{i+1}$, we just continue building the path from h_{i+1} . Otherwise, (h_i, h_{i+1}) is an edge of T_{i+1} which, by Lemma 7, must be a WN, WE, or NE edge in square S_{i+1} . Since h_i and l_i lie on the boundary of S_{i+1} , since h_i is high and l_i is low, and because $w - x_{h_i} < y_{h_i} - h$ which is in turn less than the side length of S_{i+1} , point h_i cannot lie on the W side of S_{i+1} (otherwise b would be inside square S_{i+1}). Thus (h_i, h_{i+1}) is a NE edge. If $w - x_{h_{i+1}} \geq y_{h_{i+1}} - h$ we stop, otherwise we continue the path construction from h_{i+1} . □

We can now prove the main theorem.

Proof of Theorem 5. The proof is by induction on the rank of the distance, with respect to the L_∞ -metric, between points of P (since P is finite there is only a finite number of distances to consider).

Let a and b be the two points of P that are the closest points, with respect to the L_∞ -metric. We assume w.l.o.g. that a has coordinates $(0, 0)$ and b has coordinates (w, h) with $0 \leq h \leq w$ and $0 < w = d_\infty(a, b)$. Since a and b are the closest points using the L_∞ -metric, the largest square having a as a southwest vertex and containing no points of P in its interior, which we call S_a , must have b on its boundary. Therefore (a, b) is an edge in T and $d_p^T(a, b) = d_p(a, b) \leq w + h \leq (1 + 2^{\frac{1}{p}})w + h$.

For the induction step, we again assume w.l.o.g. that a has coordinates $(0, 0)$ and b has coordinates (w, h) with $0 \leq h \leq w$ and $0 < w = d_\infty(a, b)$. If (a, b) is an edge in T , we can bound $d_p^T(a, b)$ just as we did in the base case. So we assume that (a, b) is not an edge in T .

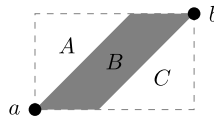


Fig. 6. Partition of $R(a, b)$ into three regions in Case 1 of the proof of Theorem 5.

Case 1. Rectangle $R(a, b)$ contains a point of P (other than a or b).

We first consider the case when there is at least one point of P , other than a and b , lying within rectangle $R(a, b)$. If there is a point $c \neq a, b$ in $R(a, b)$ such that $0 \leq y_c \leq x_c$, $0 < x_c$, $0 \leq h - y_c \leq w - x_c$, and $0 < w - x_c$ (i.e., c lies in the region B shown in Fig. 6) then, because $d_\infty(a, c) = x_c < w = d_\infty(a, b)$ and $d_\infty(c, b) = w - x_c < w = d_\infty(a, b)$, we can apply induction to get $d_p^T(a, c) \leq (1 + 2^{\frac{1}{p}})x_c + y_c$ and $d_p^T(c, b) \leq (1 + 2^{\frac{1}{p}})(w - x_c) + h - y_c$ and use these to obtain the desired bound for $d_p^T(a, b)$.

We now assume that there is no point inside region B . If there is still a point in $R(a, b)$ then there must be one, say c , that is on the boundary of S_a , the square we defined in the base case, or S_b , defined as the largest square having b as a northeast vertex and containing no points of P in its interior. W.l.o.g., we assume the former and thus there is an edge $(a, c) \in T$ such that either $y_c > x_c > 0$, i.e. c is inside region A shown in Fig. 6, or $h - y_c > w - x_c \geq 0$ and $y_c > 0$, i.e. c is inside region C . (Strict inequalities $x_c > 0$ in the former case and $y_c > 0$ in the latter case follow from the fact that a and c cannot lie on the same side of an empty square S_a by our general position assumption). Either way, $d_p^T(a, c) = d_p(a, c) \leq x_c + y_c$. If c is in region A , because $0 \leq h - y_c \leq w - x_c$, $0 < w - x_c$, and $d_\infty(c, b) = w - x_c < w = d_\infty(a, b)$, by induction we also have that $d_p^T(c, b) \leq (1 + 2^{\frac{1}{p}})(w - x_c) + (h - y_c)$. Then

$$\begin{aligned} d_p^T(a, b) &\leq d_p^T(a, c) + d_p^T(c, b) \\ &\leq x_c + y_c + (1 + 2^{\frac{1}{p}})(w - x_c) + (h - y_c) \\ &\leq (1 + 2^{\frac{1}{p}})w + h. \end{aligned}$$

In the second case, since $0 \leq w - x_c < h - y_c$ and $d_\infty(c, b) = h - y_c < w = d_\infty(a, b)$, by induction we have that $d_p^T(c, b) \leq (1 + 2^{\frac{1}{p}})(h - y_c) + (w - x_c)$. Then

$$\begin{aligned} d_p^T(a, b) &\leq d_p^T(a, c) + d_p^T(c, b) \\ &\leq x_c + y_c + (1 + 2^{\frac{1}{p}})(h - y_c) + (w - x_c) \\ &\leq (1 + 2^{\frac{1}{p}})w + h, \end{aligned}$$

where the last inequality follows from $h \leq w$.

Case 2. Rectangle $R(a, b)$ contains no point of P (other than a and b).

If no square S_1, S_2, \dots, S_{k-1} is inductive, $d_p^T(a, b) \leq (1 + 2^{\frac{1}{p}})w + h$ by Lemma 8. Otherwise, let S_i be the first inductive square in the sequence and suppose, for now, that h_i is the inductive point of S_i . By Lemma 7, $x_{h_i} > 0$. By Lemma 9, there is a j , $i \leq j \leq k$, such that $h_i, h_{i+1}, h_{i+2}, \dots, h_j$ is a path in T of length, with respect to the L_p -metric, at most $(x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j})$ and such that $w - x_{h_j} \geq y_{h_j} - h \geq 0$ and, by Lemma 7, $x_{h_j} \geq x_{x_{h_i}}$. Either $h_j = b$ or $0 < w - x_{h_j}$ and we can apply induction to bound $d_p^T(h_j, b)$ since $d_\infty(h_j, b) = w - x_{h_j} < w = d_\infty(a, b)$; in both cases we have $d_p^T(h_j, b) \leq (1 + 2^{\frac{1}{p}})(w - x_{h_j}) + (y_{h_j} - h)$. Putting all this together with Lemma 8, we get:

$$\begin{aligned} d_p^T(a, b) &\leq d_p^T(a, h_i) + d_p^T(h_i, h_j) + d_p^T(h_j, b) \\ &\leq (1 + 2^{\frac{1}{p}})x_{h_i} - (y_{h_i} - h) + (x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j}) + (1 + 2^{\frac{1}{p}})(w - x_{h_j}) + (y_{h_j} - h) \\ &\leq (1 + 2^{\frac{1}{p}})w. \end{aligned}$$

If l_i is the inductive point of S_i , $x_{l_i} > 0$ by Lemma 7. By Lemma 9 there is a j , $i \leq j \leq k$, such that $l_i, l_{i+1}, l_{i+2}, \dots, l_j$ is a path in T of length, with respect to the L_p -metric, at most $(x_{l_j} - x_{l_i}) + (y_{l_j} - y_{l_i})$ and such that $w - x_{l_j} \geq h - y_{l_j} \geq 0$ and, by Lemma 7, $x_{l_j} \geq x_{l_i}$. Just as in the previous case, either $l_j = b$ or $w - x_{l_j} > 0$ and we can apply induction to bound $d_p^T(l_j, b)$ since $d_\infty(l_j, b) = w - x_{l_j} < w = d_\infty(a, b)$. Putting all this together with Lemma 8, we get:

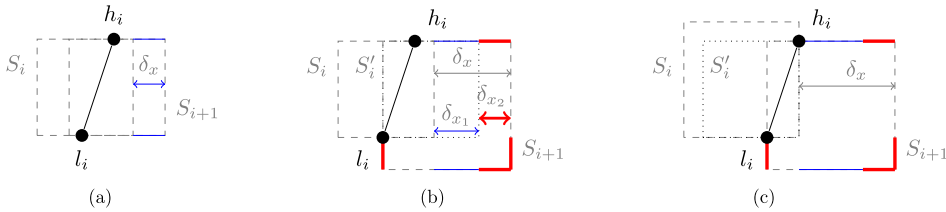


Fig. 7. The three cases in the proof of Lemma 10. In each case, the difference $P_{S_{i+1}}(h_i, l_i) - P_{S_i}(h_i, l_i)$ is shown to be at most $4\delta_x$, where $\delta_x = \delta_{i+1} - \delta_i$.

$$\begin{aligned}
 d_p^T(a, b) &\leq d_p^T(a, l_i) + d_p^T(l_i, l_j) + d_p^T(l_j, b) \\
 &\leq (1 + 2^{\frac{1}{p}})x_i + y_i + (x_j - x_i) + (y_j - y_i) + (1 + 2^{\frac{1}{p}})(w - x_j) + (h - y_j) \\
 &\leq (1 + 2^{\frac{1}{p}})w + h. \quad \square
 \end{aligned}$$

4.3. Squares with potential and the proof of the Crossing Lemma

What remains to be done is to prove Lemma 8. To do this, we need to develop some further terminology and tools. Let δ_i , for $1 \leq i \leq k$, be the horizontal distance between point a and the E side of square S_i , respectively. We also set $\delta_0 = 0$.

Definition 2. A square S_i has a potential if

$$d_p^T(a, h_i) + d_p^T(a, l_i) + P_{S_i}(h_i, l_i) \leq 4\delta_i$$

where $P_{S_i}(h_i, l_i)$ is the length of the path when moving from h_i to l_i along the sides of S_i , clockwise. Note this distance is the same for all L_p -metrics ($p \in [1, \infty]$).

Lemma 10. If (a, b) is not an edge of T and if rectangle $R(a, b)$ contains no point of P other than a and b then S_1 has a potential. Furthermore, for any $1 \leq i < k$, if S_i has a potential but is not inductive then S_{i+1} has a potential.

Proof. By Lemma 7, a lies inside the W side of S_1 and δ_1 is the side length of square S_1 . Then $d_p^T(a, h_1) + d_p^T(a, l_1) + P_{S_1}(h_1, l_1)$ is bounded by the perimeter of square S_1 which is $4\delta_1$.

Now assume that S_i , for $1 \leq i < k$, has a potential but is not inductive. Squares S_i and S_{i+1} both contain points l_i and h_i . Because S_i is not inductive, edge (l_i, h_i) must be steep and thus $\Delta_x(l_i, h_i) < \Delta_y(l_i, h_i)$. To simplify the argument that follows, we assume that l_i is “to the left” of h_i , i.e., $x_{l_i} < x_{h_i}$. The case when $x_{l_i} > x_{h_i}$ can be shown using an equivalent argument.

By Lemma 7, $T_i = \Delta(h_{i-1}, h_i, l_{i-1} = l_i)$ or $T_i = \Delta(h_{i-1} = h_i, l_{i-1}, l_i)$ and there has to be a side of S_i between the sides on which l_i and h_i lie, when moving clockwise from l_i to h_i . Using the constraints on the position of h_i and l_i within S_i from Lemma 7 and using the assumptions that (l_i, h_i) is steep and that $x_{l_i} < x_{h_i}$, we deduce that l_i must be on the S side and h_i must be on the N or E side of S_i .

If h_i is on the N side of S_i then, because $x_{l_i} < x_{h_i}$, h_i must also be on the N side of S_{i+1} . There are then two possibilities for the position of l_i within square S_{i+1} . One is that l_i is on the S side of S_{i+1} and S_{i+1} is obtained from S_i by a horizontal translation of length $\delta_x = \delta_{i+1} - \delta_i$ as shown in Fig. 7(a). Then

$$P_{S_{i+1}}(h_i, l_i) - P_{S_i}(h_i, l_i) = 2\delta_x = 2(\delta_{i+1} - \delta_i).$$

The other possibility is that l_i is on the W side of S_{i+1} . Let S'_i be the square with l_i as its SW corner and obtained by a horizontal translation of S_i , let δ_{x_1} be the length of this translation, and let δ_{x_2} be the difference between the side length of S'_i and S_{i+1} , as illustrated in Fig. 7(b). Then $\delta_{x_1} + \delta_{x_2} = \delta_{i+1} - \delta_i$ and

$$\begin{aligned}
 P_{S_{i+1}}(h_i, l_i) - P_{S_i}(h_i, l_i) &= 2\delta_{x_1} + 4\delta_{x_2} \\
 &\leq 4(\delta_{i+1} - \delta_i). \tag{1}
 \end{aligned}$$

If h_i is on the E side of S_i then, let S'_i be the square that shares its SE corner with S_i and with h_i in its NE corner, as shown in Fig. 7(c). Since (l_i, h_i) is steep, l_i lies on the S side of S'_i . Using the analysis from the previous case, we obtain $P_{S_{i+1}}(h_i, l_i) - P_{S'_i}(h_i, l_i) \leq 4(\delta_{i+1} - \delta_i)$ and since $P_{S'_i}(h_i, l_i) = P_{S_i}(h_i, l_i)$ we deduce that inequality (1) holds in all cases.

Since S_i has a potential, we obtain:

$$d_p^T(a, h_i) + d_p^T(a, l_i) + P_{S_{i+1}}(h_i, l_i) \leq 4\delta_{i+1}.$$

Assume $T_{i+1} = \Delta(h_i, h_{i+1}, l_i = l_{i+1})$; in other words, (h_i, h_{i+1}) is an edge of T with h_{i+1} lying somewhere on the boundary of S_{i+1} between h_i and l_i , when moving clockwise from h_i to l_i . By the triangular inequality, $d_p(h_i, h_{i+1}) \leq P_{S_{i+1}}(h_i, h_{i+1})$ and we have that:

$$\begin{aligned} d_p^T(a, h_{i+1}) + d_p^T(a, l_{i+1}) + P_{S_{i+1}}(h_{i+1}, l_{i+1}) &\leq d_p^T(a, h_i) + d_p(h_i, h_{i+1}) + d_p^T(a, l_i) + P_{S_{i+1}}(h_{i+1}, l_i) \\ &\leq d_p^T(a, h_i) + d_p^T(a, l_i) + P_{S_{i+1}}(h_i, l_i) \\ &\leq 4\delta_{i+1}. \end{aligned}$$

Thus S_{i+1} has a potential. The argument for the case when $T_{i+1} = \Delta(h_i = h_{i+1}, l_i, l_{i+1})$ is symmetric. \square

Lemma 11. *If square S_i has a potential and $c = h_i$ or $c = l_i$ is an eastern point in S_i then*

$$d_p^T(a, c) \leq 2x_c.$$

Proof. W.l.o.g., assume $c = h_i$. Since h_i is eastern, $\delta_i = x_c = x_{h_i}$. Because S_i has a potential, either $d_p^T(a, h_i) \leq 2x_{h_i}$ or $d_p^T(a, l_i) + P_{S_i}(h_i, l_i) \leq 2x_{h_i}$. In the second case, because edge (l_i, h_i) is in T , by the triangular inequality we obtain $d_p^T(a, h_i) \leq d_p^T(a, l_i) + d_p(l_i, h_i) \leq 2x_{h_i}$. \square

Definition 3. Let $1 \leq j \leq k$. The *maximal high path ending at h_j* and the *maximal low path ending at l_j* are defined as follows:

- If h_j is eastern in S_j , the *maximal high path ending at h_j* is simply h_j ; otherwise, it is the path h_i, h_{i+1}, \dots, h_j such that h_{i+1}, \dots, h_j are not eastern in, respectively, S_{i+1}, \dots, S_j and either $i = 0$ or h_i is eastern in S_i .
- If l_j is eastern in S_j , the *maximal low path ending at l_j* is simply l_j ; otherwise, it is the path l_i, l_{i+1}, \dots, l_j such that l_{i+1}, \dots, l_j are not eastern in, respectively, S_{i+1}, \dots, S_j and either $i = 0$ or l_i is eastern in S_i .

Note that by Lemma 7 all edges on maximal high path h_i, h_{i+1}, \dots, h_j are WN edges and thus the path length is bounded by $(x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$. Similarly, all edges in maximal low path l_i, l_{i+1}, \dots, l_j are WS edges and the length of the path is at most $(x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$.

We now have the tools to prove Lemma 8.

Proof of Lemma 8. If rectangle $R(a, b)$ is empty then, by Lemma 7, b is eastern in S_k . Thus, by Lemma 10 and Lemma 11, if no square S_1, \dots, S_{k-1} is inductive then $d_p^T(a, b) \leq 2w \leq (1 + 2^{\frac{1}{p}})w + h$.

Assume now that there is at least one inductive square in the sequence of squares S_1, \dots, S_{k-1} . Let S_j be the first inductive square and assume, for now, that h_j is the inductive point in S_j . By Lemma 10, every square S_i , for $i \leq j$, has a potential. Since (l_j, h_j) is gentle, it follows that $d_p(l_j, h_j) \leq 2^{\frac{1}{p}}(x_{h_j} - x_{l_j})$. Let $l_i, l_{i+1}, \dots, l_{j-1} = l_j$ be the maximal low path ending at l_j . Note that $d_p^T(l_i, l_j) \leq (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$. Either $l_i = l_0 = a$ or l_i is an eastern point in square S_i that has a potential and Lemma 11 applies; either way, we have that $d_p^T(a, l_i) \leq 2x_{l_i}$. Putting all this together, we get

$$\begin{aligned} d_p^T(a, h_j) + (y_{h_j} - h) &\leq d_p^T(a, l_i) + d_p^T(l_i, l_j) + d_p(l_j, h_j) + y_{h_j} - h \\ &\leq 2x_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j}) + 2^{\frac{1}{p}}(x_{h_j} - x_{l_j}) + y_{h_j} - h \\ &\leq 2^{\frac{1}{p}}x_{h_j} + x_{l_j} + (y_{h_j} - y_{l_j}) + (y_{l_i} - h) \\ &\leq (1 + 2^{\frac{1}{p}})x_{h_j} \end{aligned}$$

where the last inequality follows from $y_{l_i} - h \leq 0$ and from $x_{l_j} + (y_{h_j} - y_{l_j}) \leq x_{h_j}$ (i.e., from the assumption that edge (l_j, h_j) is gentle).

If, instead, $c = l_j$ is the inductive point in inductive square S_j , let $h_i, h_{i+1}, \dots, h_{j-1} = h_j$ be the maximal high path ending at h_j . Then $d_p^T(h_i, h_j) \leq (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$. Just as in the first case, we have that $d_p^T(a, h_i) \leq 2x_{h_i}$ and

$$\begin{aligned} d_p^T(a, l_j) - y_{l_j} &\leq d_p^T(a, h_i) + d_p^T(h_i, h_j) + d_p(h_j, l_j) - y_{l_j} \\ &\leq 2x_{h_i} + (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i}) + 2^{\frac{1}{p}}(x_{l_j} - x_{h_j}) - y_{l_j} \\ &\leq 2^{\frac{1}{p}}x_{l_j} + x_{h_j} + (y_{h_j} - y_{l_j}) \\ &\leq (1 + 2^{\frac{1}{p}})x_{l_j}, \end{aligned}$$

where the last inequality follows from $x_{h_j} + (y_{h_j} - y_{l_j}) \leq x_{l_j}$, i.e., from the assumption that (h_j, l_j) is gentle. \square

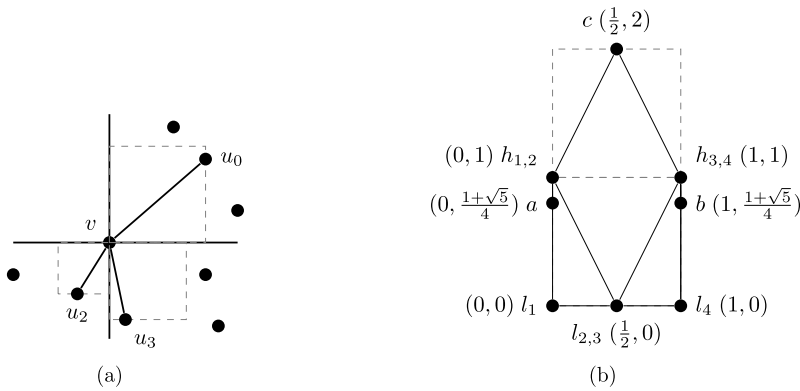


Fig. 8. (a) Construction of Yao_4^∞ . (b) A degenerate example of graph Yao_4^∞ with stretch factor greater than 2.618. This example can be modified into one that has all points in general position by moving the points slightly so that they appear in the order $l_{2,3}, l_1, l_4, a, b, h_{3,4}, h_{1,2}, c$ when ordered by y -coordinate and in the order $a, l_1, h_{1,2}, l_{2,3}, c, h_{3,4}, l_4, b$ when ordered by x -coordinate.

5. Conclusion and open problems

The proof of **Theorem 5** is constructive, and we summarize now the algorithm that is implicit in the proof. The algorithm constructs a path in the L_∞ -Delaunay triangulation T on P between a pair of points a and b .

If edge $(a, b) \in T$ the path is simply edge (a, b) . If rectangle $R(a, b)$ contains a point c of P ($c \neq a, b$), then we recursively construct a path from a to c and another from c to b and we return the concatenation of the two. If rectangle $R(a, b)$ contains no point of P (other than a and b), we consider the triangles T_1, T_2, \dots, T_k that segment $[ab]$ intersects and the associated squares S_1, S_2, \dots, S_k . If none of these squares is inductive, we return the shorter of path $a, h_1, h_2, \dots, h_{k-1}, b$ and path $a, l_1, l_2, \dots, l_{k-1}, b$. Otherwise, we find the smallest j such that S_j is inductive with inductive point c .

If $c = l_j$, we find the largest $i < j$ such that h_i is eastern in S_i ($i = 0$ if none of h_1, h_2, \dots, h_{j-1} is eastern in, respectively, S_1, \dots, S_{j-1}); we also find the smallest j' such that $j \leq j' \leq k$ and $0 \leq h - y_{l_{j'}} \leq w - x_{l_{j'}}$ (where $w = d_\infty(a, b) = \max\{\Delta_x(a, b), \Delta_y(a, b)\}$ and $h = \min\{\Delta_x(a, b), \Delta_y(a, b)\}$). The path returned is the concatenation of:

1. the shorter of path a, h_1, h_2, \dots, h_i and path $a, l_1, l_2, \dots, l_i, h_i$,
2. path $h_i, h_{i+1}, \dots, h_{j-1}, h_j, l_j, l_{j+1}, \dots, l_{j'-1}, l_{j'}$, and
3. the path from $l_{j'}$ to b , obtained recursively.

If the eastern point c in S_j is h_j , equivalent steps are taken.

We conclude this paper by bringing up several questions and open problems that are raised by the results we obtained. We return here to the usual assumption that the edge weight function is the L_2 -metric.

5.1. Delaunay triangulations based on polygon distance functions

The L_1 -Delaunay triangulation is the first type of Delaunay triangulation that was shown to be a spanner [5]. Progress on the spanning properties of the TD-Delaunay and the L_2 -Delaunay triangulations soon followed (as discussed in Section 1). In this paper, we determine the precise stretch factor of an L_1 -Delaunay triangulation and close the problem for good. We believe that our proof techniques can be extended and that they will lead, yet again, to new insights on the stretch factor of other types of Delaunay triangulations. For example, let P_k denote the convex distance function defined by a regular k -gon. We observe that the exact stretch factor of P_k -Delaunay triangulations is known for $k = 3, 4$ since P_3 is the triangular distance function of [6], and P_4 is nothing else than the (L_1 - or) L_∞ -metric. Determining the stretch factor of P_k -Delaunay triangulations for larger k would undoubtedly be an important step towards understanding the stretch factor of Delaunay triangulations.

5.2. The stretch factor of Yao_4^∞

Bose et al. [13] have recently defined and studied the stretch factor of Yao_4^∞ , a subgraph of the L_∞ -Delaunay triangulation on a set of points P in the plane. To describe this subgraph, we define a cone to be the region in the plane between two rays that emanate from the same point. With every point u of P we associate four disjoint 90° cones emanating from u : they are defined by the translation of the x - and y -axis from the origin to point u . The graph Yao_4^∞ is locally constructed by connecting each point of P to its closest neighbor – according to the L_∞ -metric – in each cone. Fig. 8(a) illustrates how the neighbors in Yao_4^∞ of a point v are chosen.

Because there is an empty square circumscribing the endpoints of every edge of Yao_4^∞ , it is a (typically proper) subgraph of the L_∞ -Delaunay triangulation on the set of points P . The lower bound of $\sqrt{4 + 2\sqrt{2}}$ on the stretch factor of the

L_∞ -Delaunay triangulation is thus also a lower bound on the stretch factor of Yao_4^∞ . However, this bound is not tight for Yao_4^∞ . We show in Fig. 8(b) the graph Yao_4^∞ on a set of points (*not* in general position for simplicity's sake) that illustrates this. In this graph, $d_2(a, b) = 1$ while

$$d_2^{\text{Yao}_4^\infty}(a, b) = \frac{3 + \sqrt{5}}{2} \approx 2.618\dots$$

which implies that the stretch factor of Yao_4^∞ on a set of points in the plane can be at least $\frac{3+\sqrt{5}}{2} > \sqrt{4 + 2\sqrt{2}}$.

Bose et al. [13] showed that the stretch factor of Yao_4^∞ is bounded from above by 8 with respect to the L_∞ -metric and $8\sqrt{2}(29 + 23\sqrt{2})$ with respect to the L_2 -metric. We leave the question of determining the exact stretch factor of Yao_4^∞ as an open problem.

5.3. Locality and routing

From a routing perspective, it is of interest to construct routes in geometric graphs that can be determined *locally* from a neighbor's coordinates only [14]. Unfortunately, the route that is constructed by our algorithm is built using *non-local* decisions. It would be interesting to know whether in the L_1 -Delaunay triangulation a route with stretch $\sqrt{4 + 2\sqrt{2}}$ can be constructed using a local routing algorithm. For TD-Delaunay triangulations, [15] showed that there is no local routing algorithm that achieves a stretch that is less than $5/\sqrt{3} \approx 2.88$, whereas the stretch factor is actually 2. They also provide a distributed routing algorithm following a path of maximal stretch $5/\sqrt{3}$. We leave open the question regarding the gap between the stretch factor of L_1 -Delaunay triangulations and the stretch that is achievable using local routing.

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References

- [1] A. Okabe, B. Boots, K. Sugihara, S.N. Chiu, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, 2nd edition, Wiley, 2000.
- [2] Ø. Hjelle, M. Dæhlen, *Triangulations and Applications*, Springer, 2006.
- [3] K.M. Alzoubi, X.-Y. Li, Y. Wang, P.-J. Wan, O. Frieder, Geometric spanners for wireless ad hoc networks, *IEEE Trans. Parallel Distrib. Syst.* 14 (4) (2003) 408–421.
- [4] P. Bose, P. Morin, Competitive online routing in geometric graphs, *Theor. Comput. Sci.* 324 (2–3) (2004) 273–288.
- [5] L.P. Chew, There is a planar graph almost as good as the complete graph, in: *Proceedings of the 2nd Annual ACM Symposium on Computational Geometry*, SoCG, 1986, pp. 169–177.
- [6] L.P. Chew, There are planar graphs almost as good as the complete graph, *J. Comput. Syst. Sci.* 39 (2) (1989) 205–219.
- [7] D.P. Dobkin, S.J. Friedman, K.J. Supowit, Delaunay graphs are almost as good as complete graphs, *Discrete Comput. Geom.* 5 (4) (1990) 399–407.
- [8] J.M. Keil, C.A. Gutwin, Classes of graphs which approximate the complete Euclidean graph, *Discrete Comput. Geom.* 7 (1) (1992) 13–28.
- [9] G. Xia, The stretch factor of the Delaunay triangulation is less than 1.998, *SIAM J. Comput.* 42 (4) (2013) 1620–1659.
- [10] P. Bose, L. Devroye, M. Löffler, J. Snoeyink, V. Verma, Almost all Delaunay triangulations have stretch factor greater than $\pi/2$, *Comput. Geom.* 44 (2) (2011) 121–127.
- [11] G. Xia, L. Zhang, Toward the tight bound of the stretch factor of Delaunay triangulations, in: *Proceedings of the 23rd Annual Canadian Conference on Computational Geometry*, CCCG, 2011.
- [12] P. Bose, P. Carmi, S. Collette, M. Smid, On the stretch factor of convex Delaunay graphs, *J. Comput. Geom.* 1 (1) (2010) 41–56.
- [13] P. Bose, M. Damian, K. Douïeb, J. O'Rourke, B. Seamone, M.H.M. Smid, S. Wührer, $\pi/2$ -angle Yao graphs are spanners, *Int. J. Comput. Geom. Appl.* 22 (1) (2012) 61–82.
- [14] P. Bose, P. Carmi, S. Durocher, Bounding the locality of distributed routing algorithms, *Distrib. Comput.* 26 (1) (2013) 39–58.
- [15] P. Bose, R. Fagerberg, A. van Renssen, S. Verdonschot, Competitive routing in the half- θ_6 -graph, in: *Proceedings of the 23rd Symposium on Discrete Algorithms*, SODA, ACM-SIAM, 2012, pp. 1319–1328.