

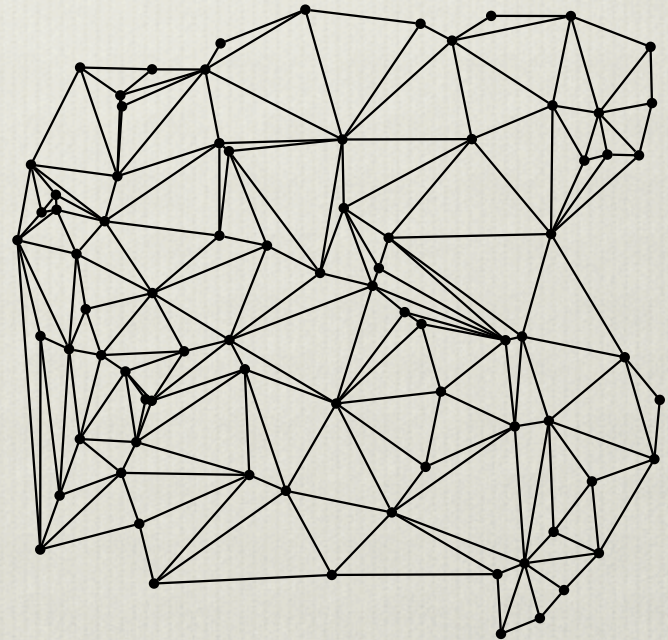
The Stretch Factor of L_1 - and L_∞ -Delaunay Triangulations

Nicolas Bonichon, Bordeaux U. - LaBRI

Cyril Gavoille, Bordeaux U. - LaBRI

Nicolas Hanusse, CNRS - LaBRI

Ljubomir Perković, DePaul U. - Chicago



Geometric spanner

P = set of points in the plane

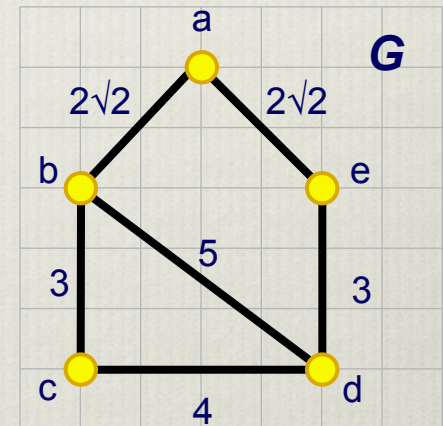
K = complete Euclidean graph on P

[weight(u,v)= $\|u-v\|_2$ for all u,v of P]

G = spanner of K (spanning subgraph)

G has **stretch** factor **s** if

$d_G(u,v) \leq \mathbf{s} \cdot d_K(u,v)$ for all u,v of P



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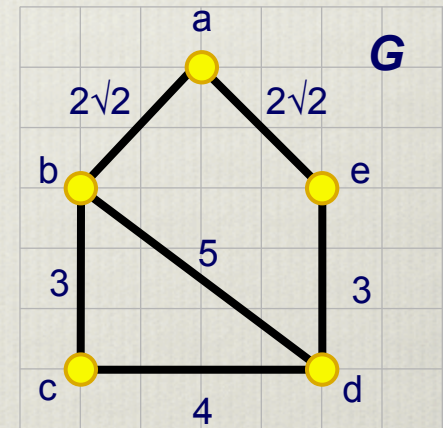
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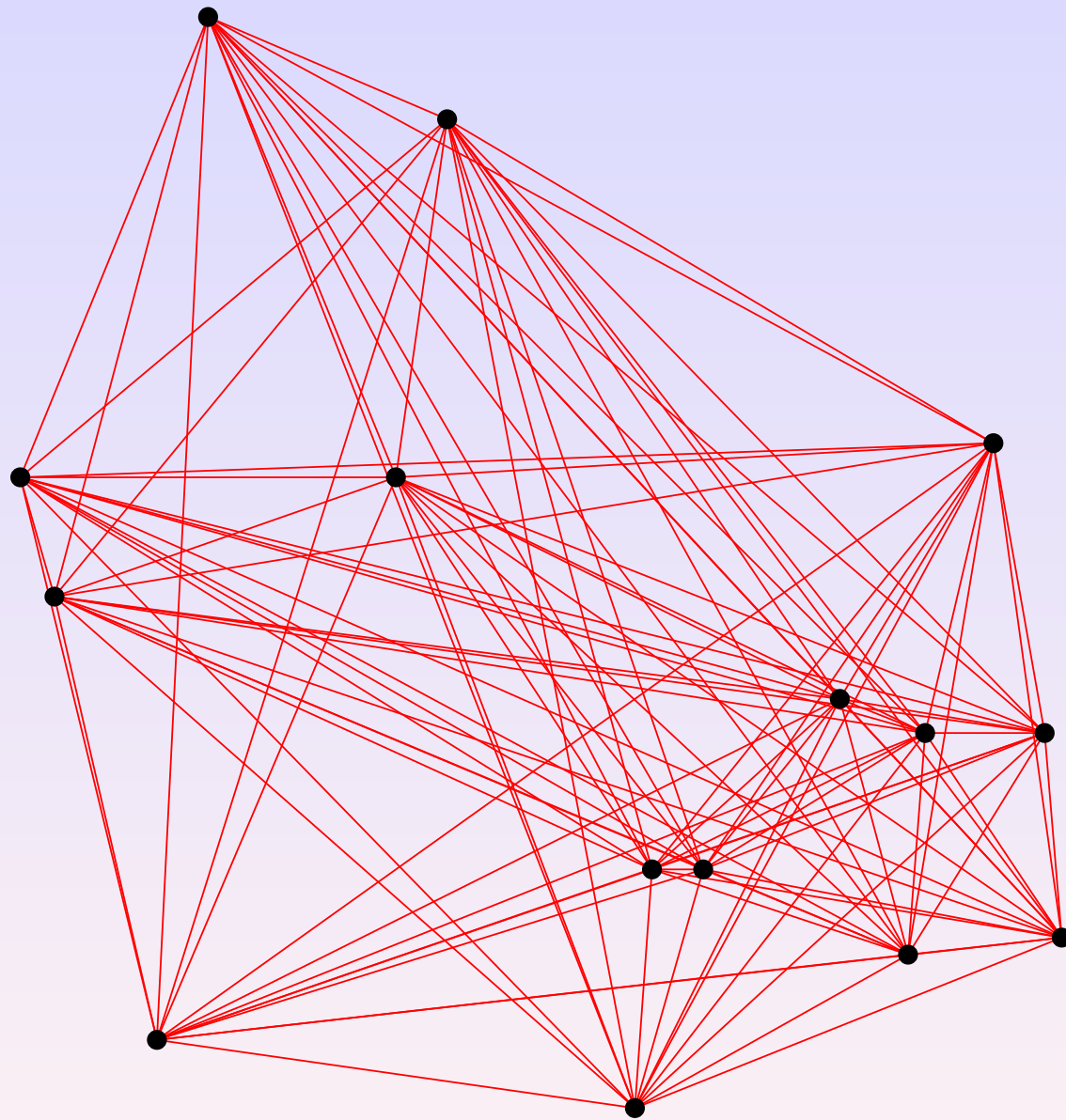
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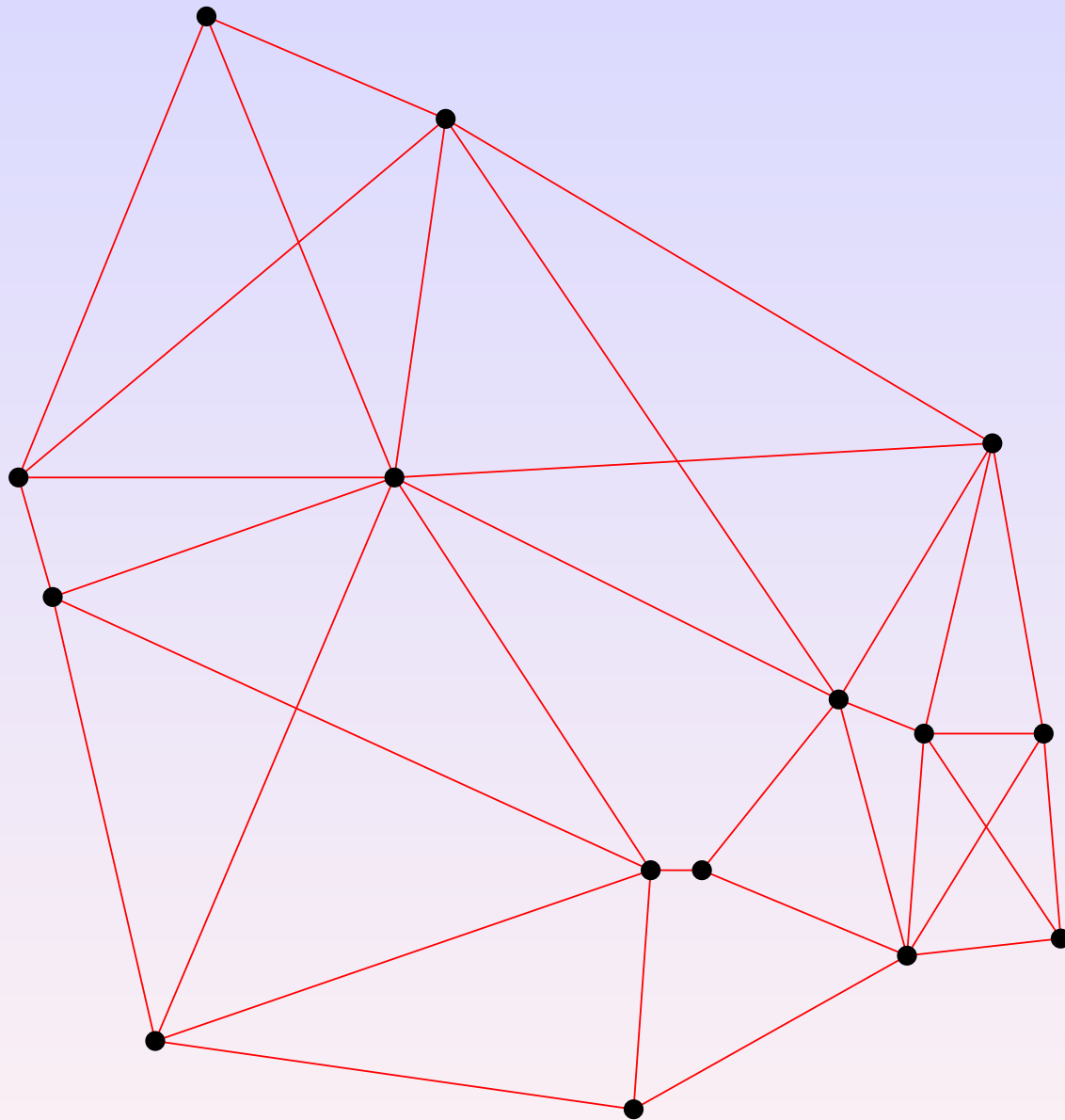
$$d_G(c,e) = 7, d_K(c,e) = 5$$

$$d_G(b,e) = 4\sqrt{2}, d_K(b,e) = 4$$

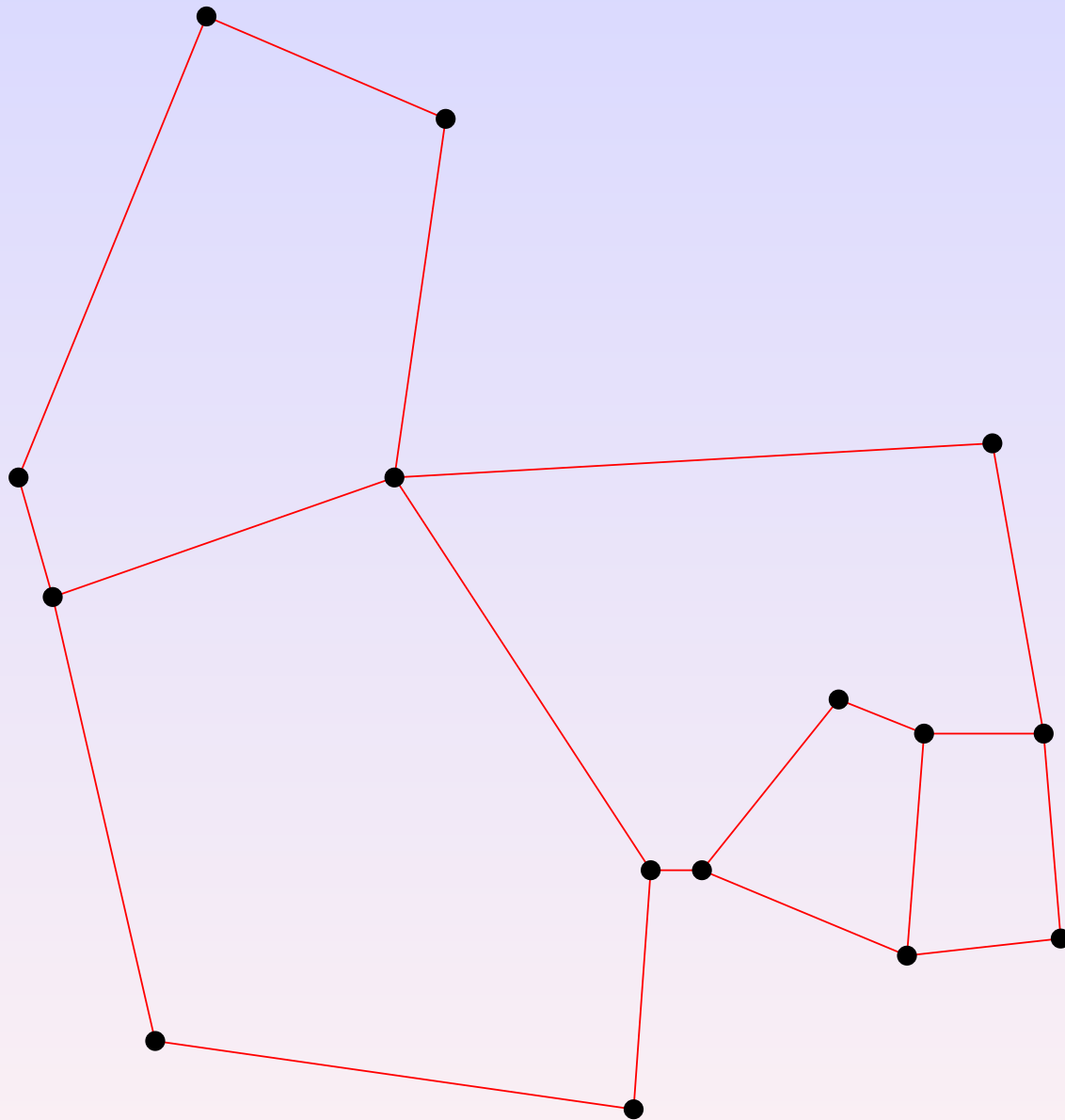
stretch factor of G is $\sqrt{2}$



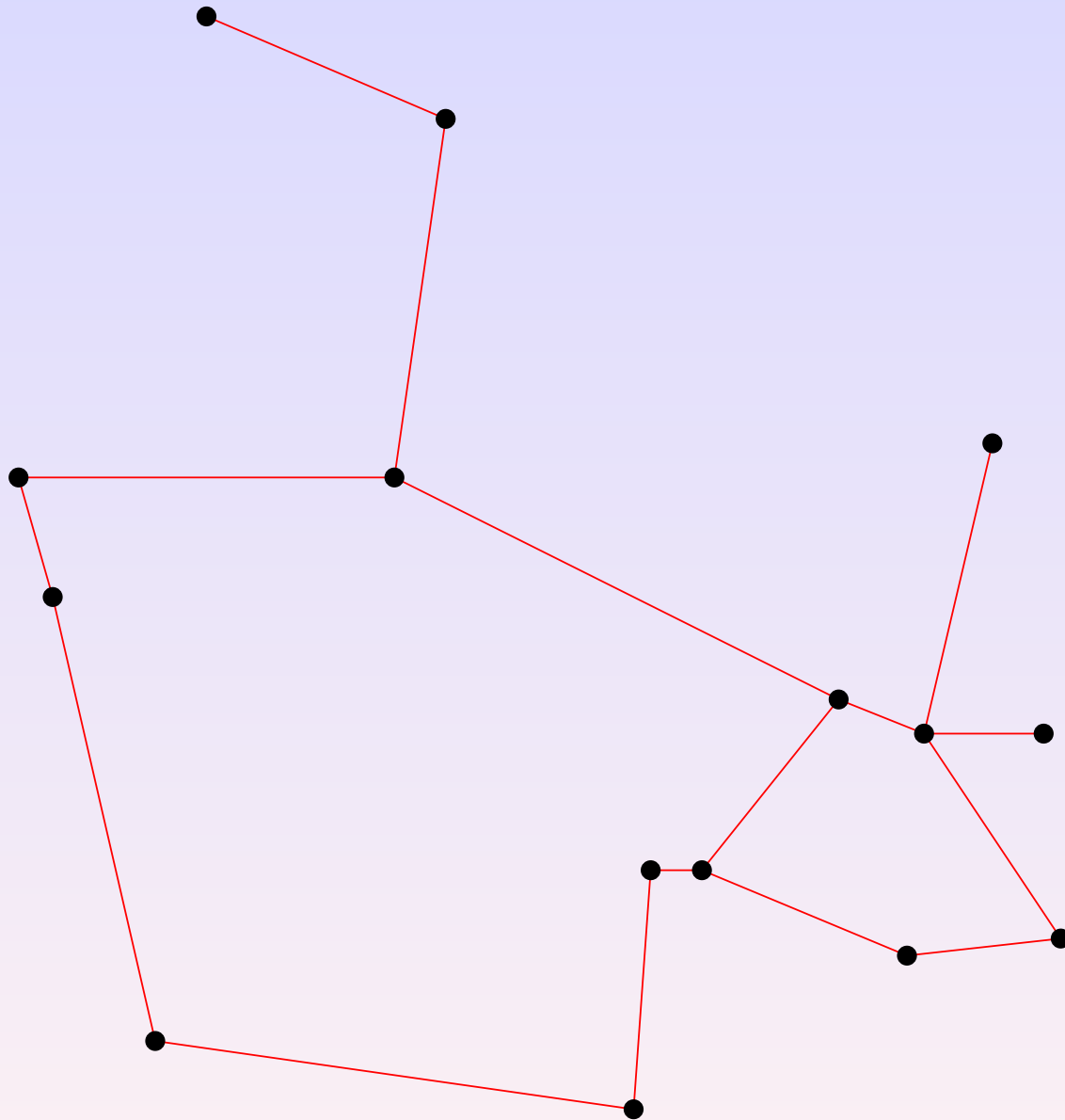
A complete Euclidian graph on 15 nodes



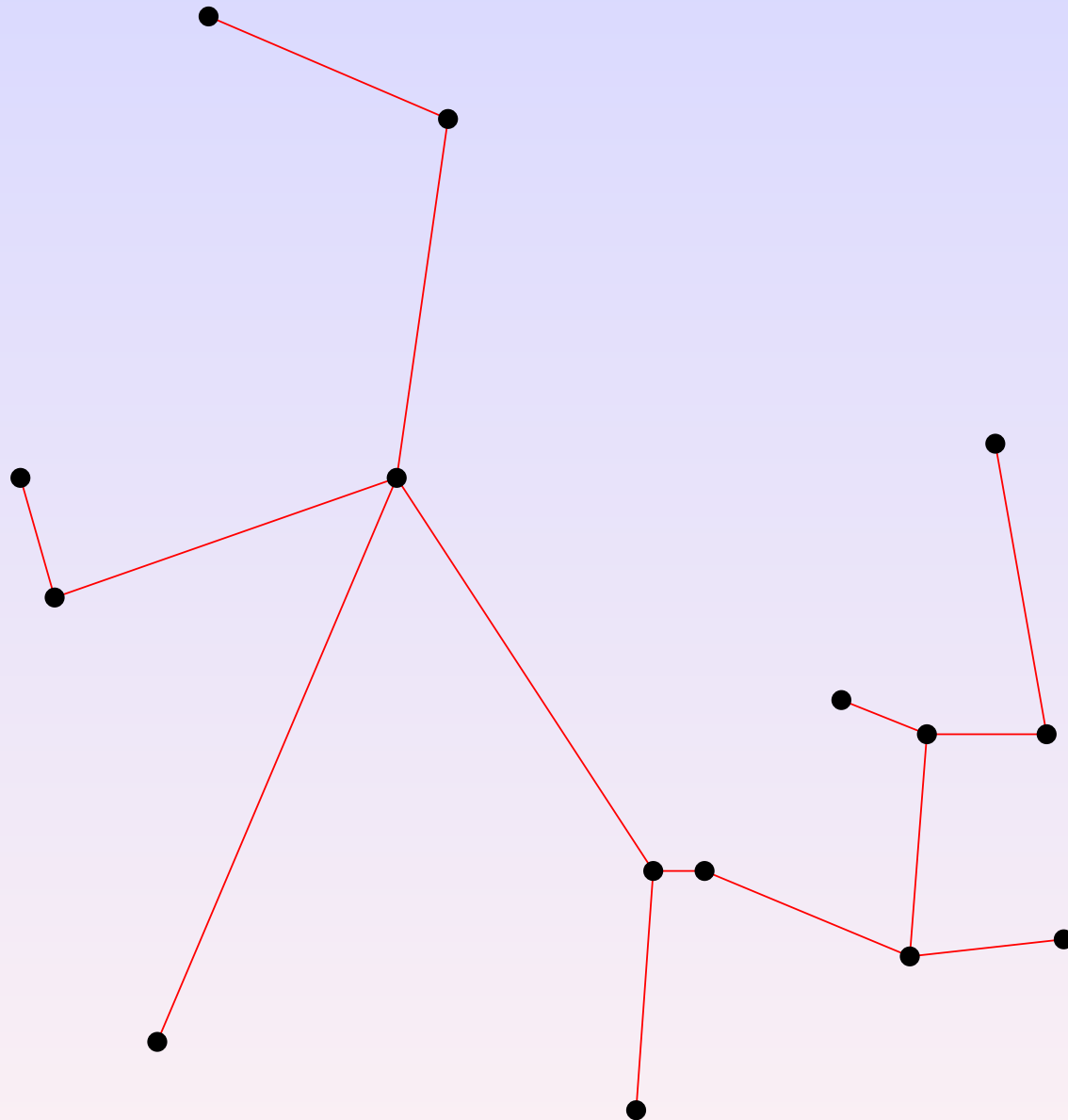
A (minimum cost) spanner with stretch 1.2



A (minimum cost) spanner with stretch 1.7



A (minimum cost) spanner with stretch 2.0



A (minimum cost) spanner with stretch 3.0

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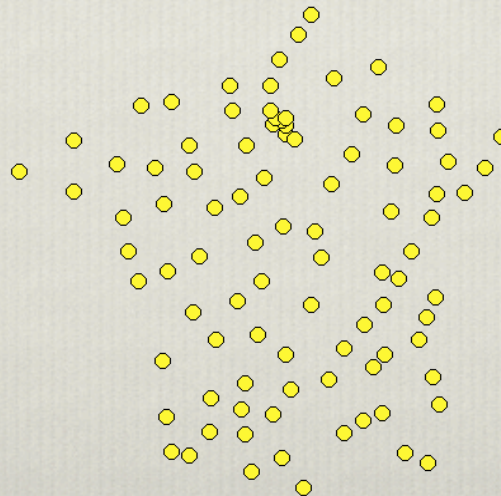
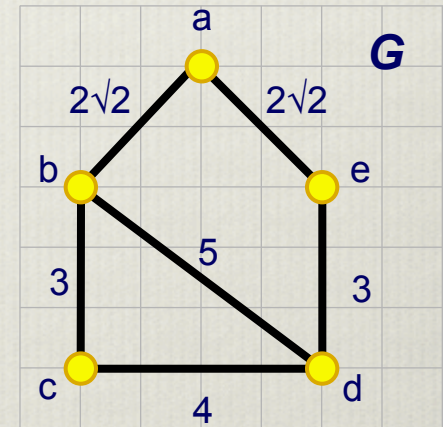
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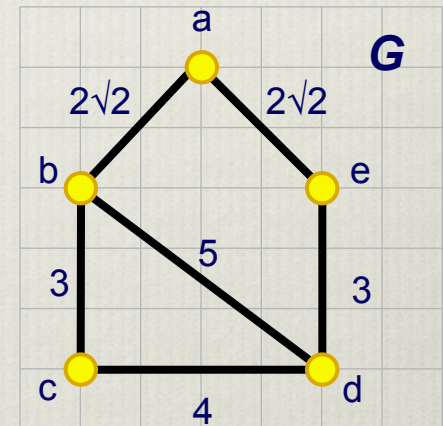
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Good spanners

- **Small stretch**
- Few edges
- **Planar**
- Small max degree
- Routable
- Computable [locally]
- ...



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stretch factor of G is $\sqrt{2}$

Delaunay Triangulation

[*Boris Delaunay '34*]

Voronoi cell

$C(p_i) = \{ u : d_K(p_i, u) \leq d_K(p_j, u) \text{ for all } s_j \neq s_i \}$

Delaunay triangulation of $P=\{p_1, p_2, \dots\}$

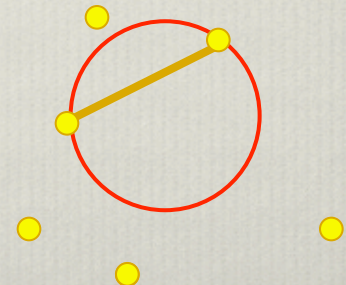
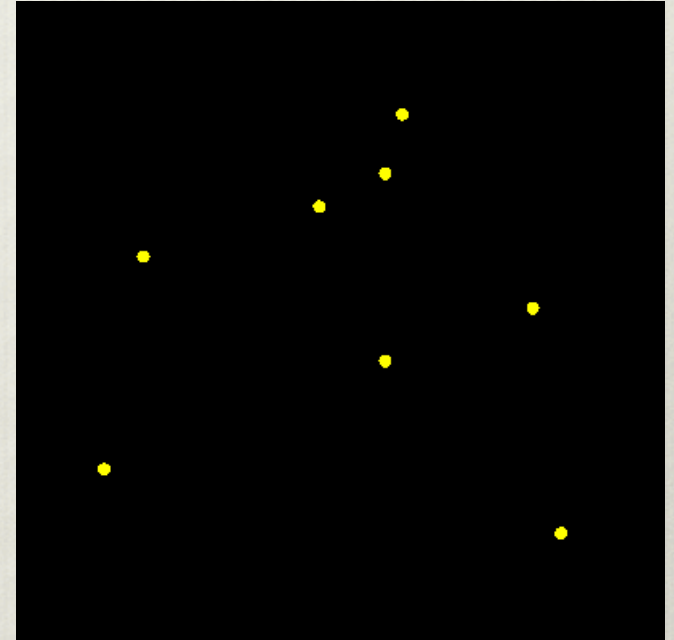
(p_i, p_j) is an edge iff $C(p_i) \cap C(p_j) \neq \emptyset$

General position

no 3 co-linear points, no 4 co-circular points

Alt. definition

(p_i, p_j) is an edge iff there is an empty circle supporting (p_i, p_j)



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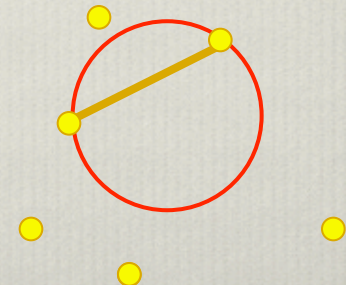
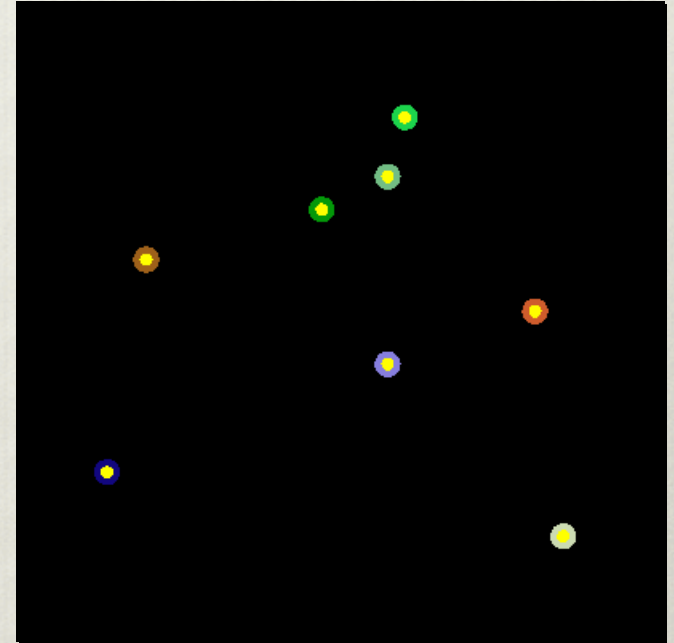
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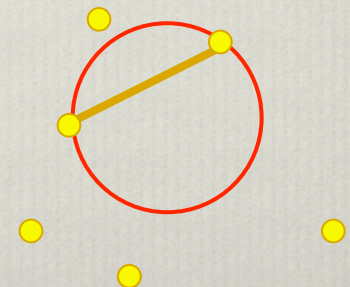
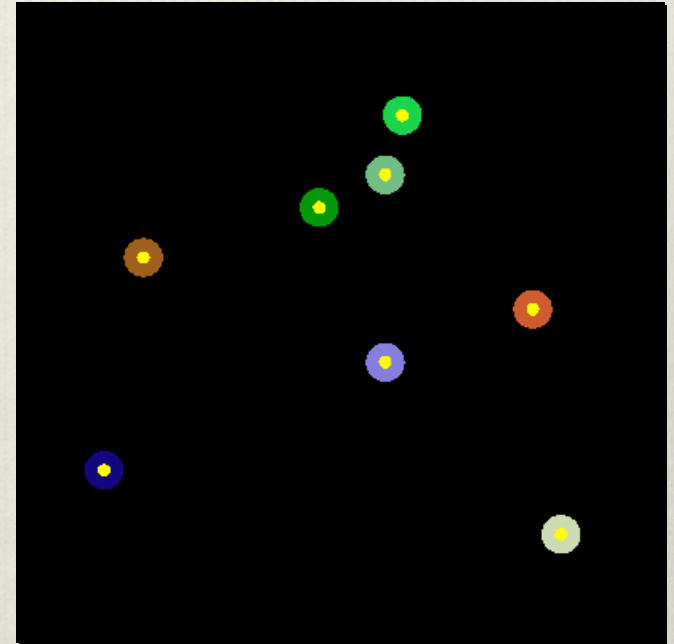
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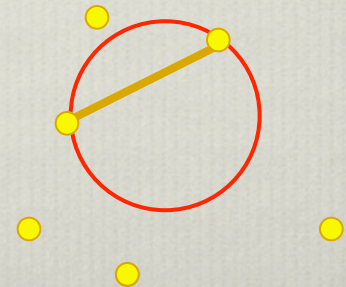
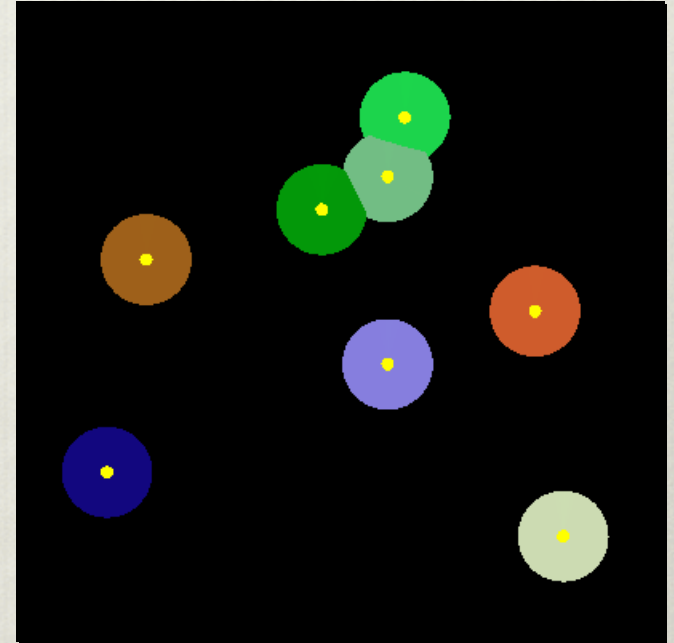
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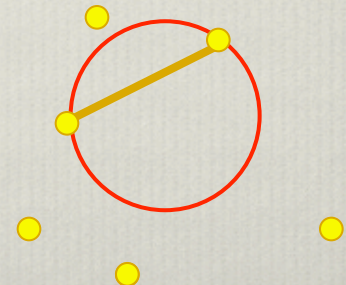
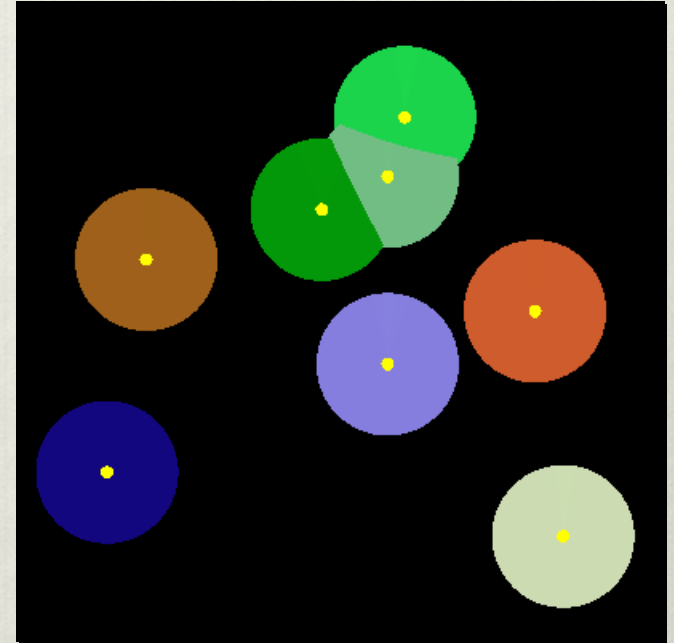
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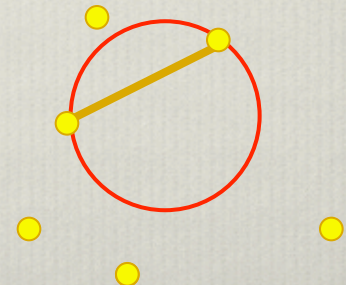
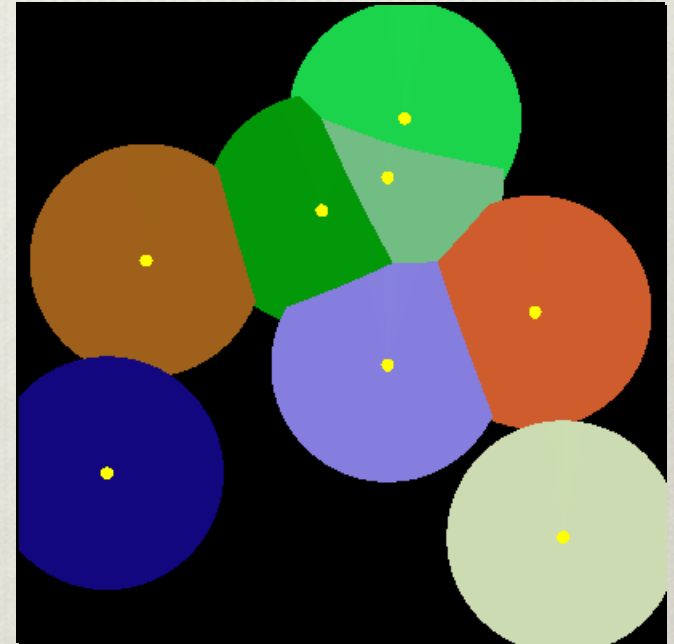
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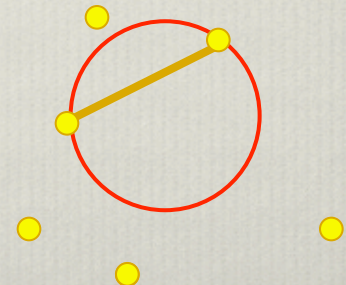
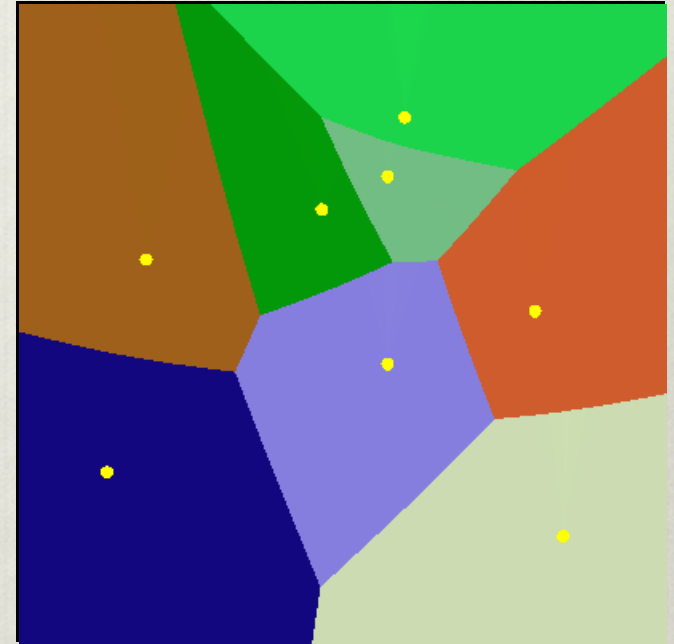
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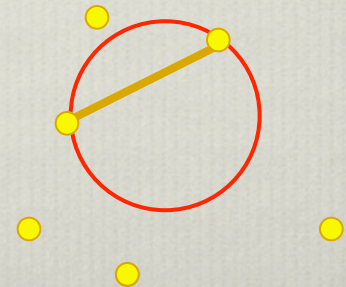
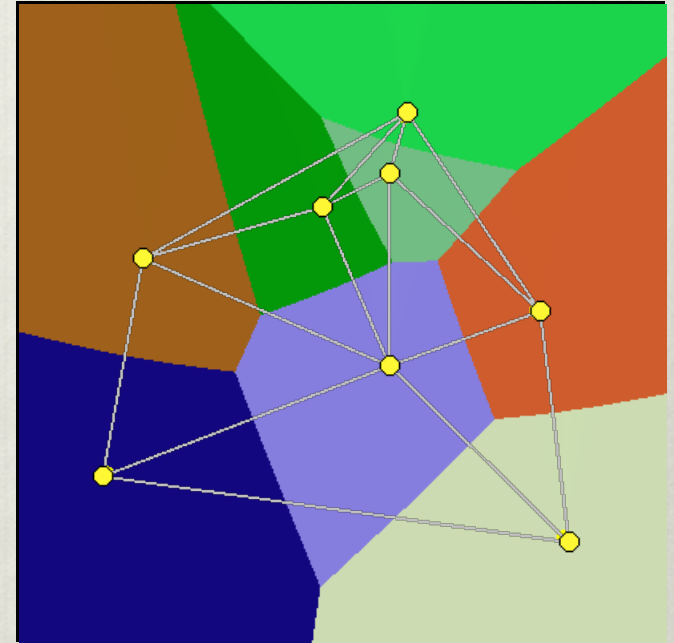
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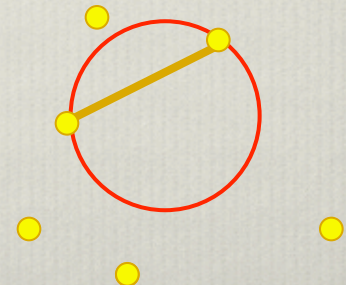
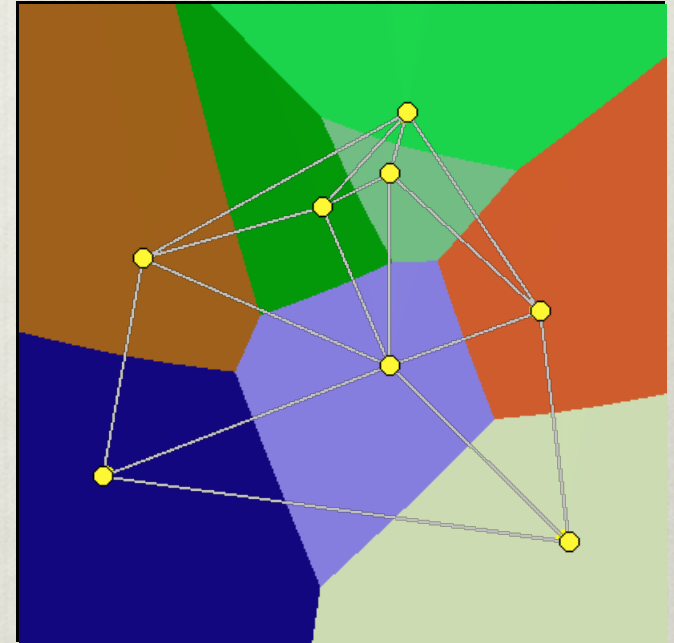
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Alt. definition

(p_i, p_j) is an edge iff there is an empty circle supporting (p_i, p_j)

→ each face is supported by an empty circle



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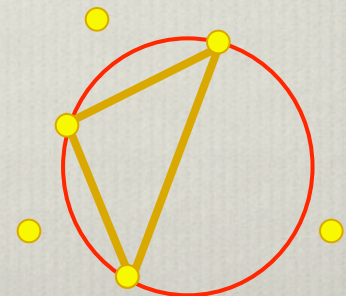
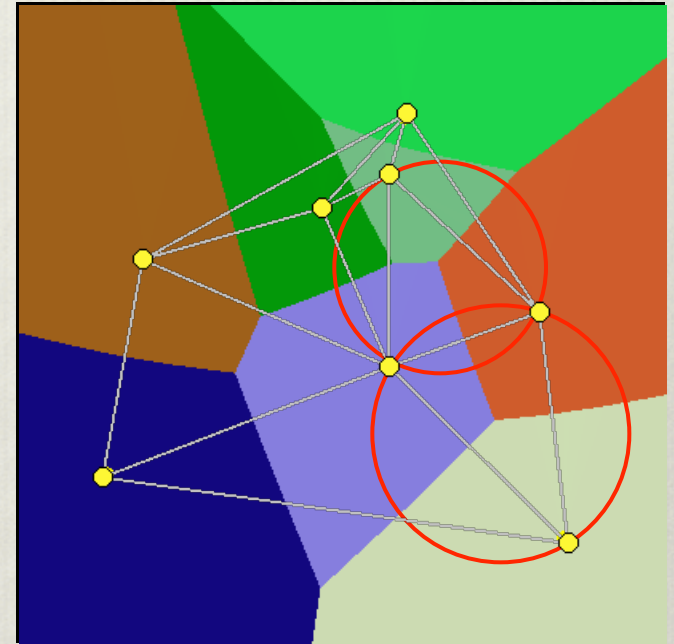
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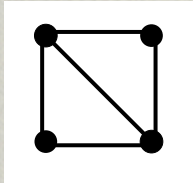
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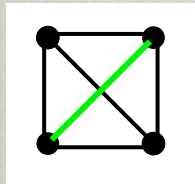
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The Stretch of Delaunay Triangulation

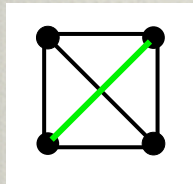


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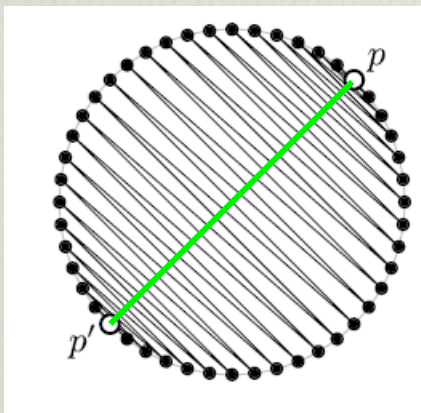


1.414

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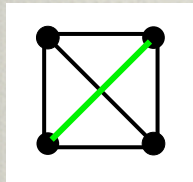


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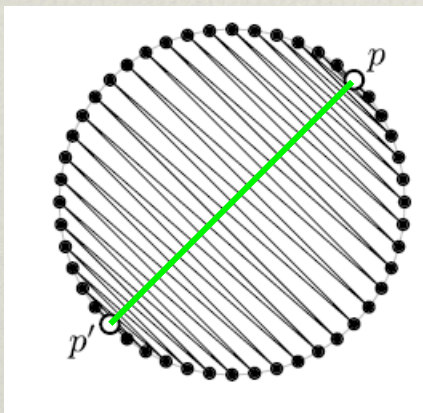


1.570

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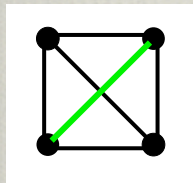
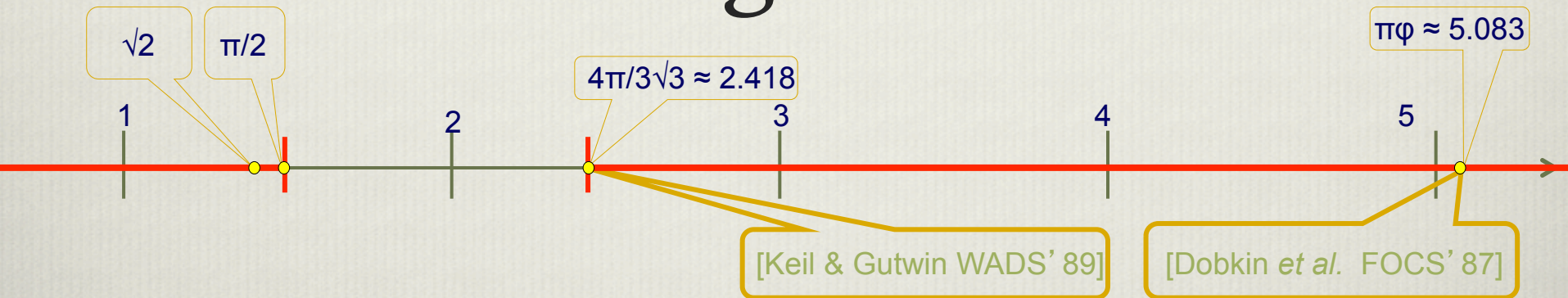


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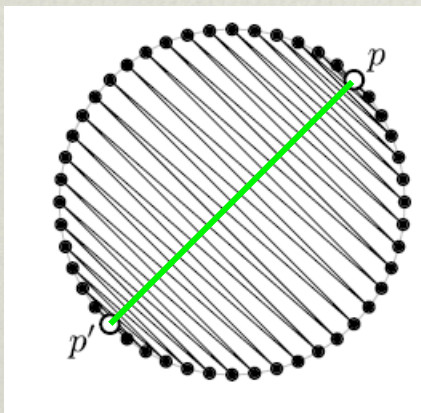


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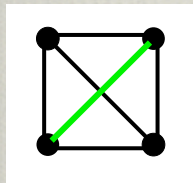
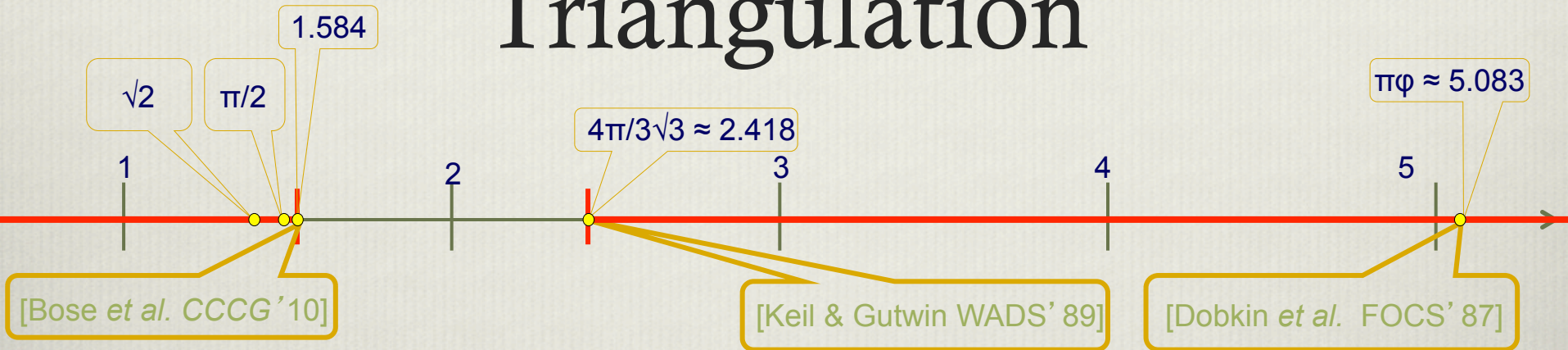


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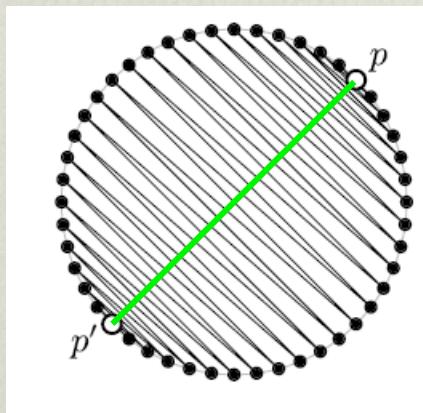


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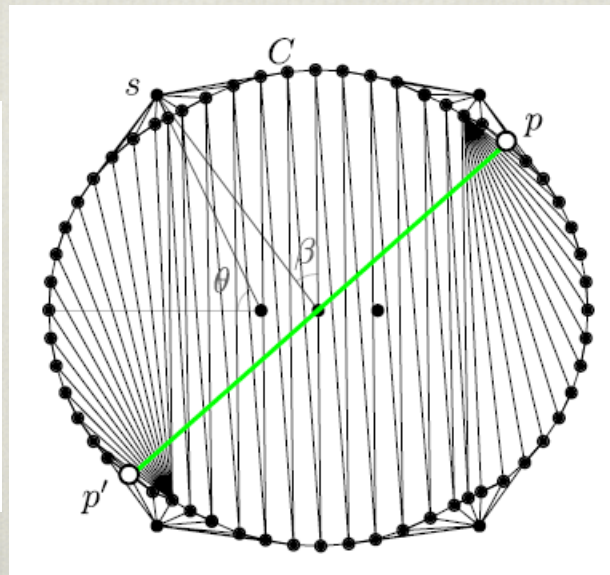
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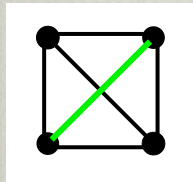
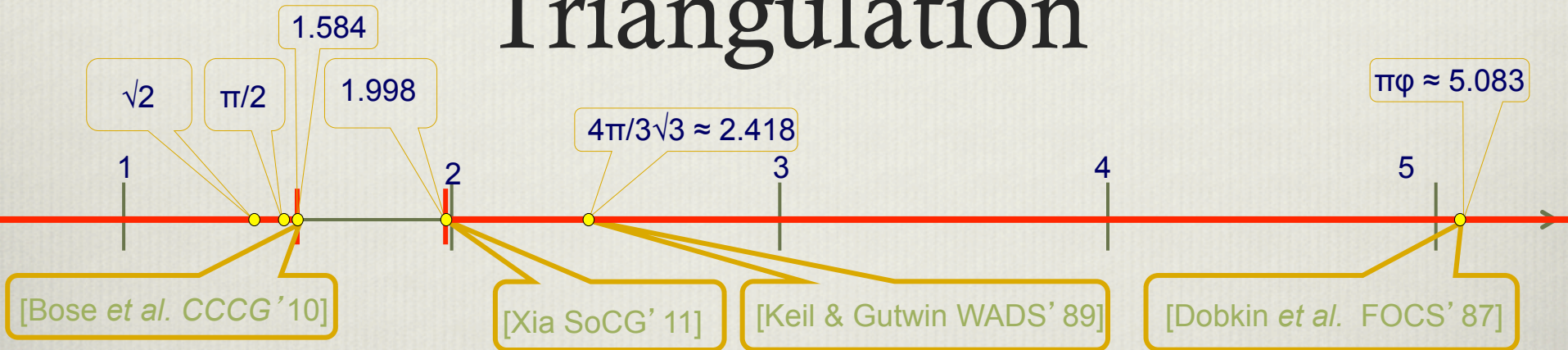


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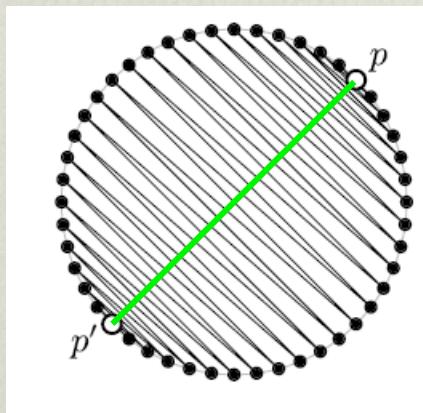


1.584

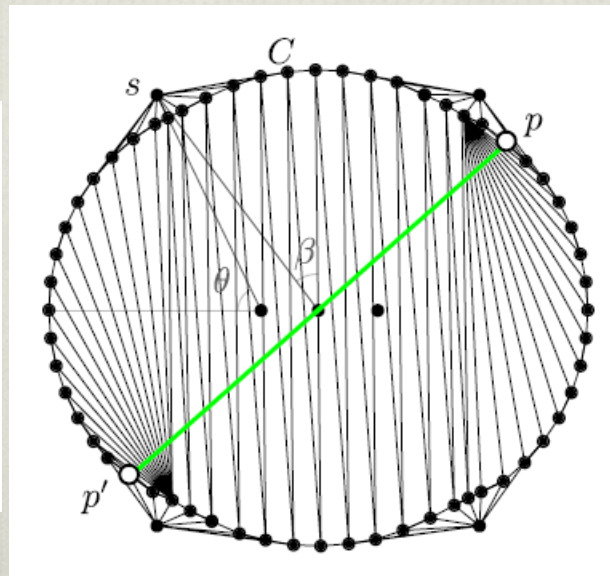
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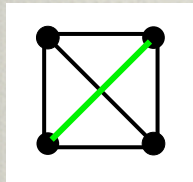
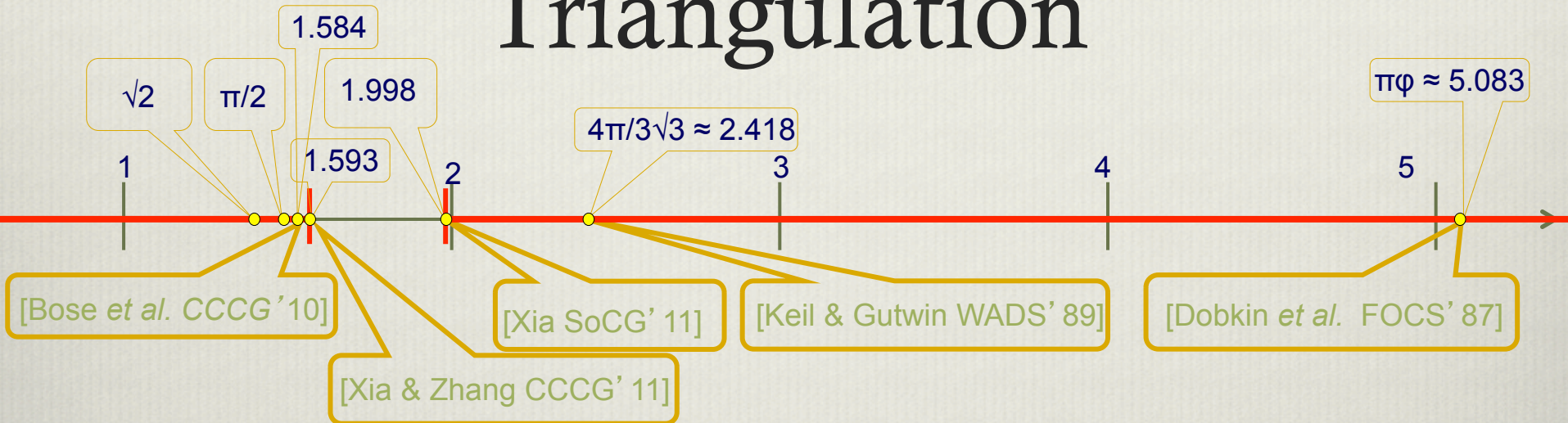


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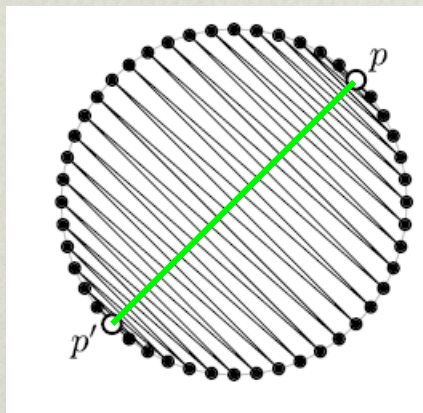


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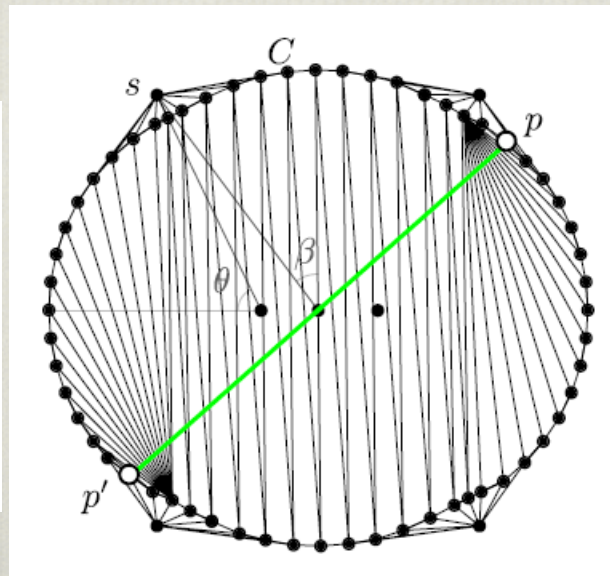
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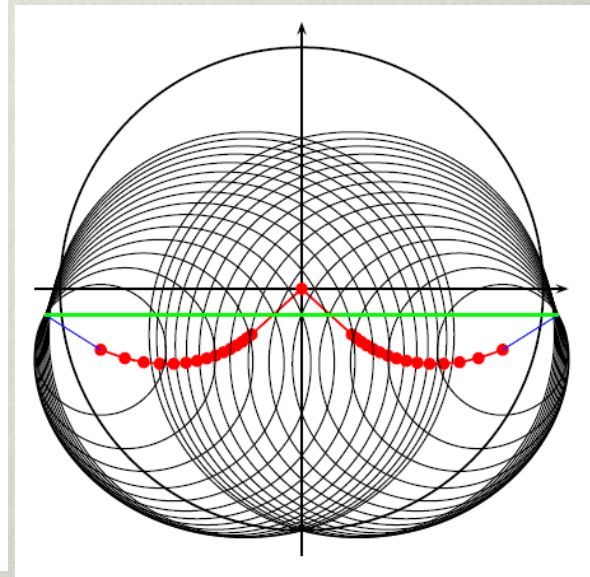
1.414



1.570



1.584



1.593



L_1 -Delaunay Triangulation

[Chew - SoCG'86]

$$d_{L_1}(u,v) = |x_u - x_v| + |y_u - y_v|$$

L_1 -Voronoi cell

$$C(p_i) = \{ u : d_{L_1}(p_i, u) \leq d_{L_1}(p_j, u) \text{ for all } p_j \neq p_i \}$$

L_1 -Delaunay triangulation of $P = \{p_1, p_2, \dots\}$

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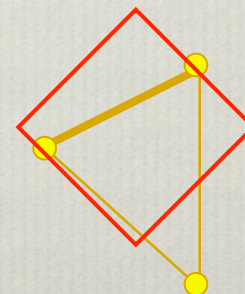
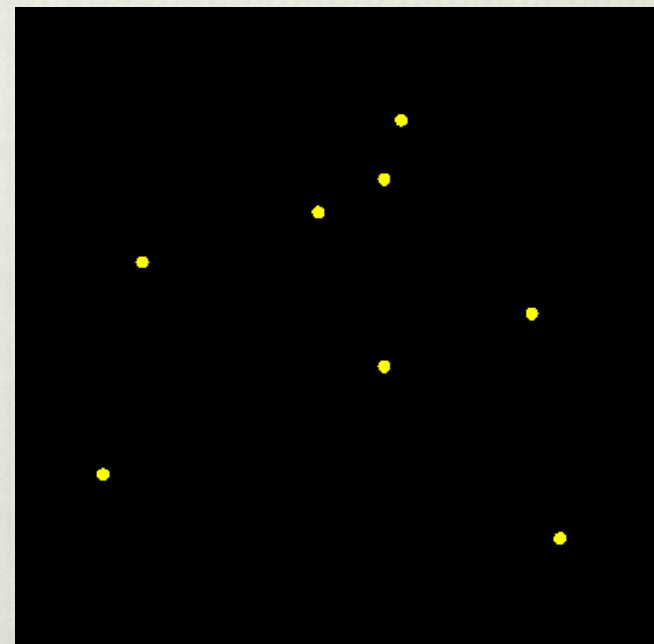
General position

no 2 points with same x- or y-coordinates

no 4 points on a square

Alt. definition

(p_i, p_j) is an edge iff there is an empty **square** supporting (p_i, p_j)





L_1 -Delaunay Triangulation

[Chew - SoCG'86]

$$d_{L_1}(u,v) = |x_u - x_v| + |y_u - y_v|$$

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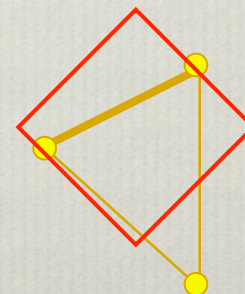
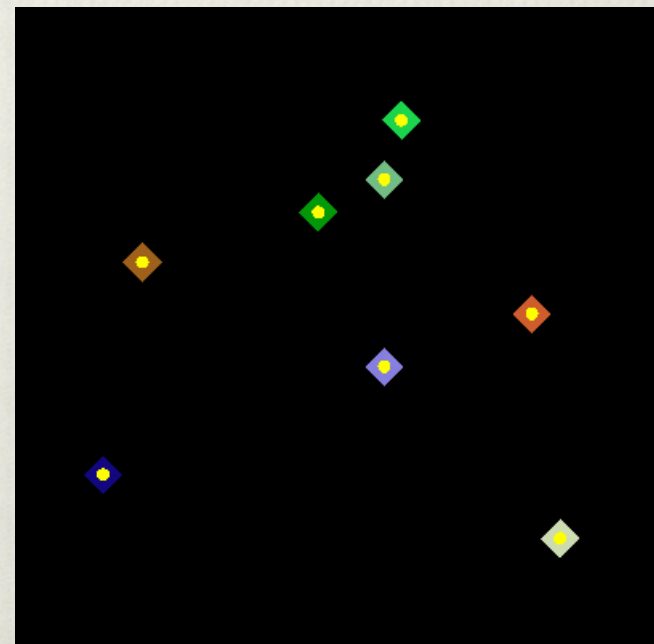
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L_1 -Delaunay Triangulation

[Chew - SoCG'86]

$$d_{L_1}(u,v) = |x_u - x_v| + |y_u - y_v|$$

L_1 -Voronoi cell

$$C(p_i) = \{ u : d_{L_1}(p_i, u) \leq d_{L_1}(p_j, u) \text{ for all } p_j \neq p_i \}$$

L_1 -Delaunay triangulation of $P = \{p_1, p_2, \dots\}$

(p_i, p_j) is an edge iff $C(p_i) \cap C(p_j) \neq \emptyset$

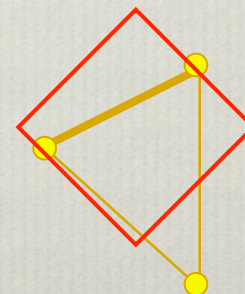
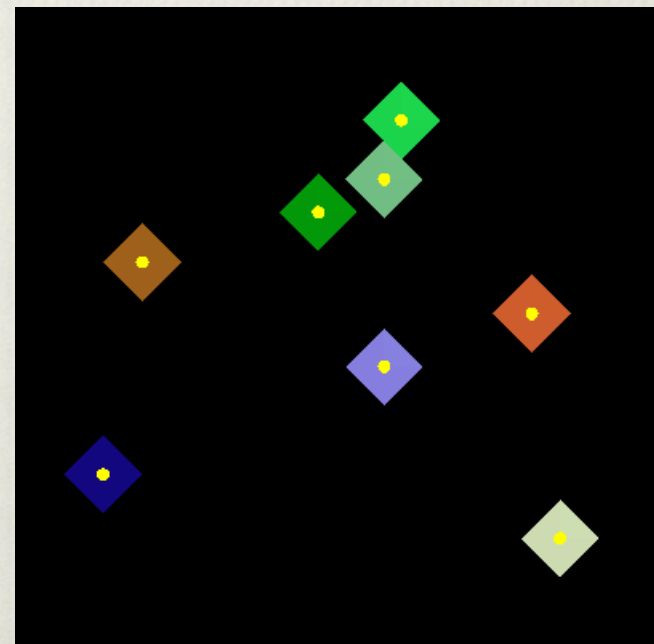
General position

no 2 points with same x- or y-coordinates

no 4 points on a square

Alt. definition

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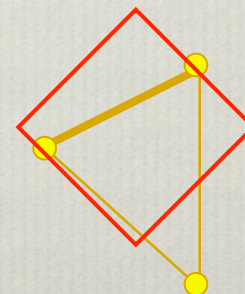
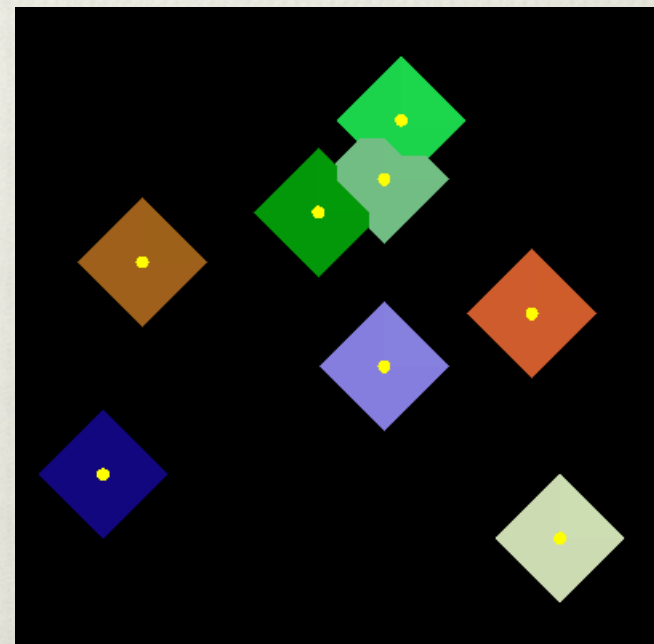
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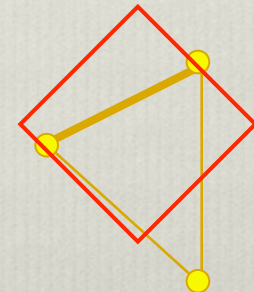
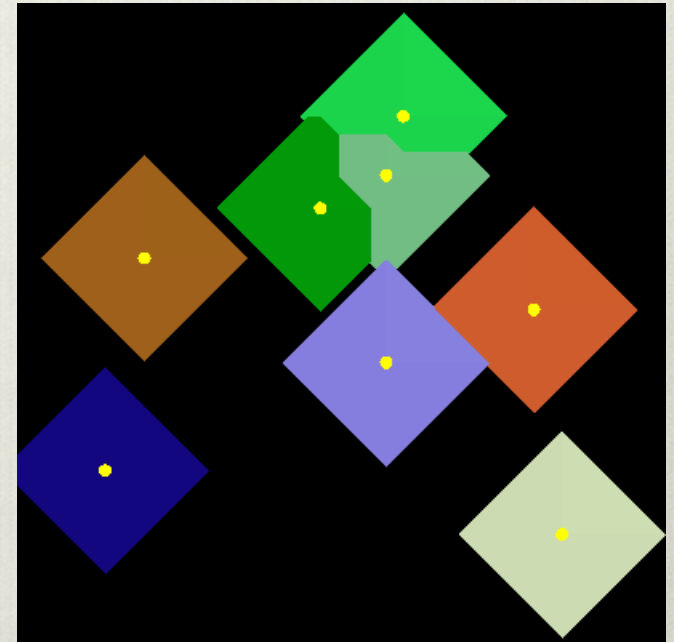
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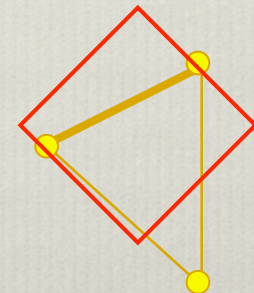
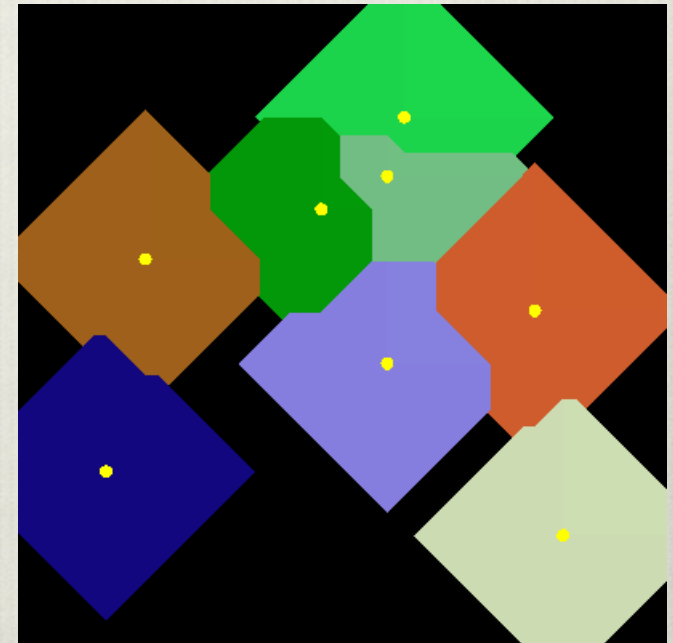
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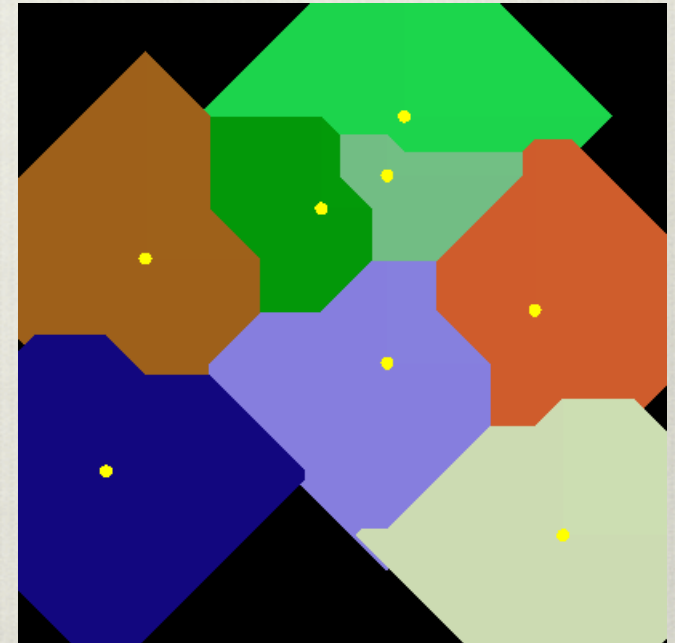
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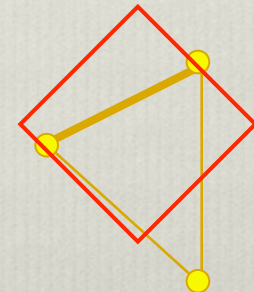
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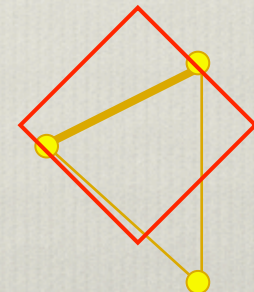
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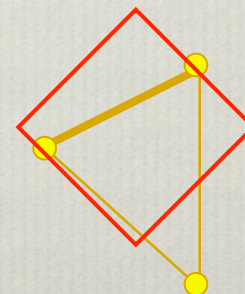
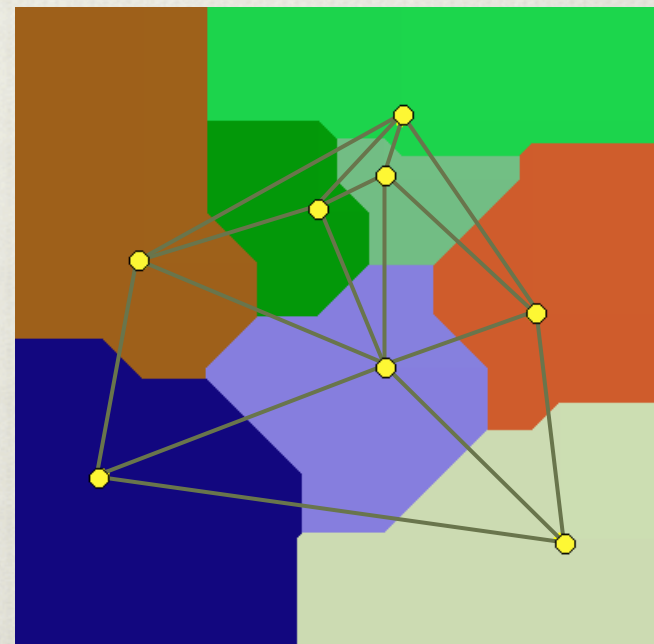
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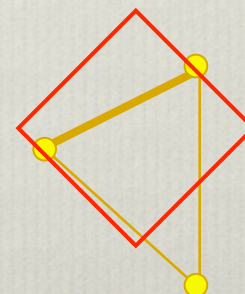
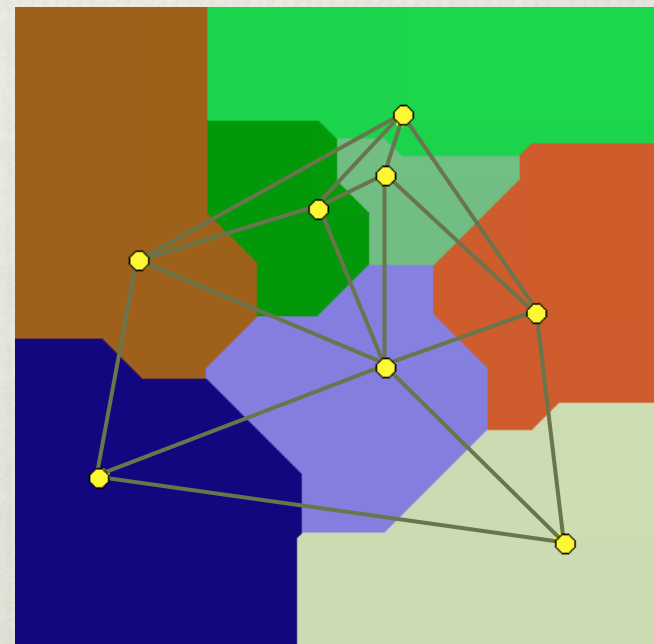
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→ each face is supported by an empty **square**





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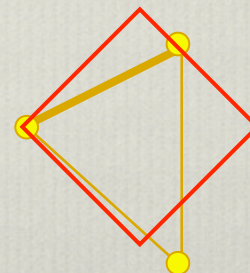
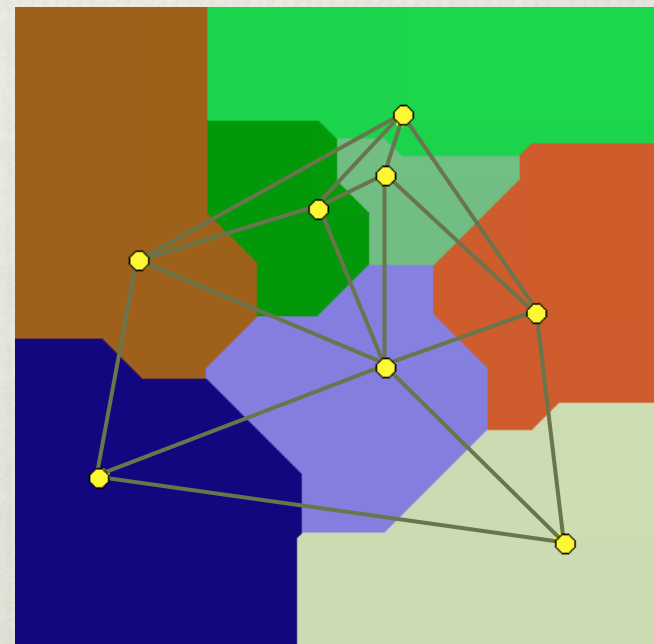
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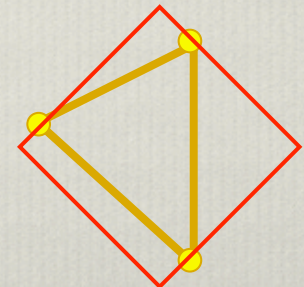
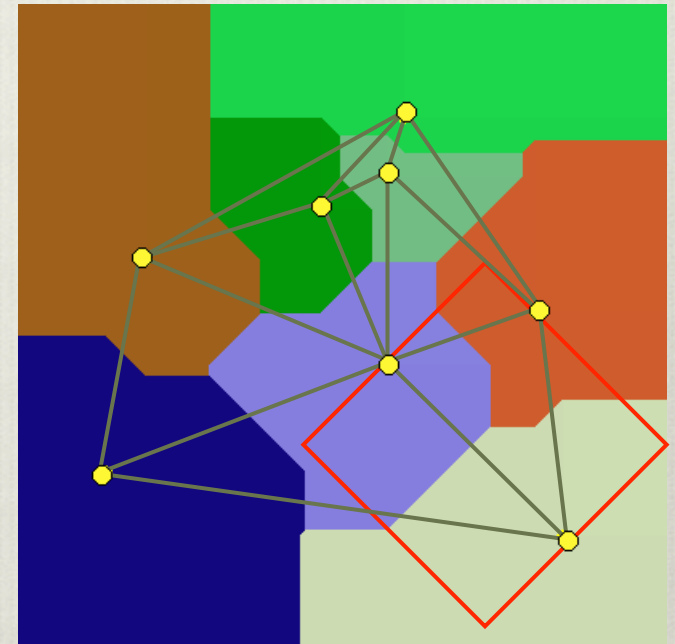
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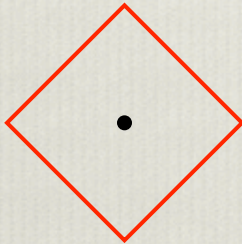


L_1 -Delaunay Triangulation vs. L_∞ -Delaunay Triangulation

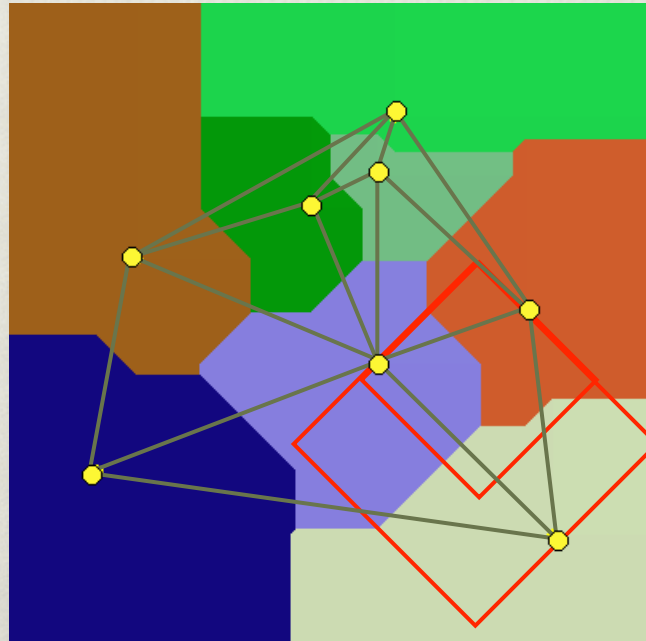
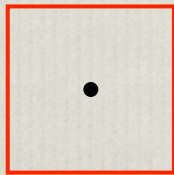
$$d_{L_1}(u,v) = |x_u - x_v| + |y_u - y_v|$$

$$d_{L_\infty}(u,v) = \max\{|x_u - x_v|, |y_u - y_v|\}$$

L_1 ball



L_∞ ball



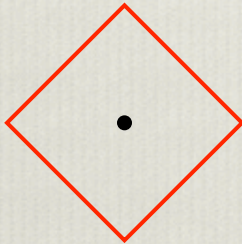


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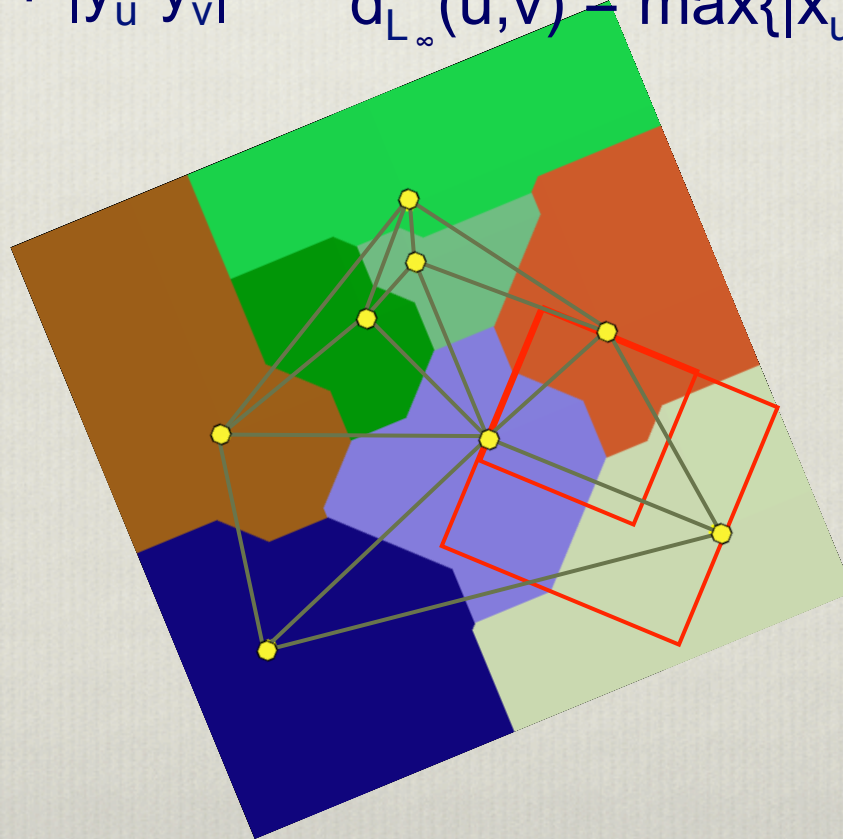
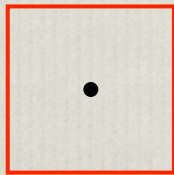
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L_1 ball



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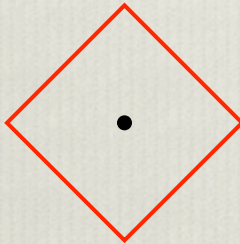


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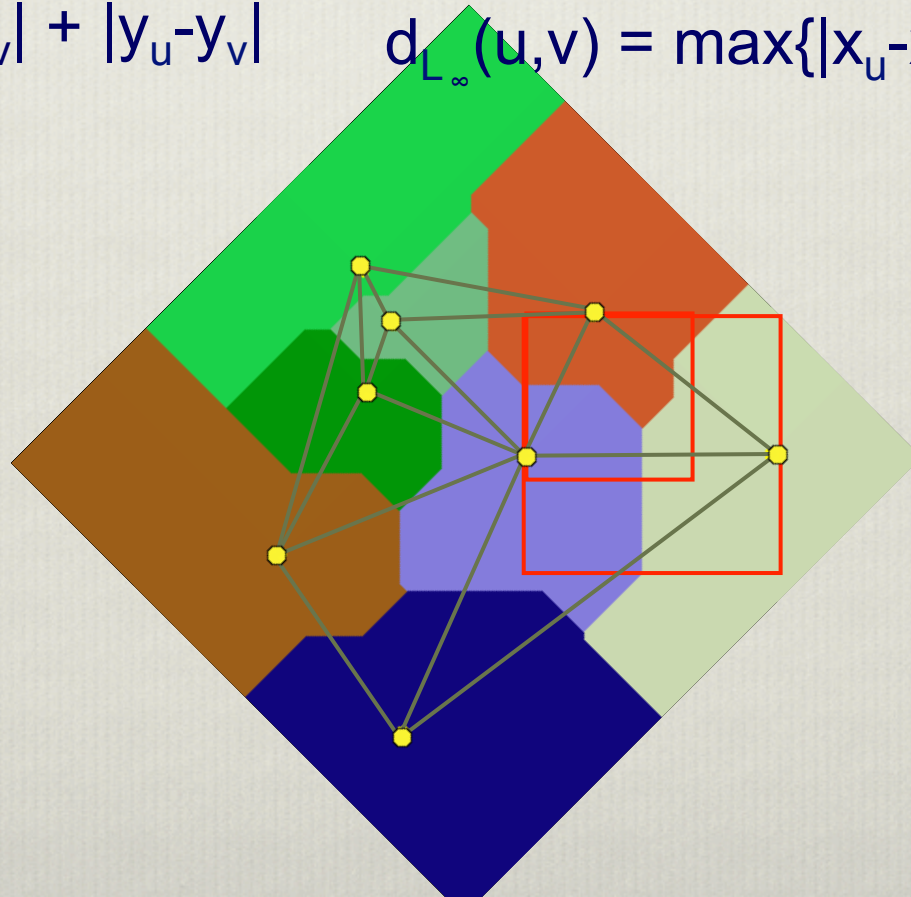
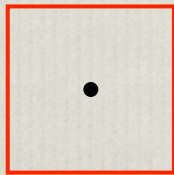
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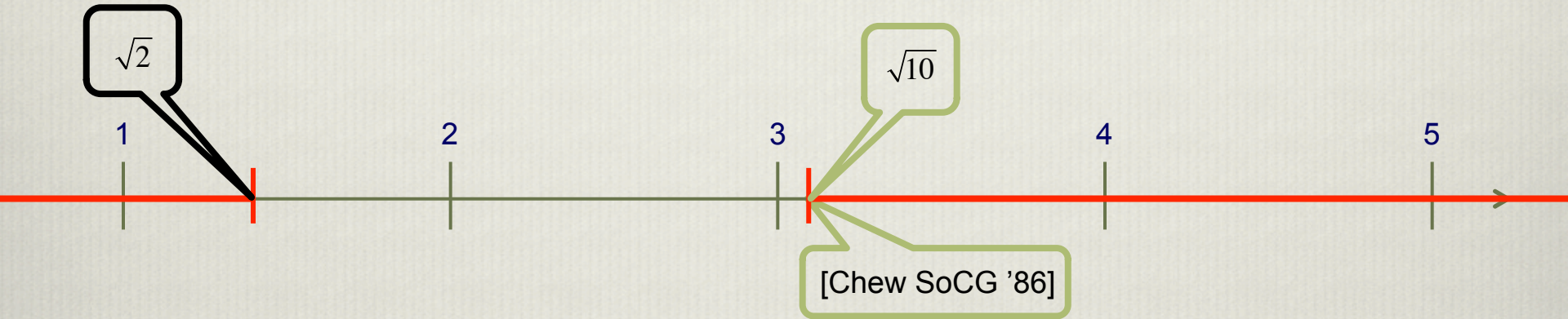
L_1 ball



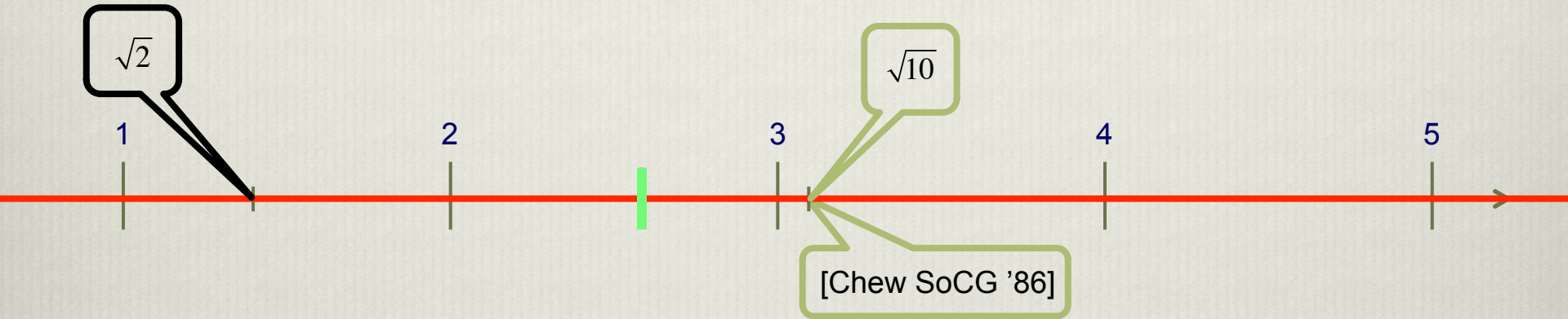
L_∞ ball



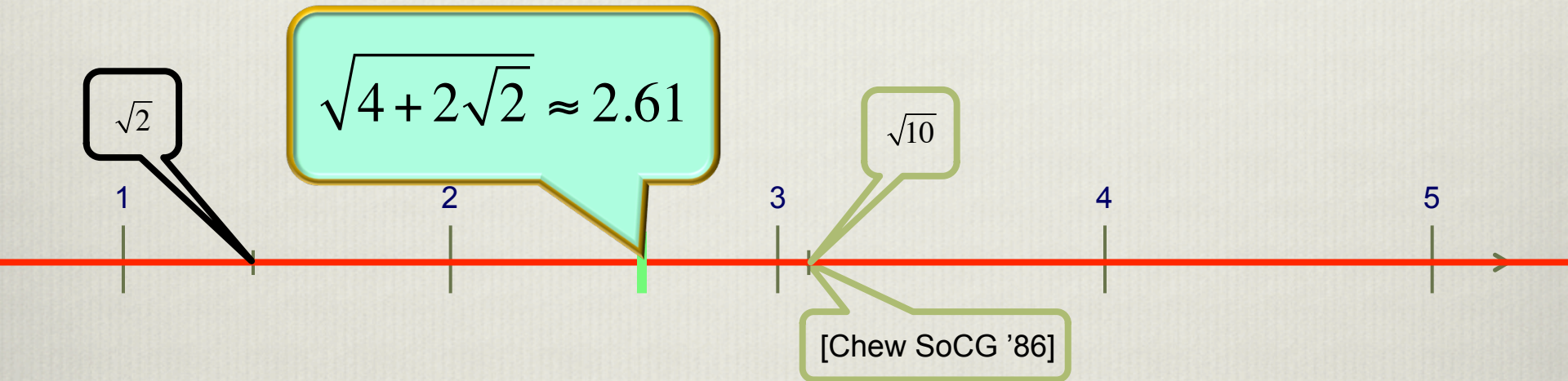
The Stretch of L_1 -Delaunay Triangulation



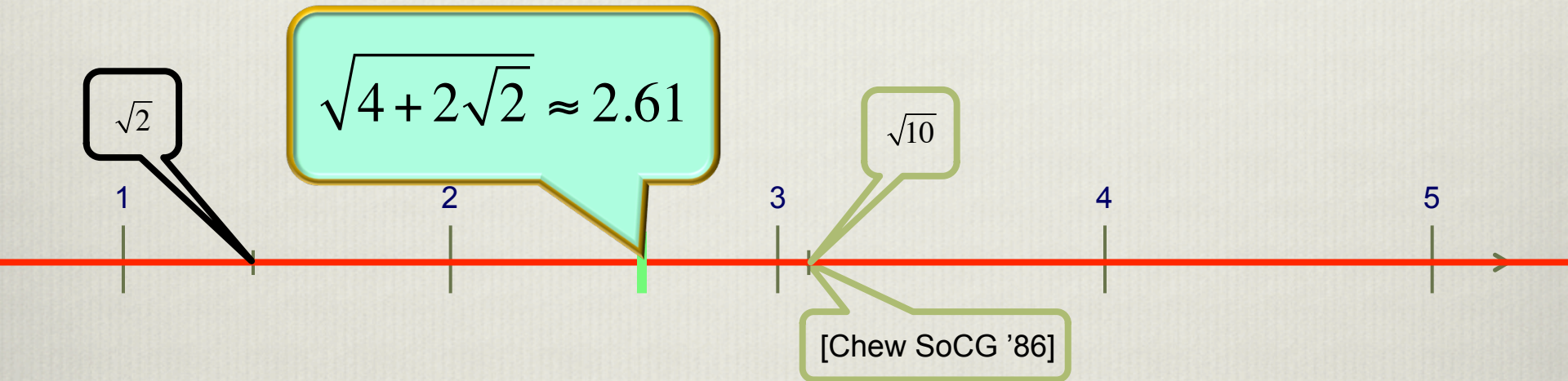
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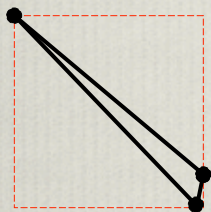


Outline

- Lower bound
- Upper Bound

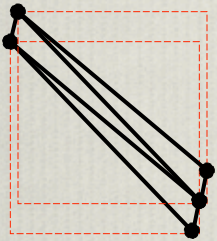


Stretch ≥ 2.61



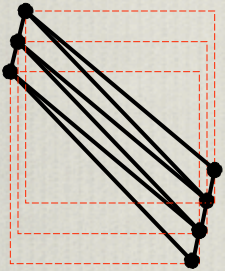


Stretch ≥ 2.61



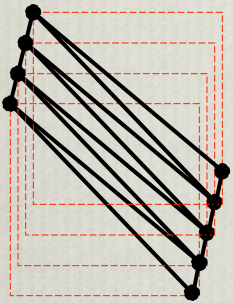


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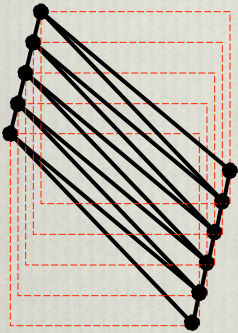


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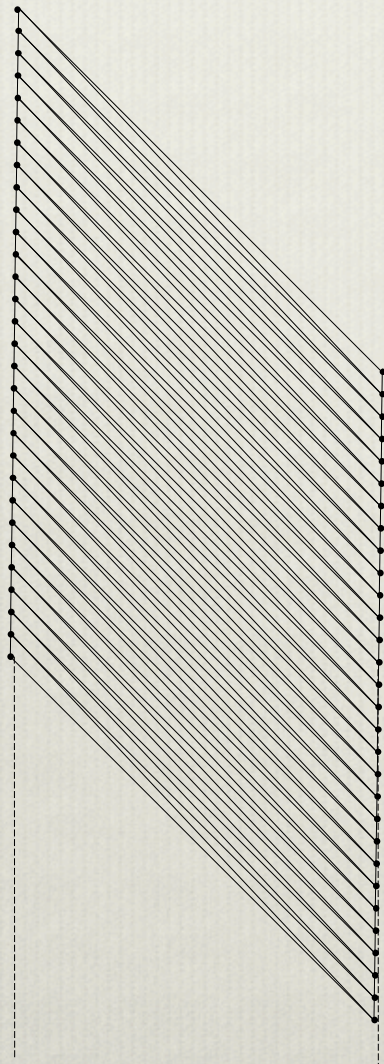
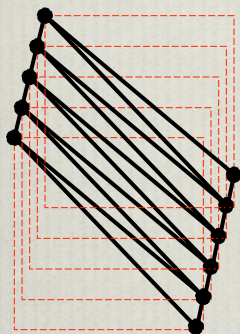


Stretch ≥ 2.61



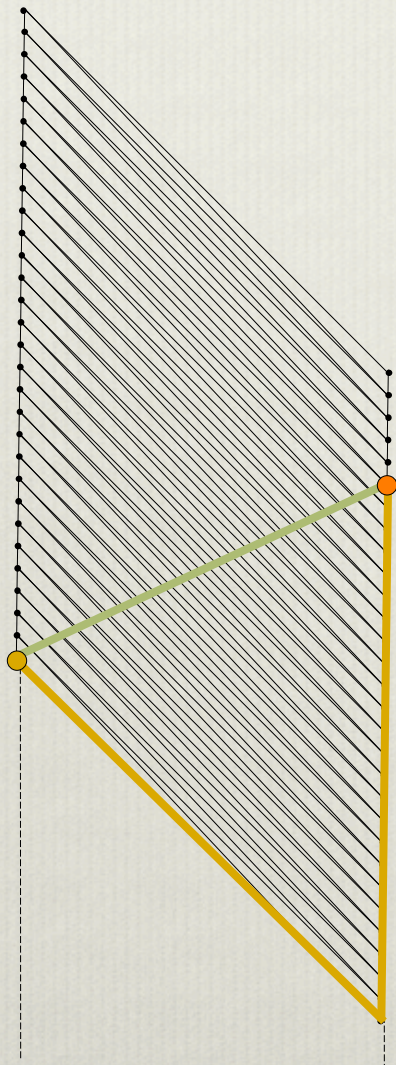
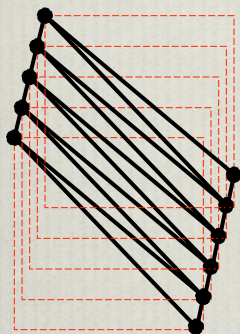


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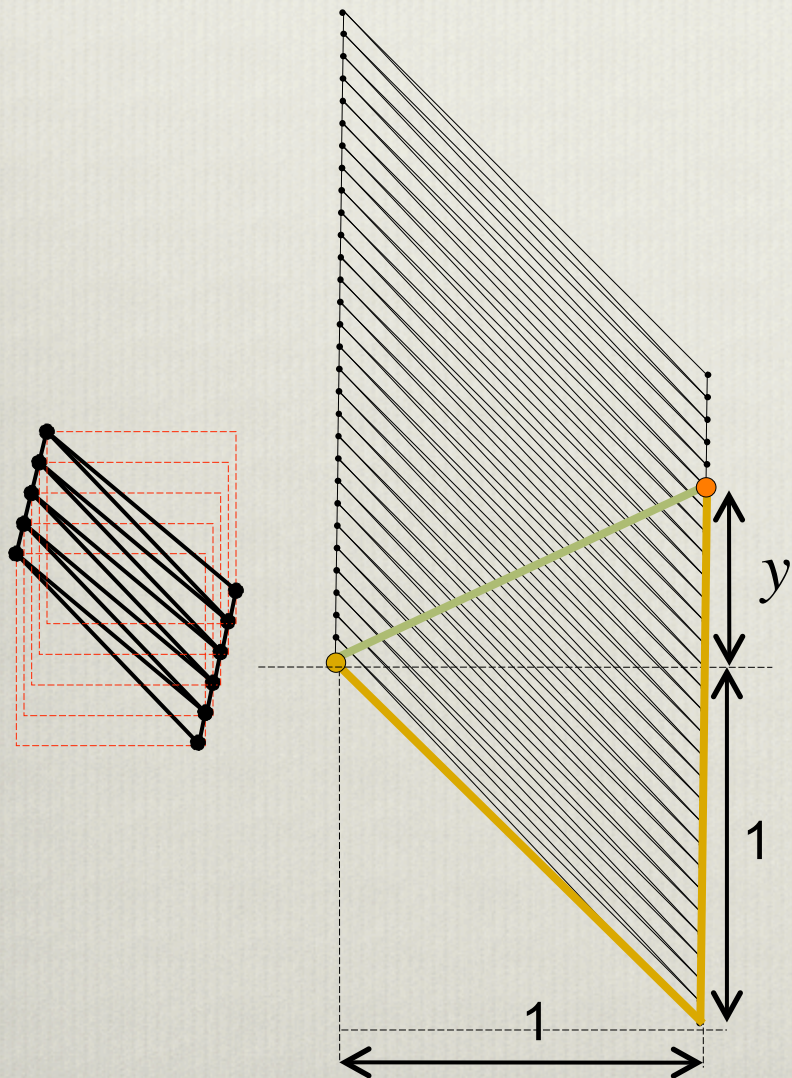
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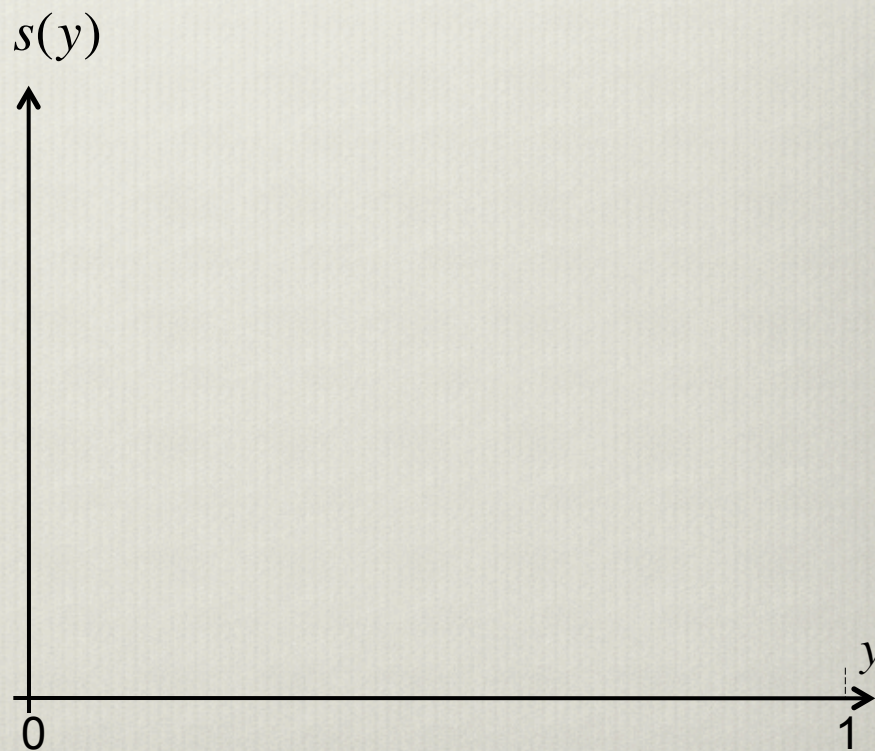
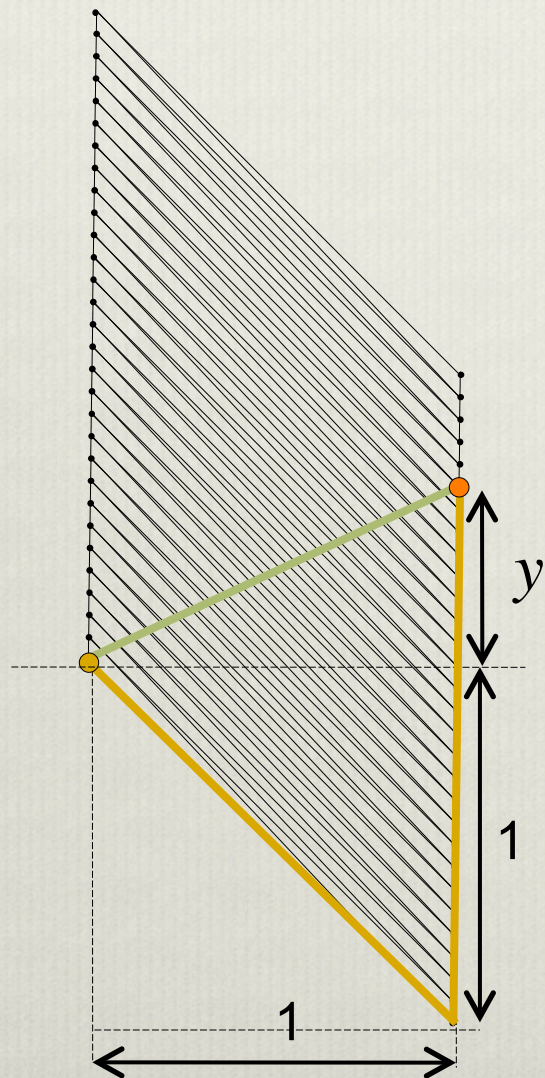
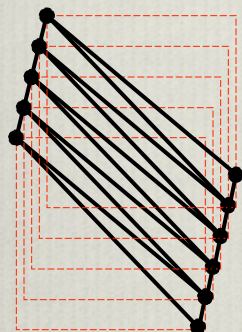
$$s(y) = \frac{1 + \sqrt{2} + y}{\sqrt{1 + y^2}} \quad 0 < y \leq 1$$





Stretch ≥ 2.61

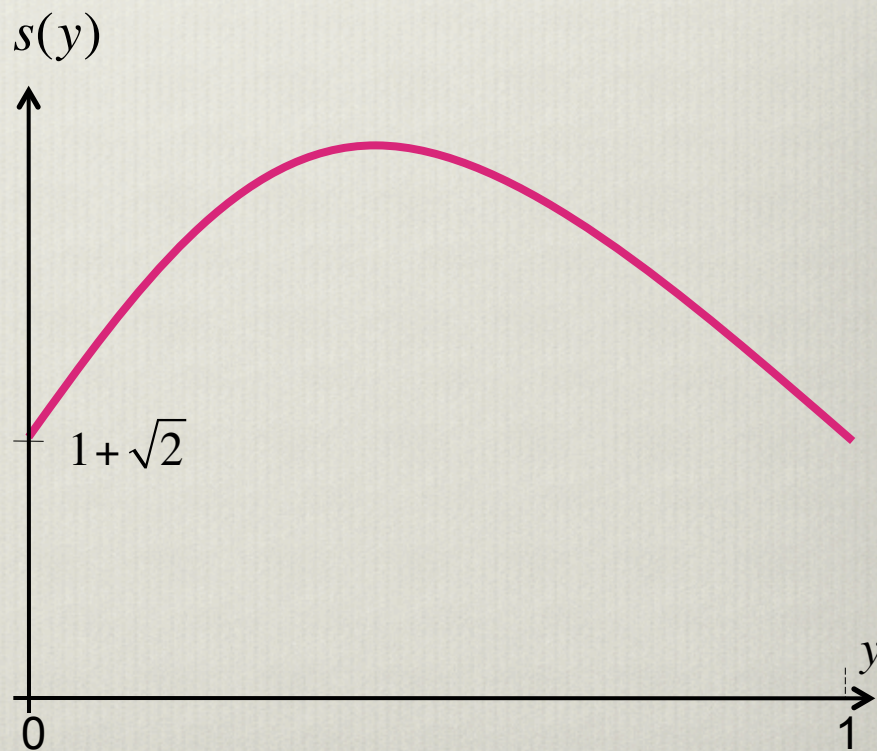
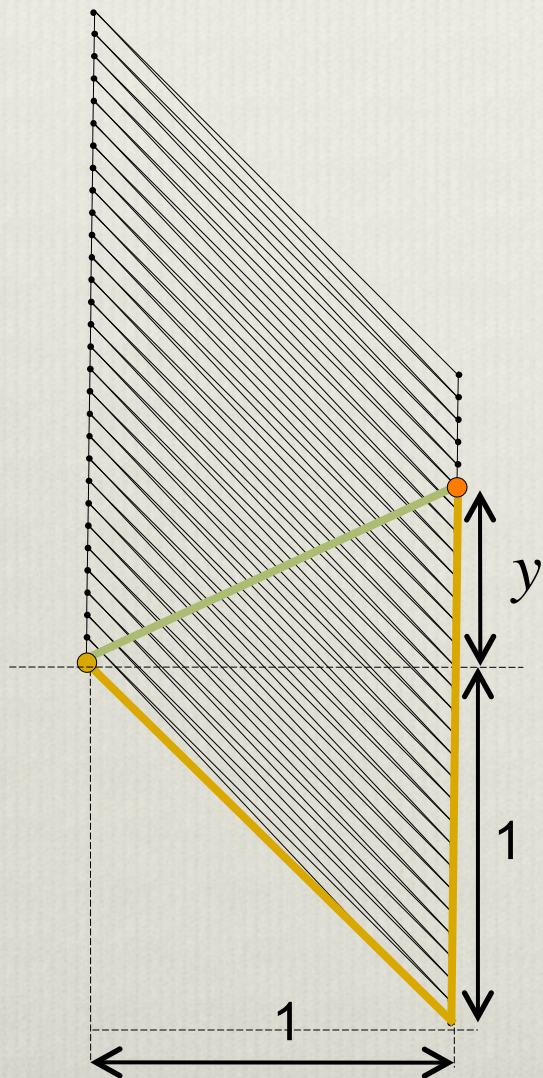
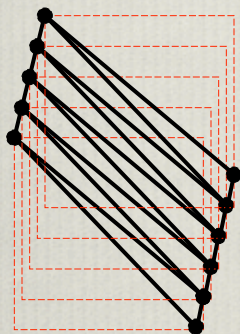
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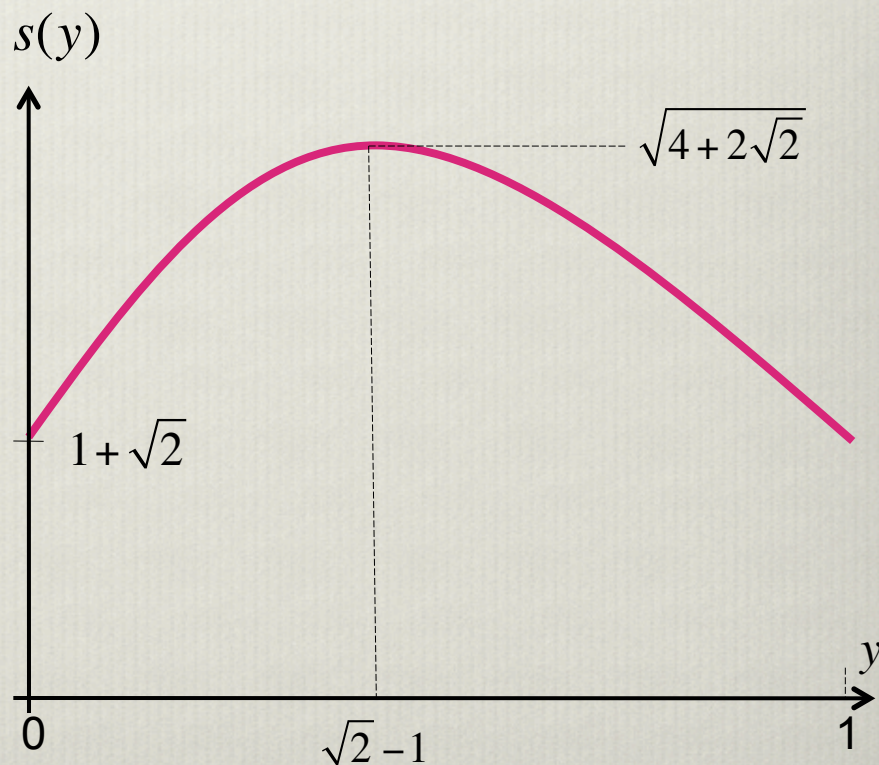
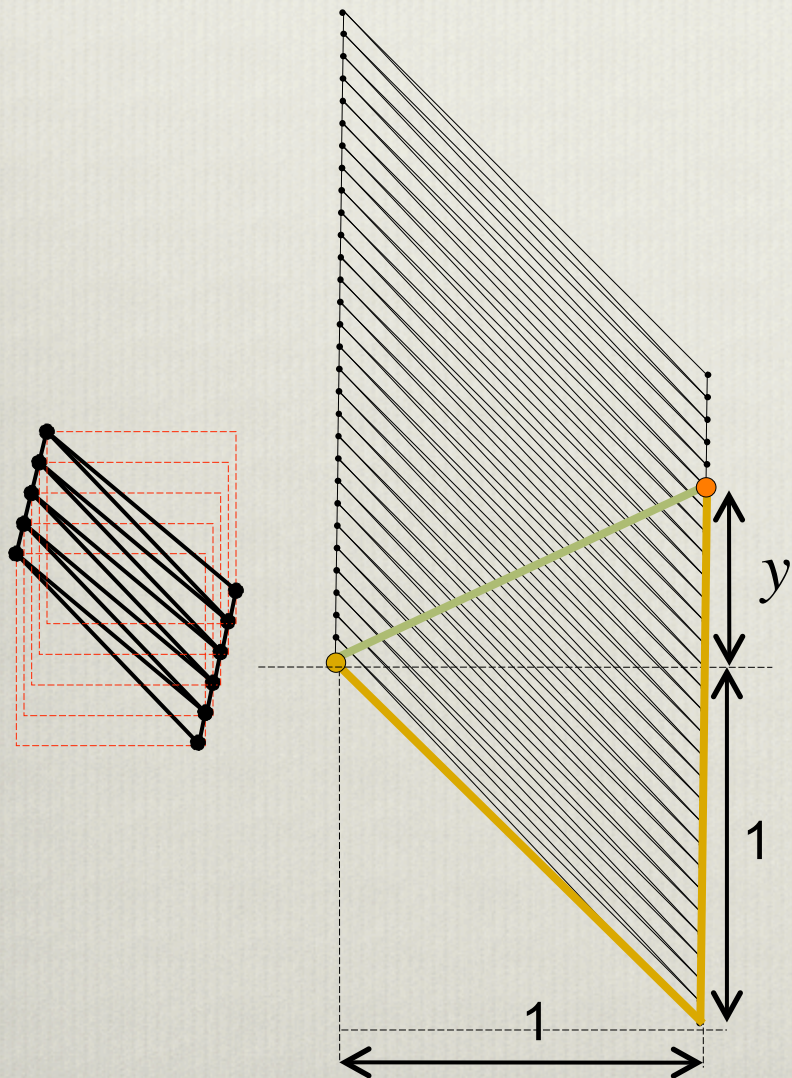
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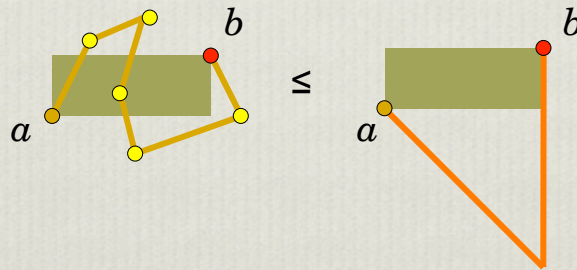




Main Theorem

Let T be the L_∞ -Delaunay triangulation on a set of points P in general position, and let a, b points of P . Let $x := \max\{|x_a - x_b|, |y_a - y_b|\}$ and $y := \min\{|x_a - x_b|, |y_a - y_b|\}$. Then,

$$d_T(a, b) \leq (1 + \sqrt{2})x + y$$

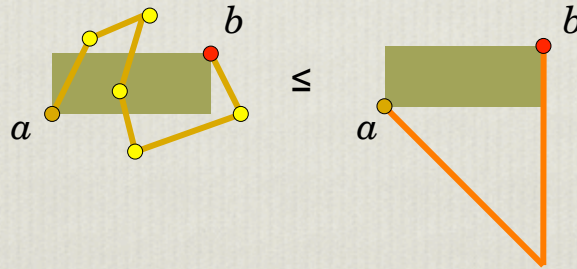




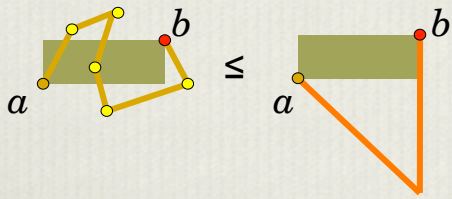
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$$d_T(a, b) \leq (1 + \sqrt{2})x + y$$



$$\frac{(1 + \sqrt{2})x + y}{\sqrt{x^2 + y^2}} \leq s(y / x) \leq \sqrt{4 + 2\sqrt{2}} \approx 2.61$$



Main Thm's Proof

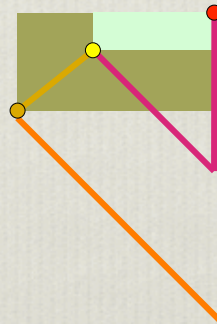
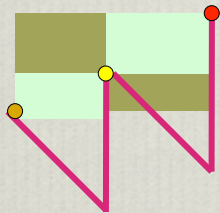
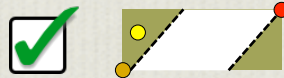
(warm up)

Proof. By induction on the width of $R(a,b)$

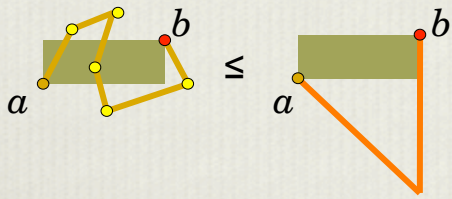
$R(a,b) =$

☒ Case 0: (a,b) edge of T (base case)

☒ Case 1: c in $R(a,b)$



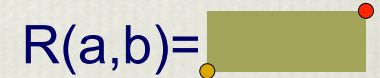
consider $R(b,a)$



Main Thm's Proof

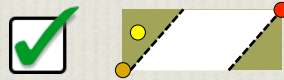
(warm up)

Proof. By induction on the width of $R(a,b)$

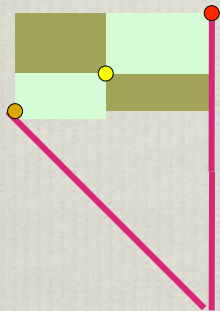


✓ Case 0: (a,b) edge of T (base case)

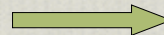
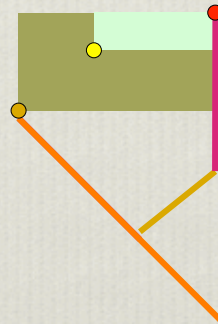
✓ Case 1: c in $R(a,b)$



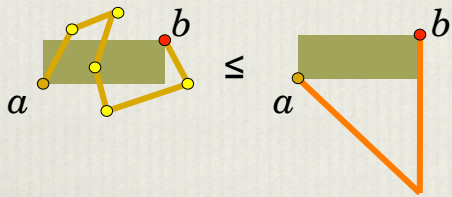
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Case 2: c not in $R(a,b)$

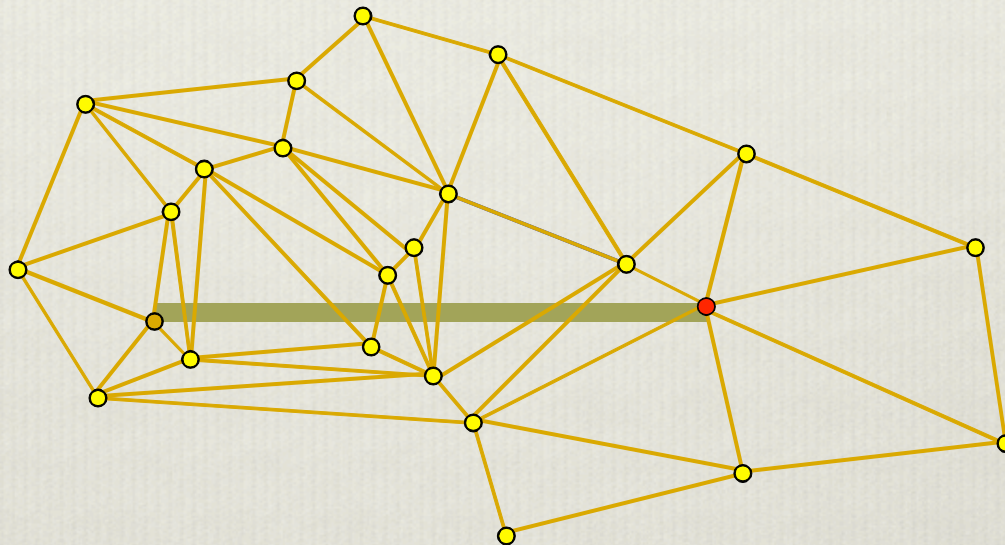


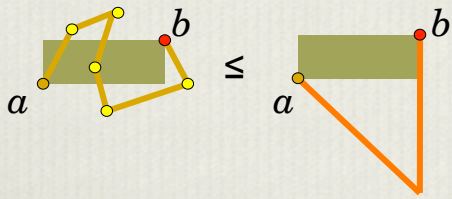
the interesting case !



Main Thm's Proof

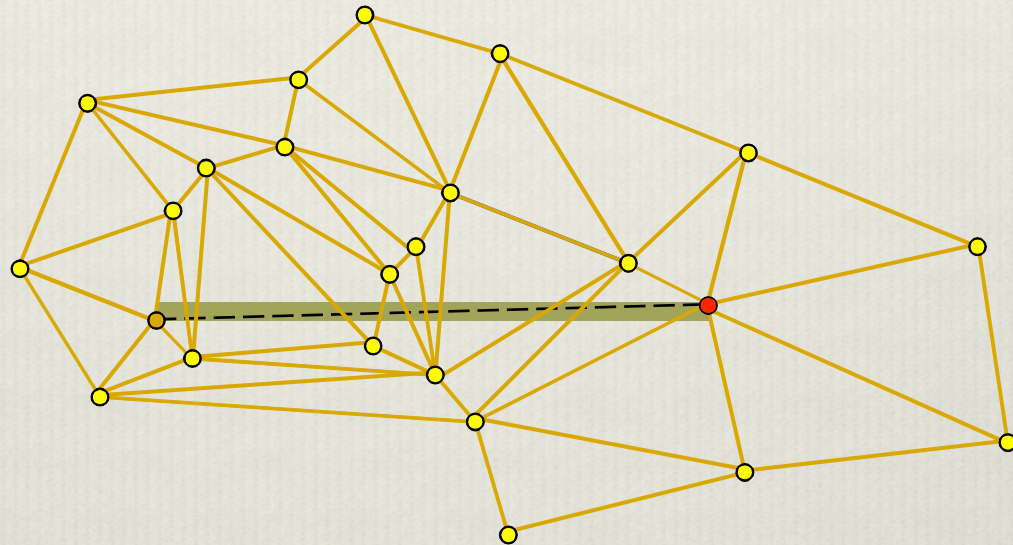
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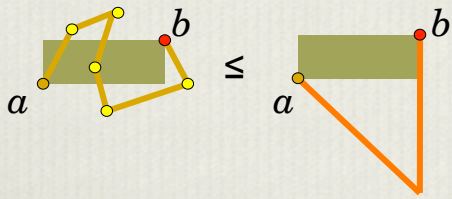




Main Thm's Proof

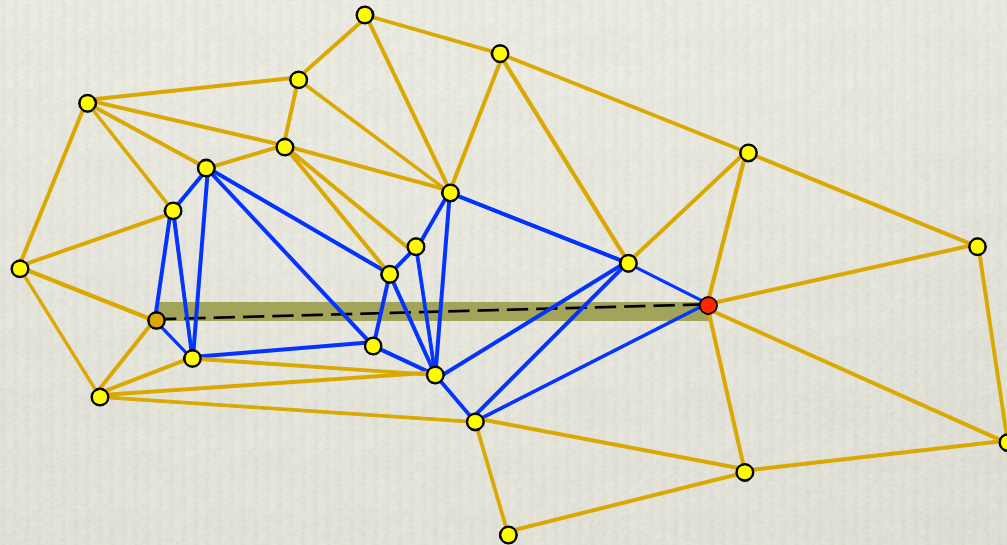
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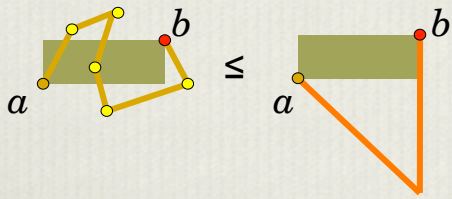




Main Thm's Proof

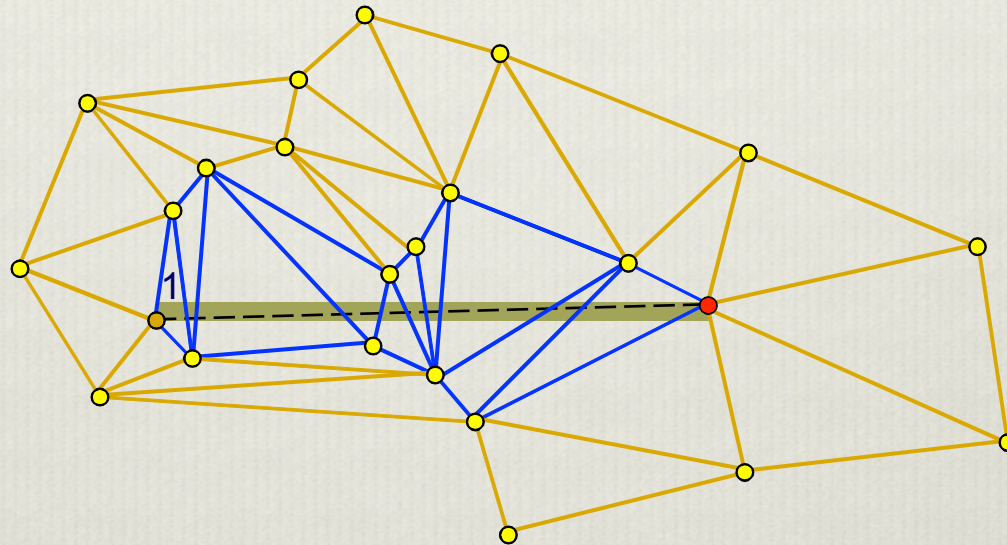
(cont'd)

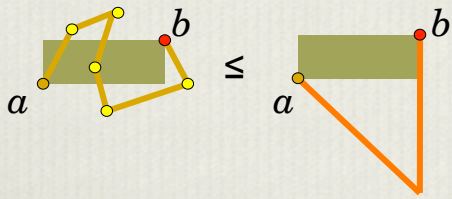




Main Thm's Proof

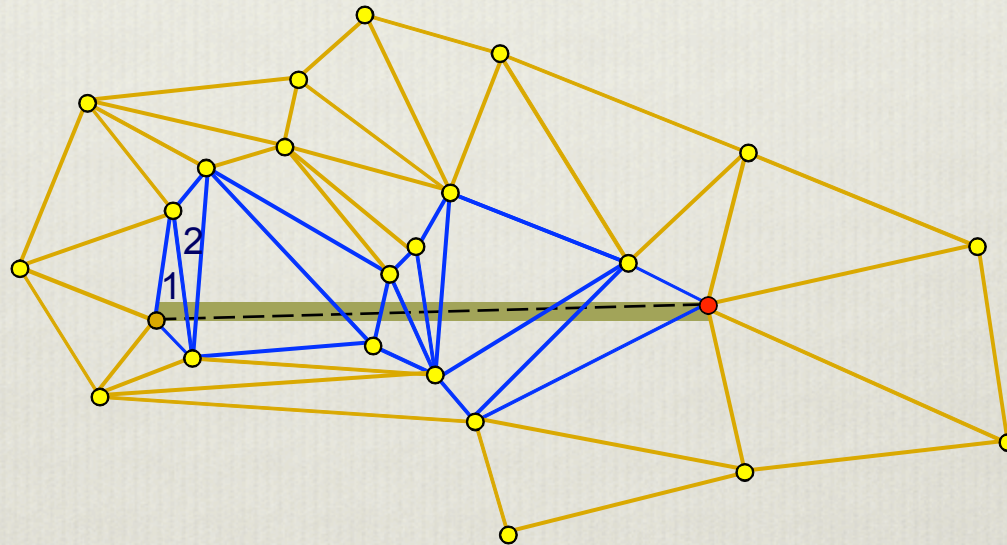
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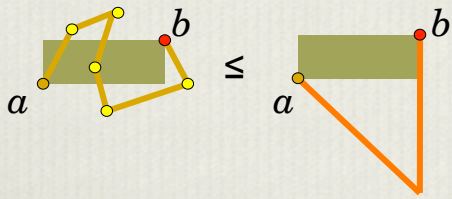




Main Thm's Proof

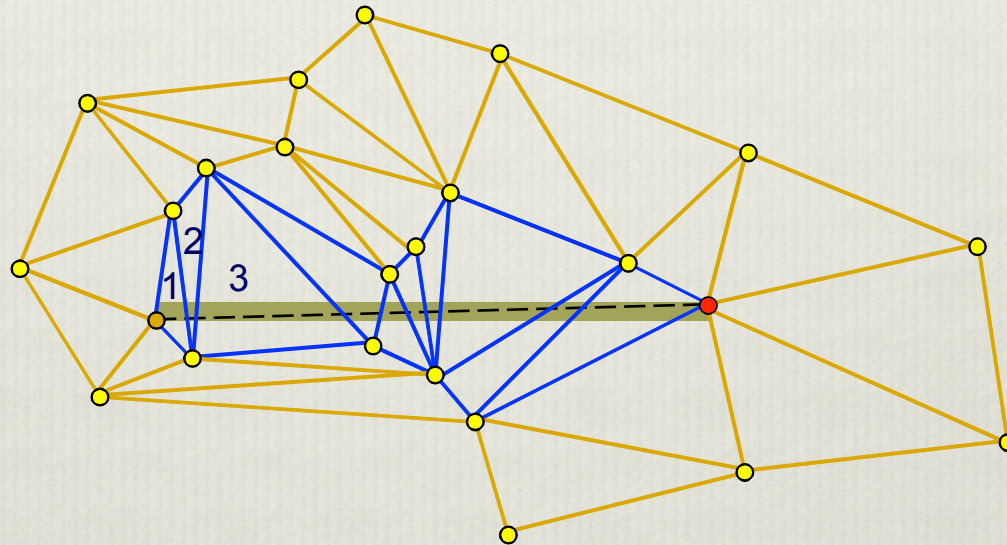
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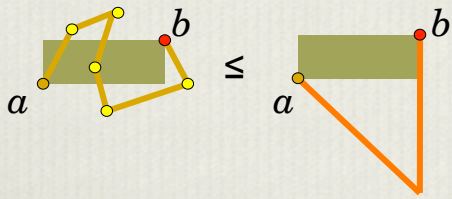




Main Thm's Proof

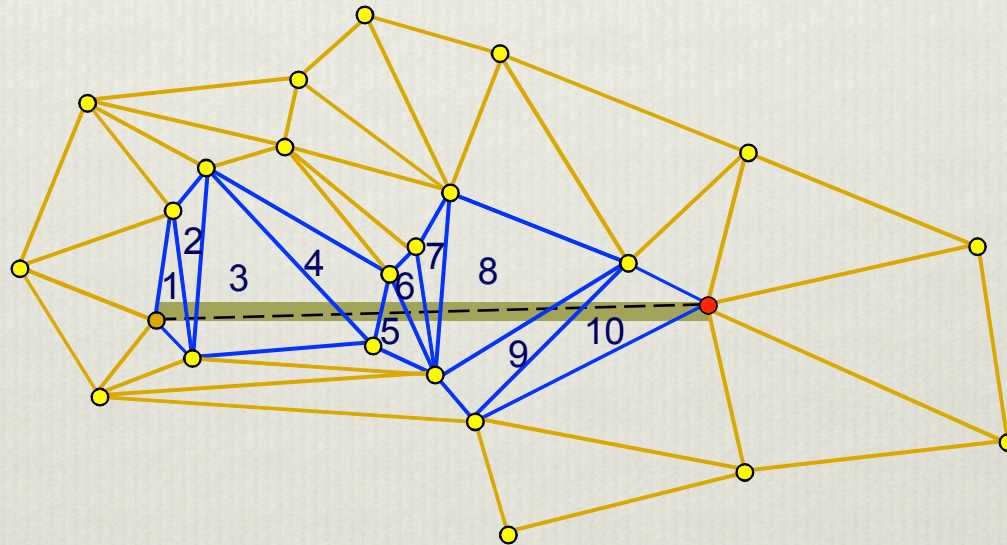
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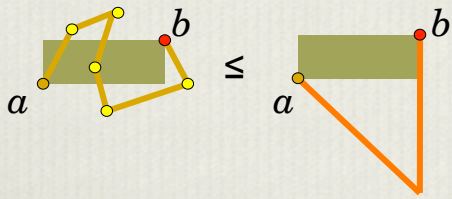




Main Thm's Proof

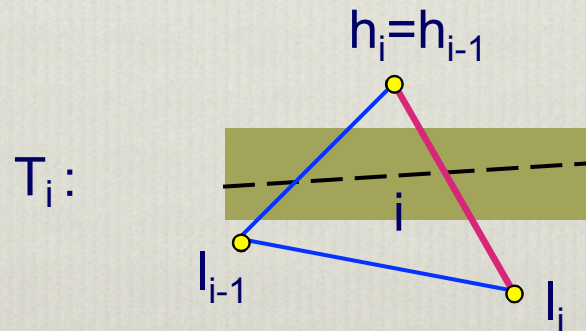
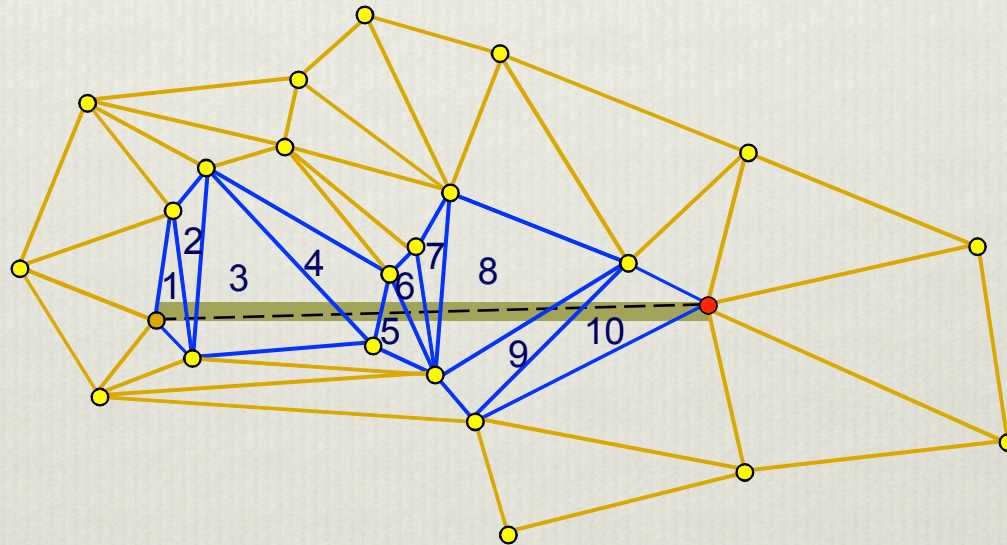
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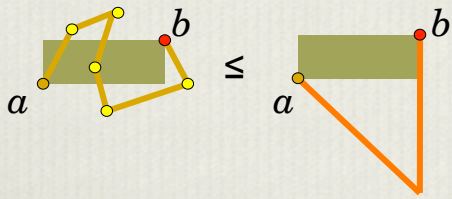




Main Thm's Proof

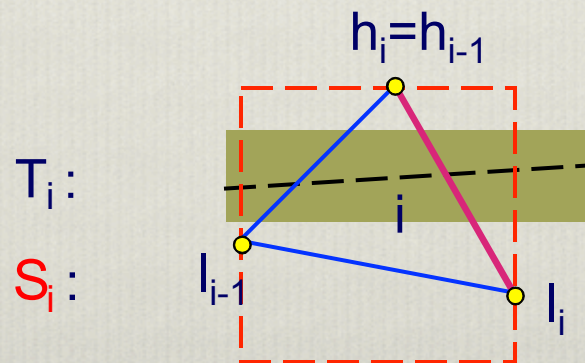
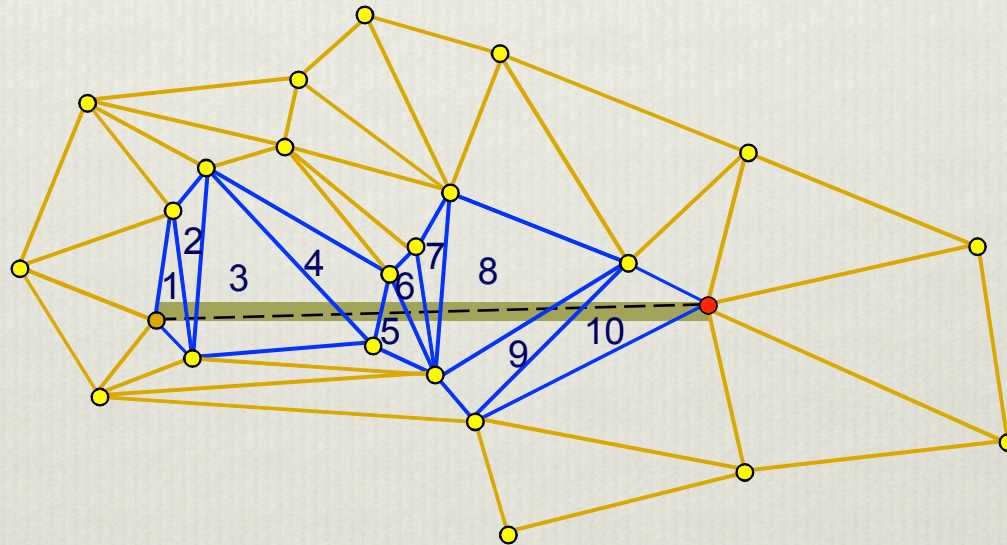
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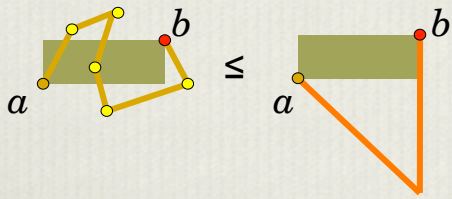




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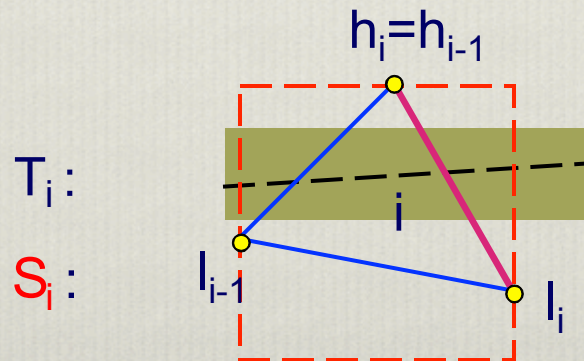
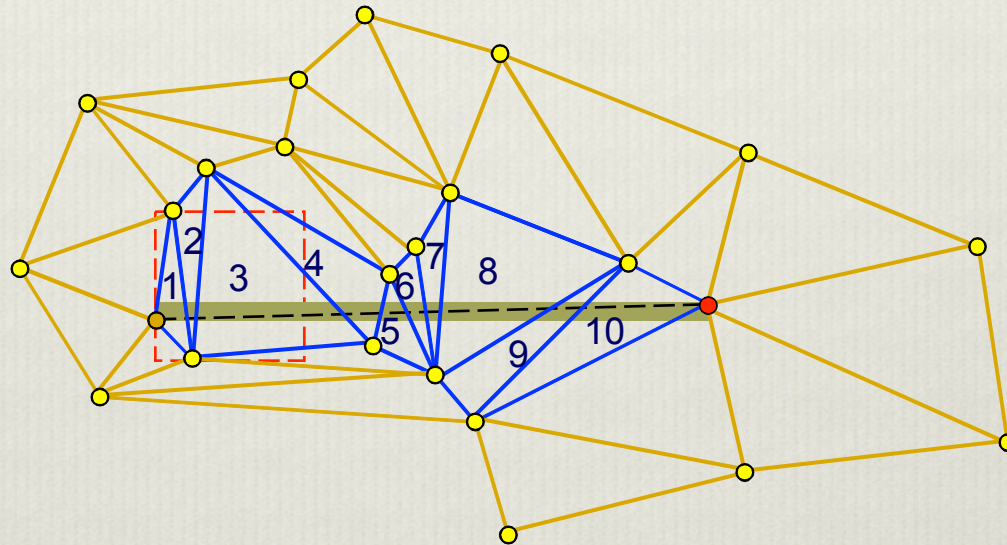
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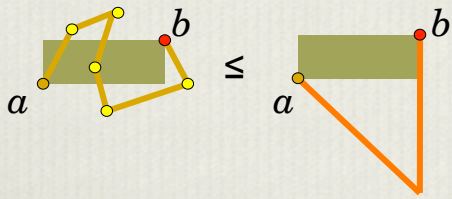




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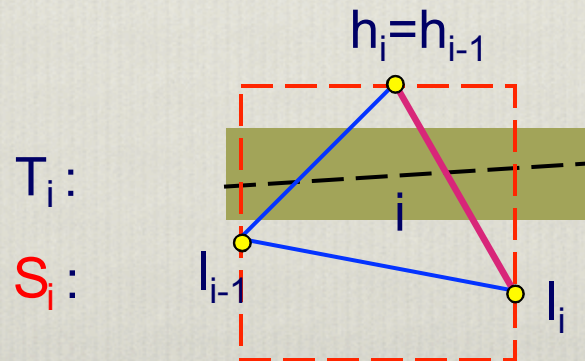
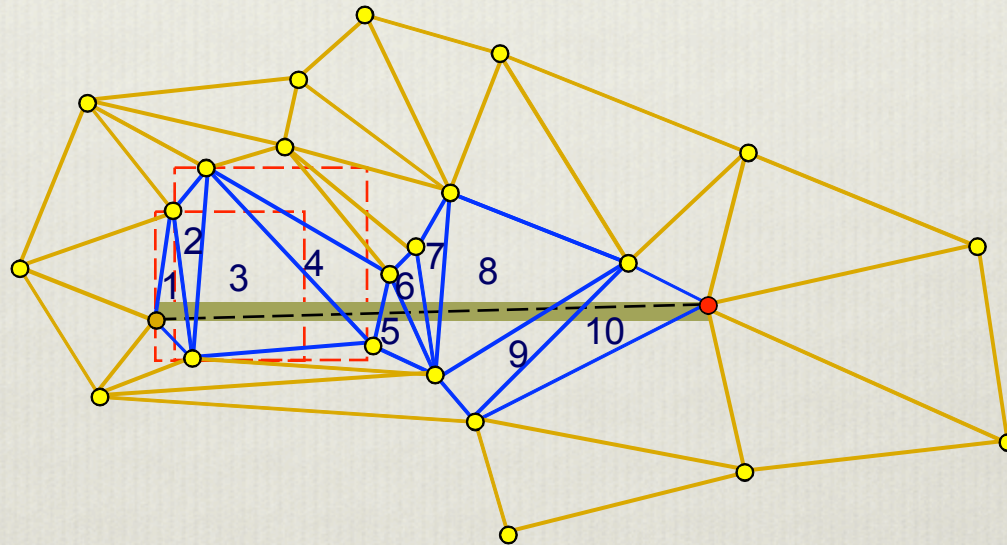
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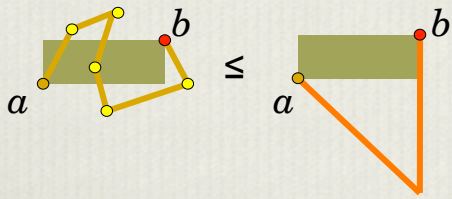




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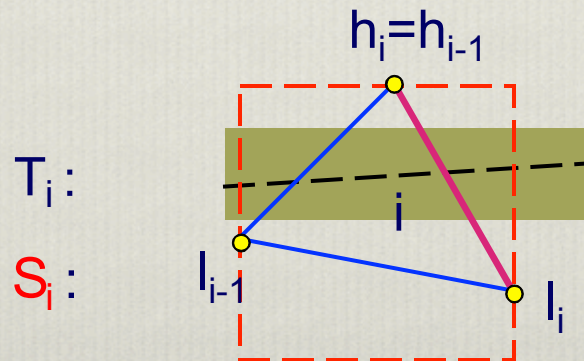
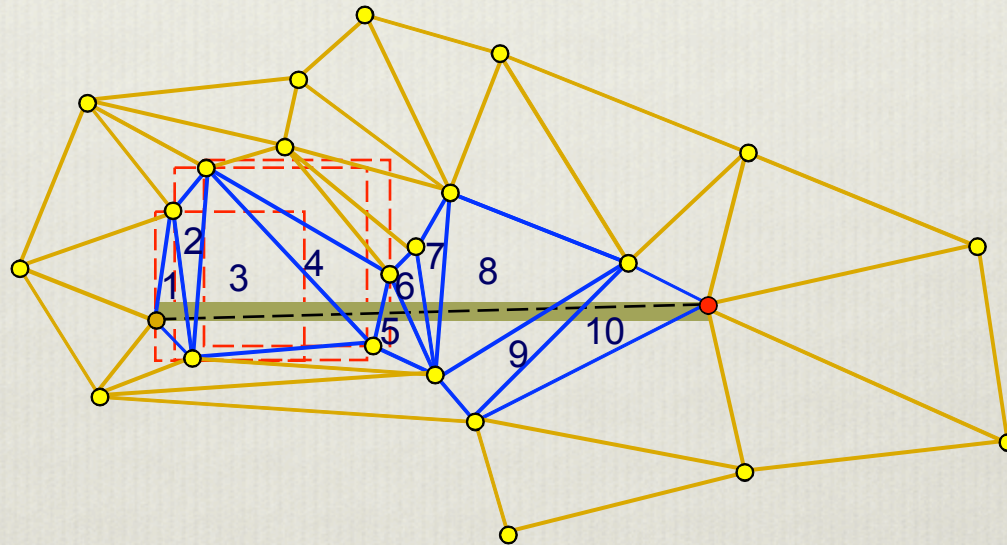
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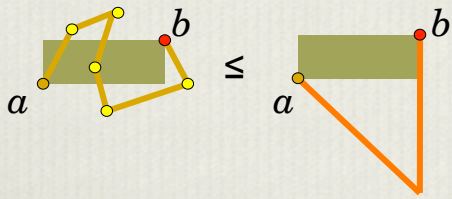




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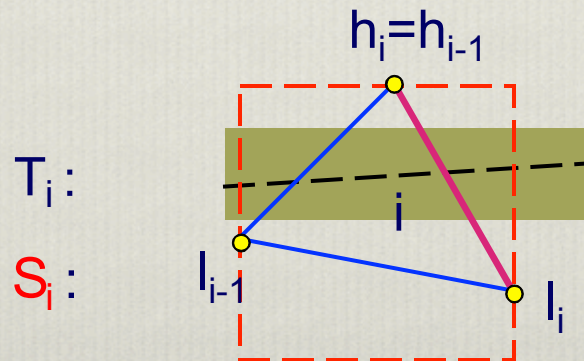
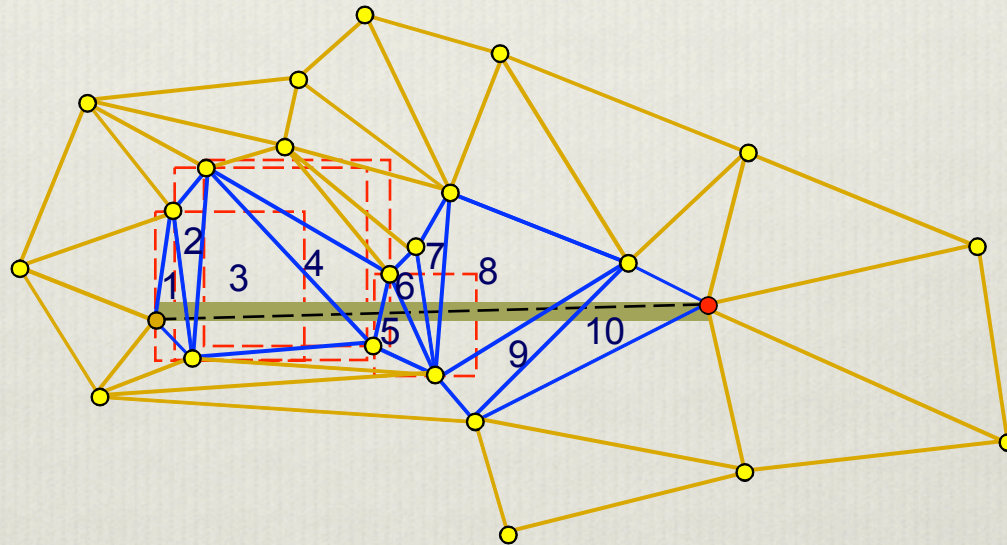
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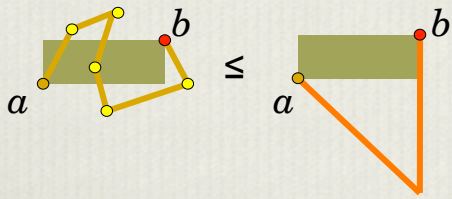




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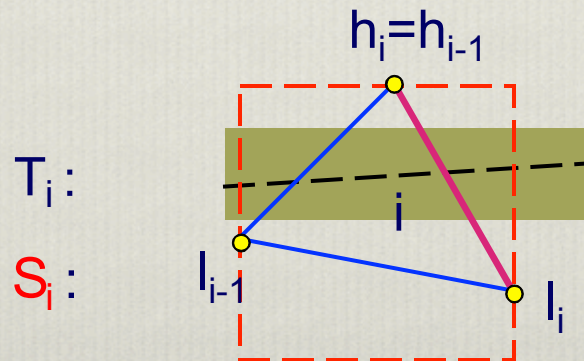
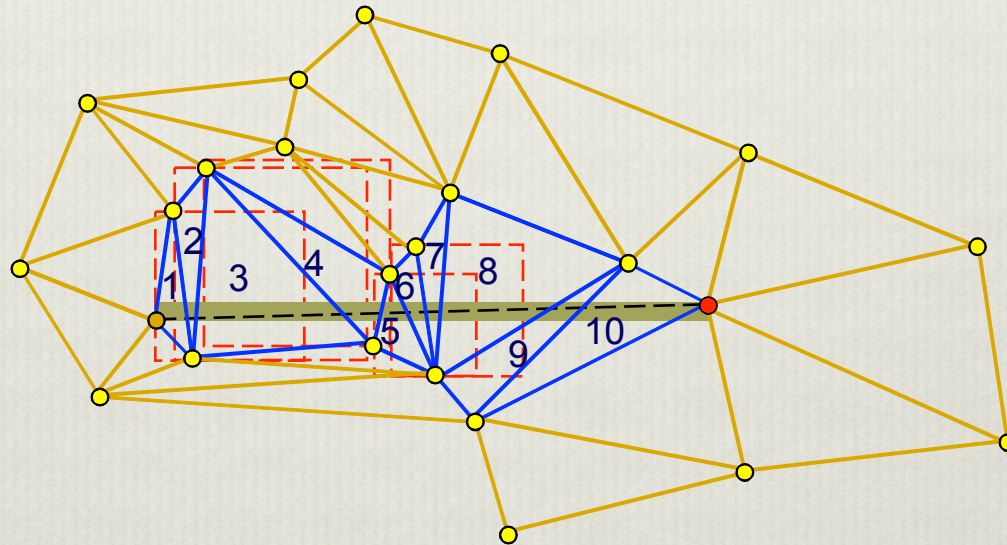
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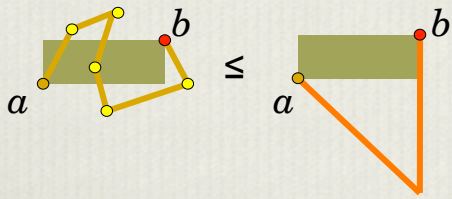




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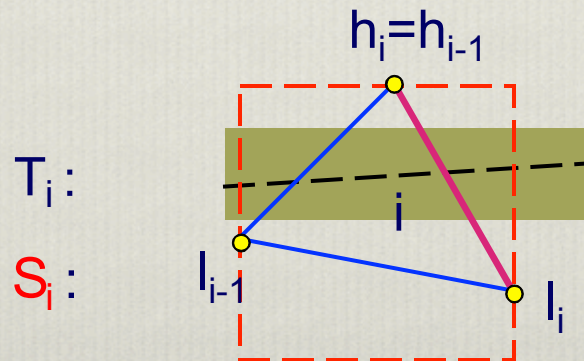
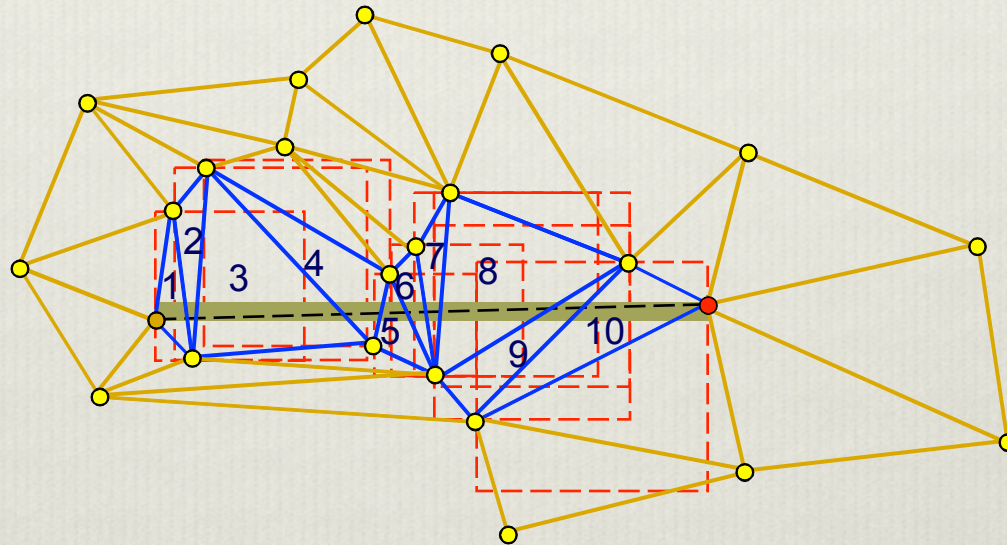
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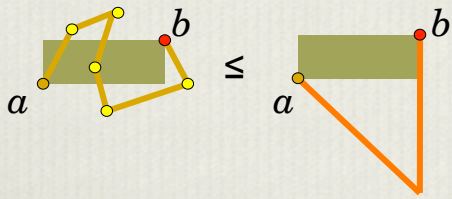




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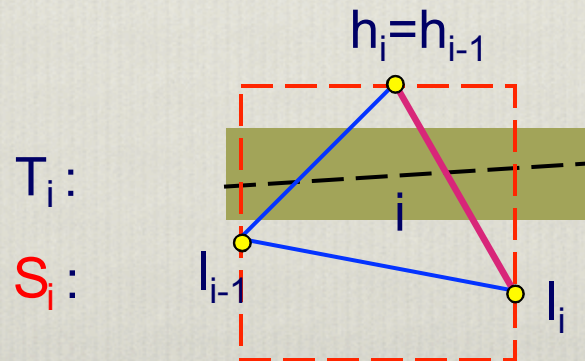
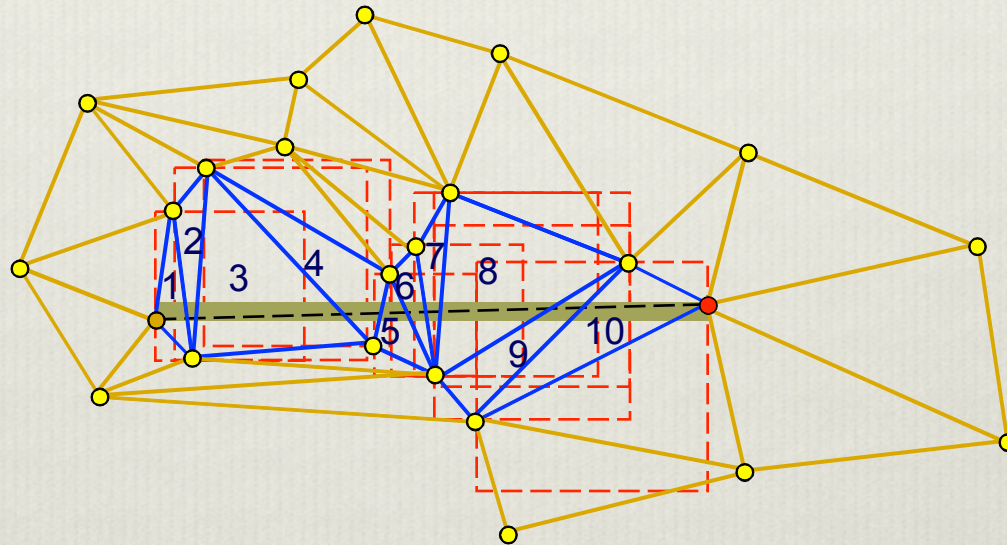
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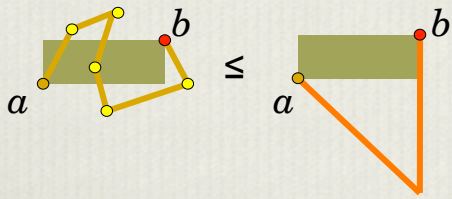




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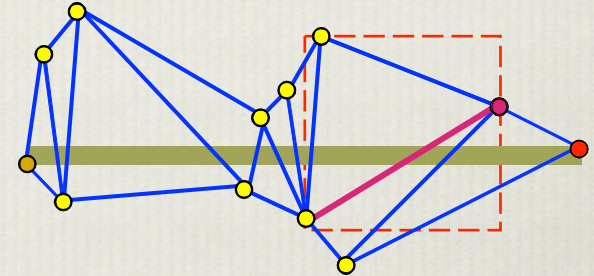
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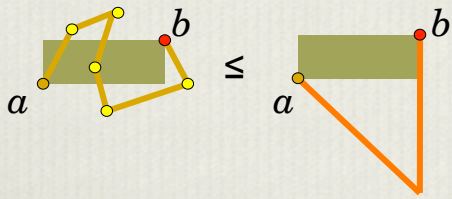




Inductive Square

Definition. S_i is **inductive** if edge (l_i, h_i) has a slope $\leq 45^\circ$



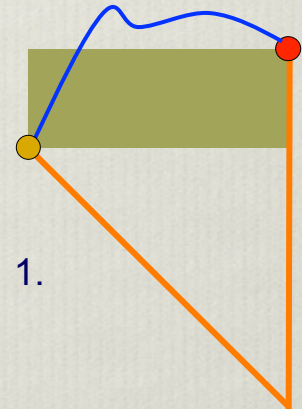
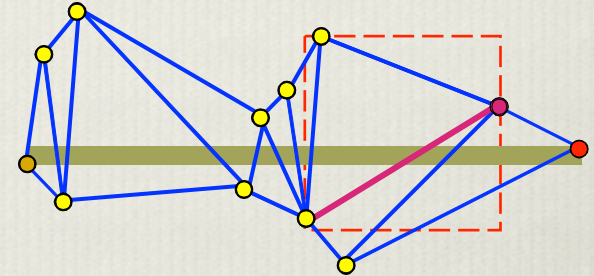


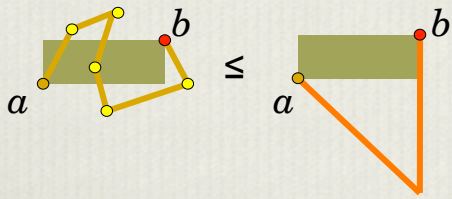
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Crux lemma: (proof in the paper)

1. There is no inductive square: $d_T(a, b) \leq (1 + \sqrt{2})x + y$



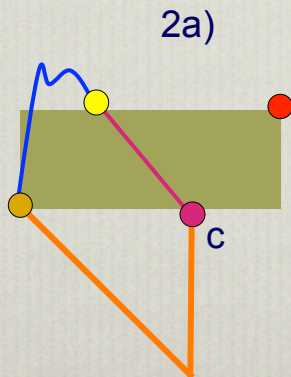
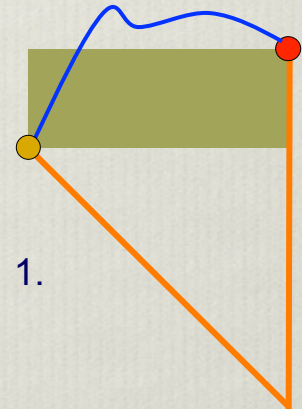
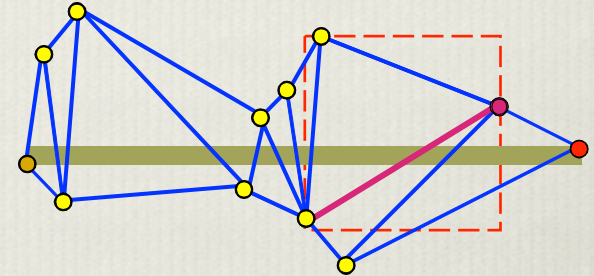


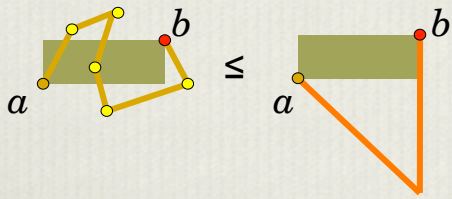
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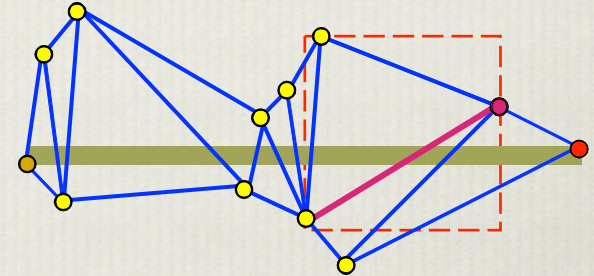
1. There is no inductive square: $d_T(a, b) \leq (1 + \sqrt{2})x + y$
2. Let S_i be the **first** inductive square:
 - a) $c = l_i$ and $d_T(a, c) \leq (1 + \sqrt{2})x_c + y_c$





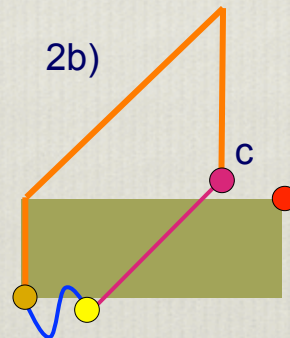
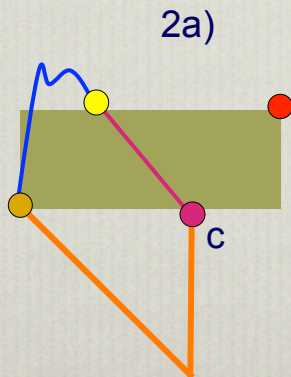
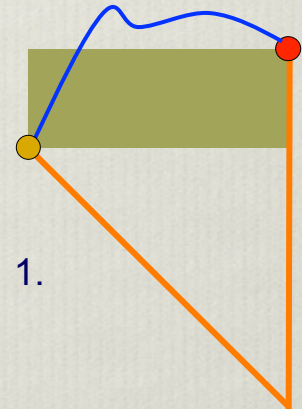
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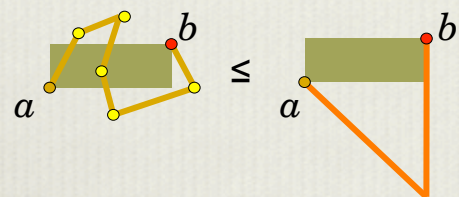
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1. There is no inductive square: $d_T(a, b) \leq (1 + \sqrt{2})x + y$
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 - a) $c = l_i$ and $d_T(a, c) \leq (1 + \sqrt{2})x_c + y_c$
 - b) $c = h_i$ and $d_T(a, c) \leq (1 + \sqrt{2})x_c + y - y_c$





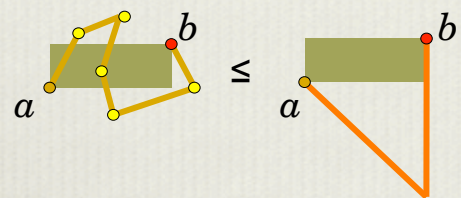
Main Thm's proof

(it's time to conclude!)

Case 2: c not in $R(a,b)$

$$y_b - y_c \leq x_b - x_c$$

$$y_b - y_c > x_b - x_c$$

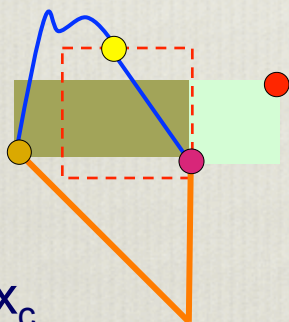


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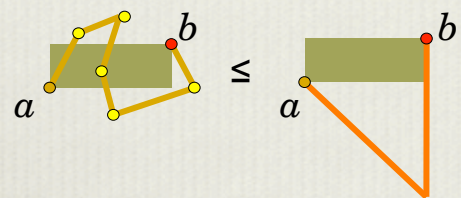
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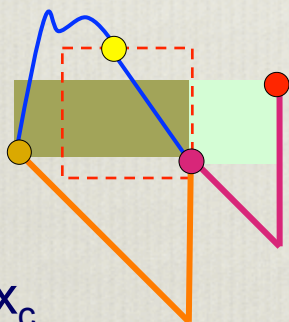


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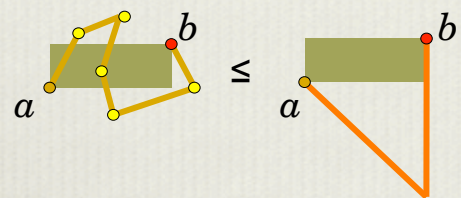
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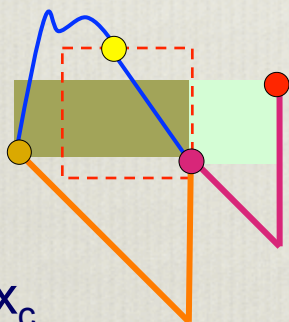


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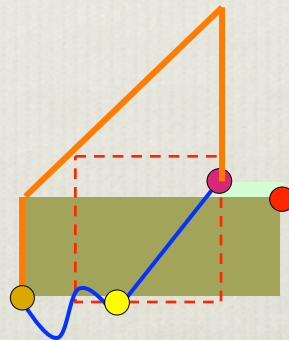
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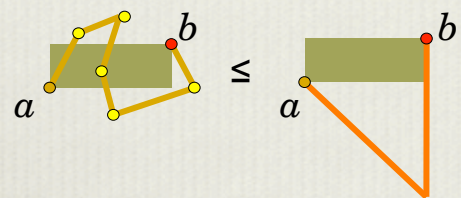
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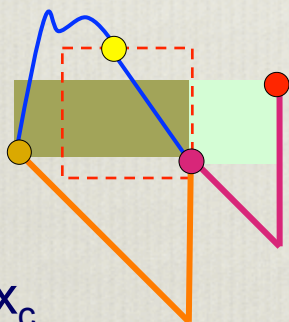


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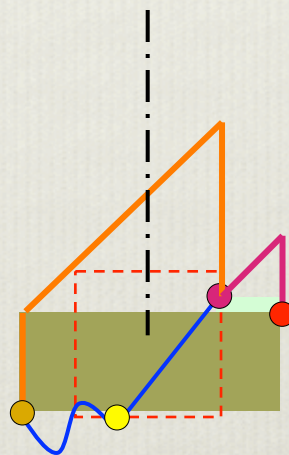
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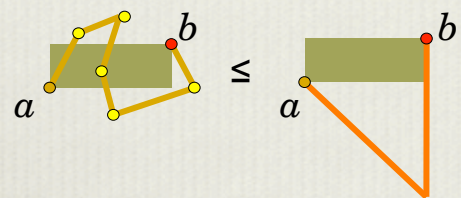
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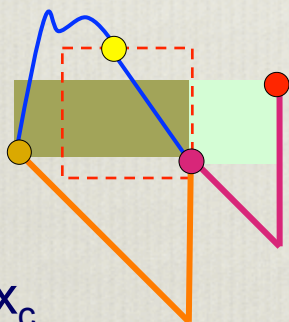


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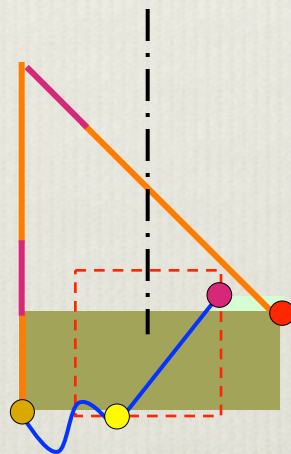
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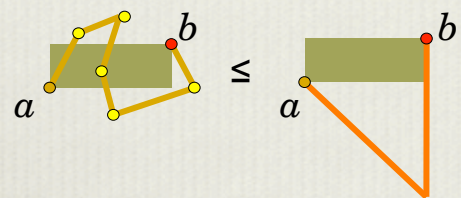
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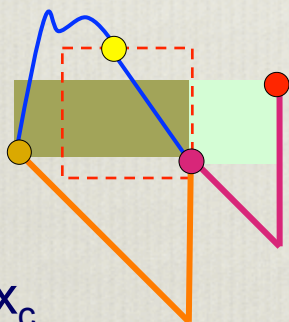


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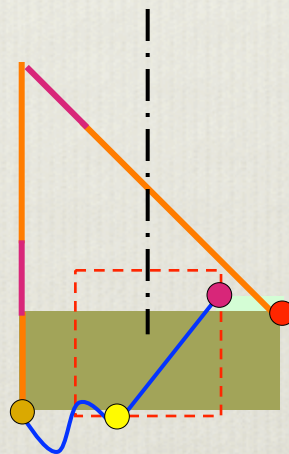
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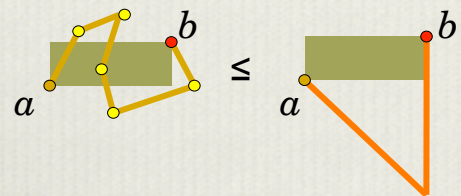
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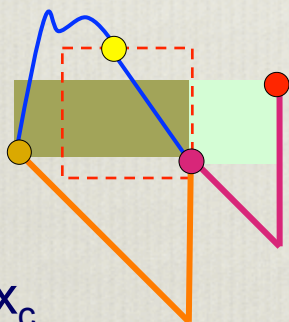


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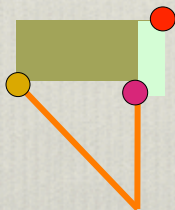
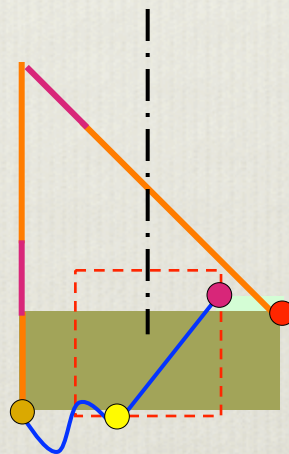
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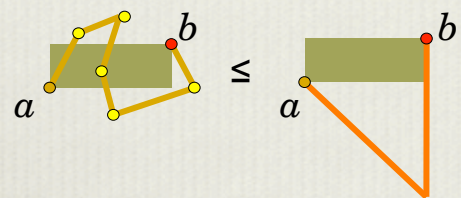
Case 2: c not in $R(a,b)$

✓ $y_b - y_c \leq x_b - x_c$



$y_b - y_c > x_b - x_c$



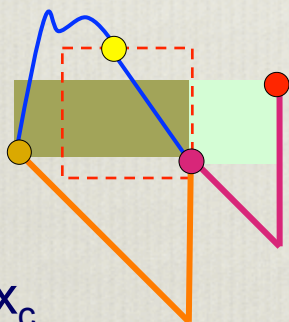


Main Thm's proof

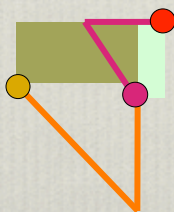
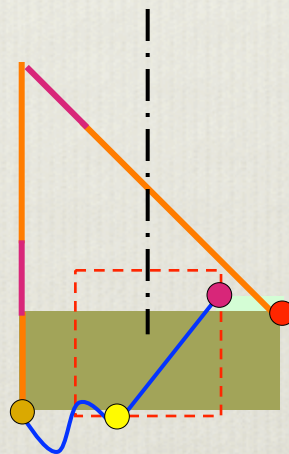
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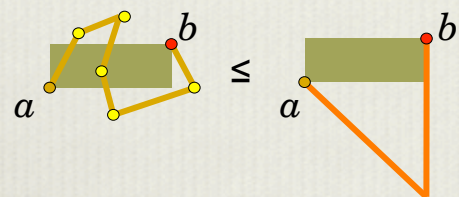
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too long ☹️

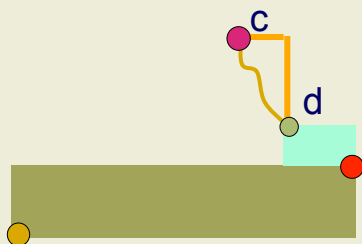


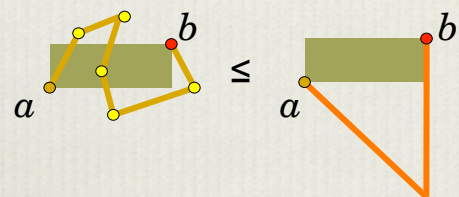
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Case 2: c not in $R(a,b)$

Monotonic lemma. If c is an inductive node, then there is d in $R(c,b)$ such that $d_T(c,d) \leq |x_c - x_d| + |y_c - y_d|$ and $|y_d - y_b| \leq |x_d - x_b|$.



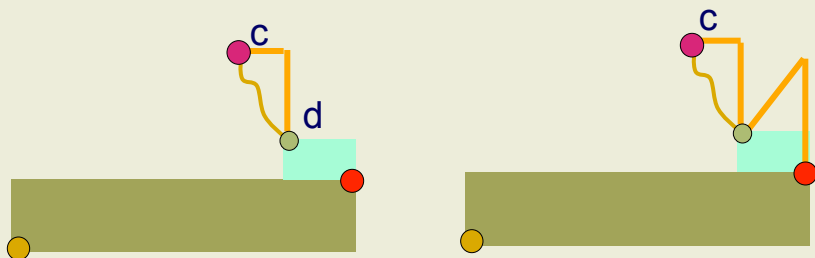


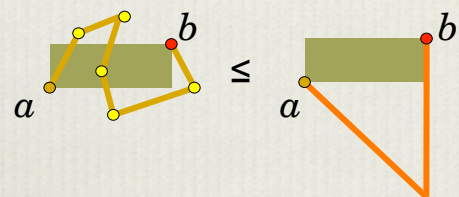
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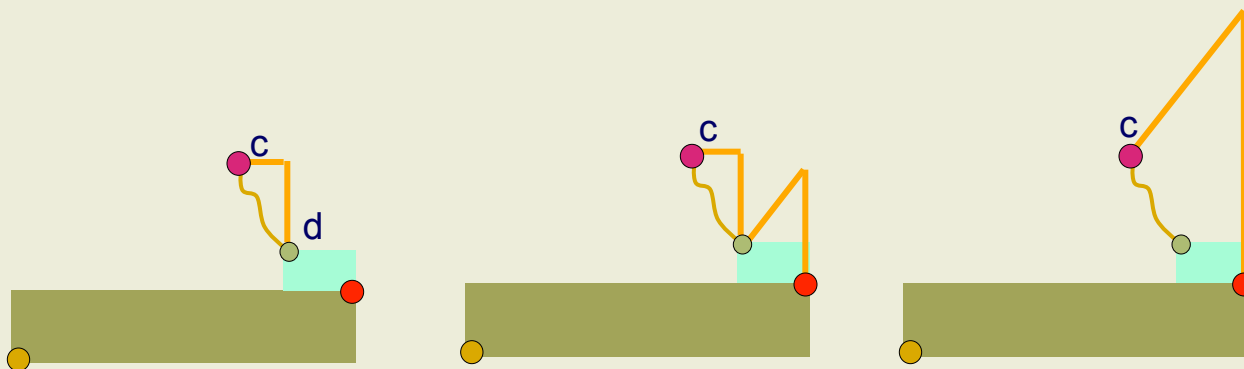


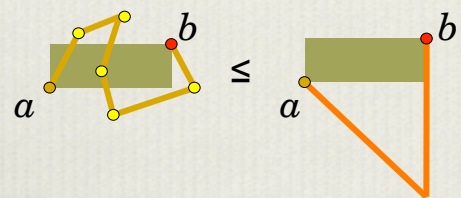
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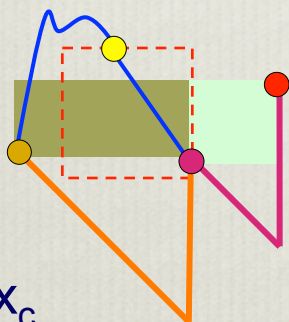


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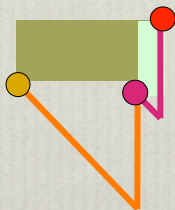
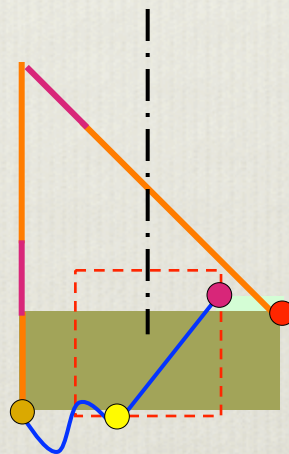
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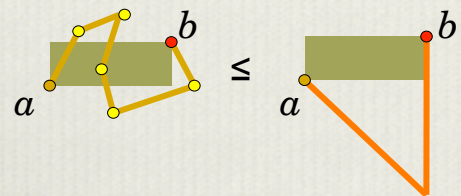
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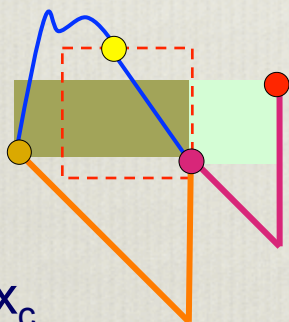


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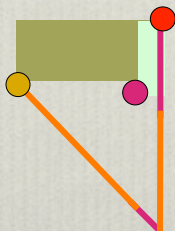
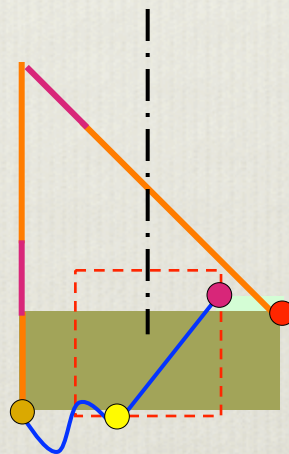
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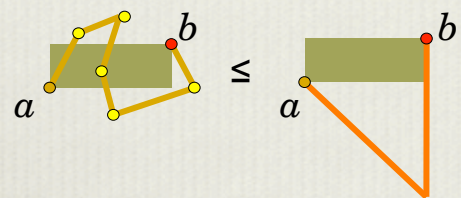
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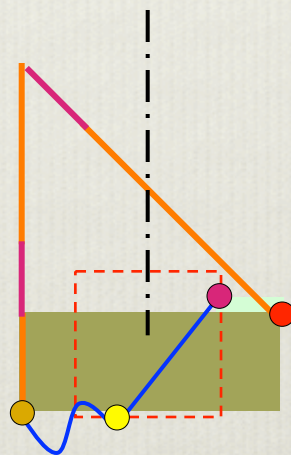
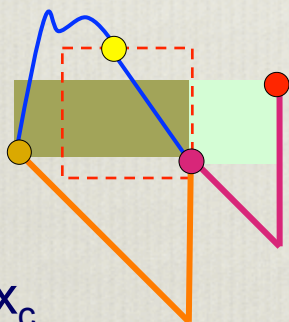


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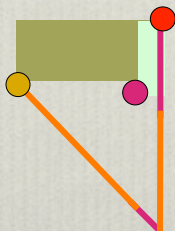
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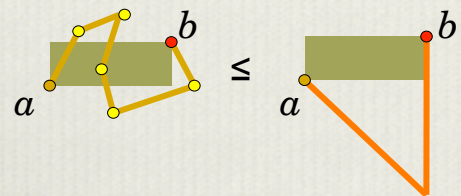
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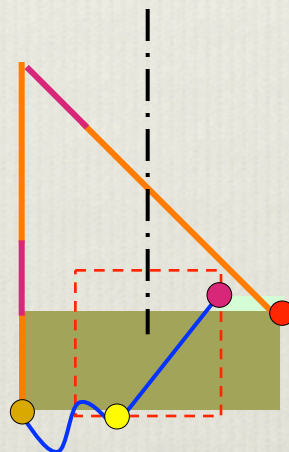
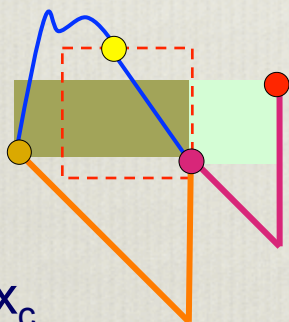


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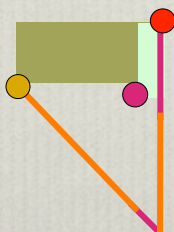
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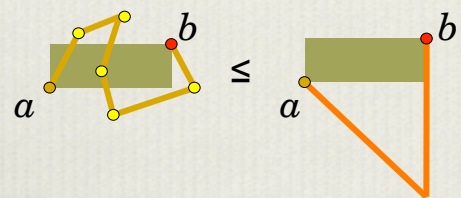
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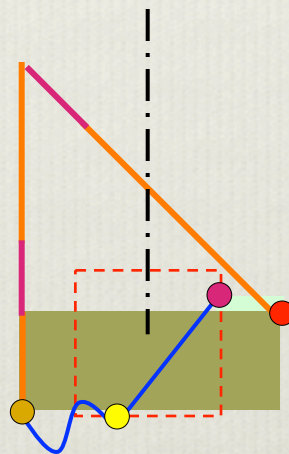
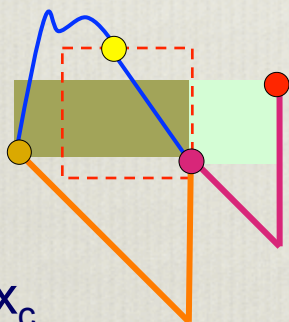
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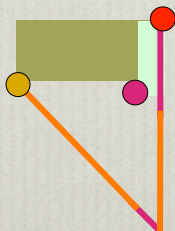
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Conclusion

Stretch for L_p -Delaunay triangulations

done for $p=1$ and $p=\infty$

Problem 1: $1 < p < \infty$? $p=2$?

Conclusion

Stretch for L_p -Delaunay triangulations

done for $p=1$ and $p=\infty$

Problem 1: $1 < p < \infty$? $p=2$?

Stretch for (regular k -gone)-Delaunay triangulations

done for $k=3$ (stretch=2) and $k=4$ (stretch ≈ 2.61)

Problem 2: $k > 4$? $k=\infty$?