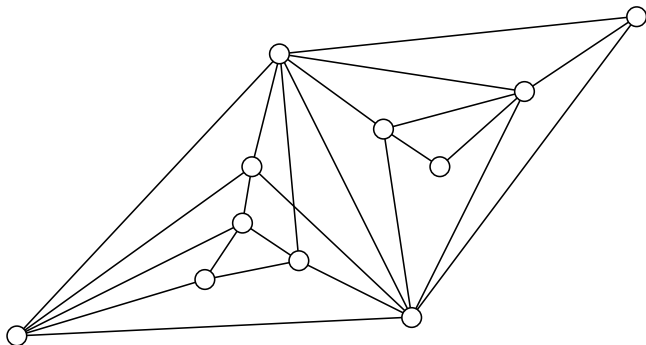


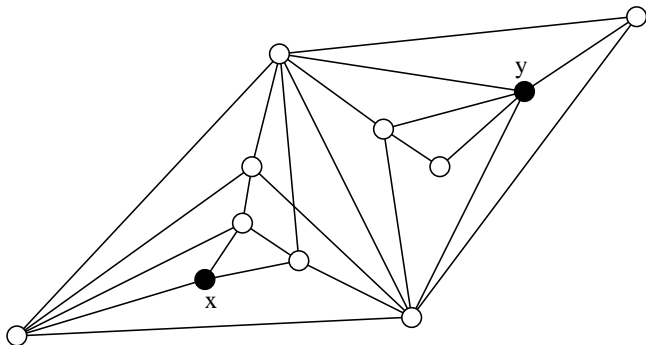
Distance labeling of permutation graphs

F. Bazzaro, C. Gavoille

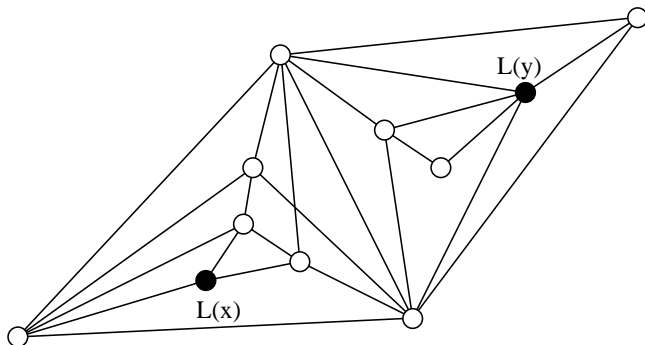
Distance labeling



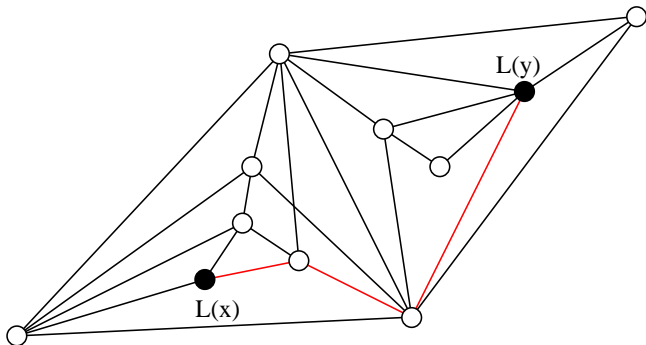
Distance labeling



Distance labeling



Distance labeling



$$\text{Dist}(L(x), L(y)) = \text{dist}_G(x, y) = 3$$

Existing results

Distance labeling schemes :

- proper interval graphs : $2 \lceil \log n \rceil$ [GP03]
- permutation graphs : $\log^2 n$ [KKP00]

Routing schemes :

- $\Delta \log n$ with an additive factor 1 [DL04]

Permutation graphs

A permutation graph $G[\pi] = (V, E)$ is defined as follows :

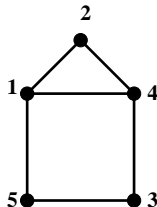
- $V = \{1, 2, \dots, n\}$
- $\{i, j\} \in E \Leftrightarrow (i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$

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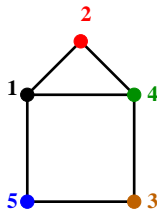
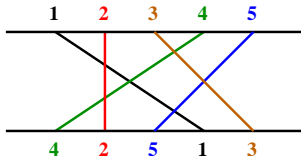
$$\pi = 4 2 5 1 3$$



Permutation graphs

A permutation graph is the intersection graph of segments between two parallel lines.

$$\pi = 42513$$



Non-separating linear extension

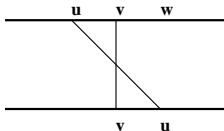
For all $u, v, w \in V$ such that:

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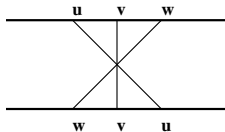
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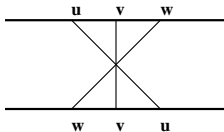
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Non-separating linear extension

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$\Rightarrow \{u, w\} \in E$

Non-separating linear extension

Lemma

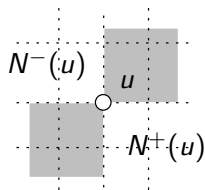
For all $u, v, w \in V$ such that $u \leq v \leq w$,

- if $\{u, w\} \in E$, then $\{u, v\} \in E$ or $\{v, w\} \in E$;
- if $\{u, w\} \notin E$, then $\{u, v\} \notin E$ or $\{v, w\} \notin E$.

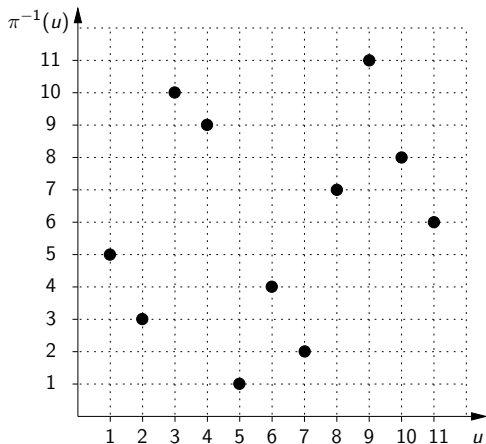
Permutation graph draw in the plane

Each vertex $i \in V$ is associated with the point of coordinates $(i, \pi^{-1}(i))$.

$$\{i, j\} \in E \Leftrightarrow (i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$$

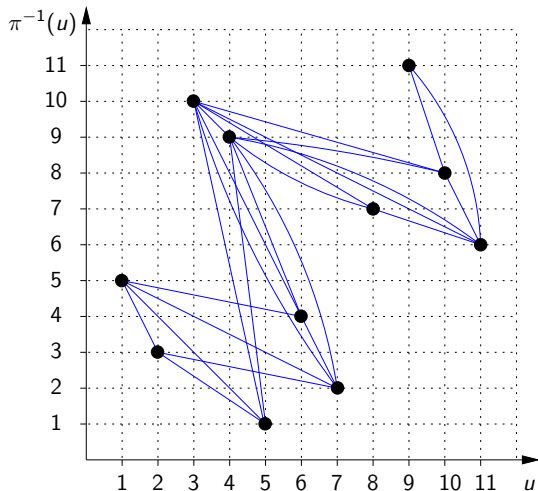


Permutation graph draw in the plane



$$\pi = 5, 7, 2, 6, 1, 11, 8, 10, 4, 3, 9$$

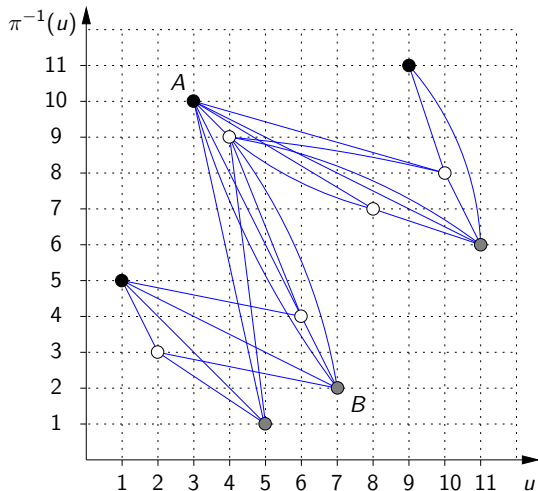
Neighbors



Particular sets

- $A = \{u \in V \mid N^-(u) = \emptyset\}$
- $B = \{u \in V \mid N^+(u) = \emptyset\}.$

Particular sets



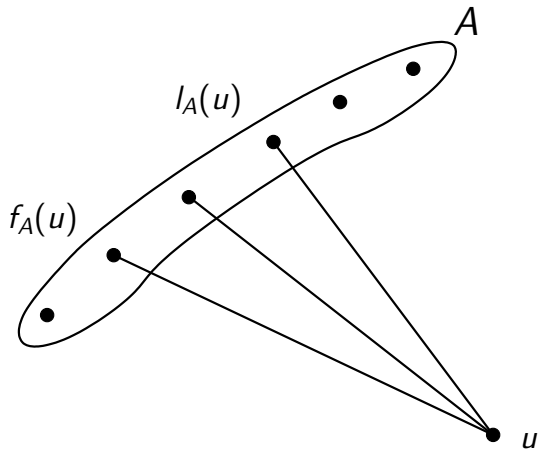
Adjacency

Lemma

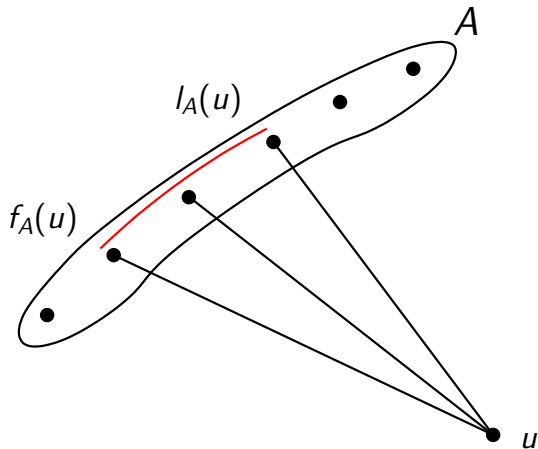
For every $u \in V$, there exist $f_a(u), l_a(u) \in A$ such that $f_a(u) \leq l_a(u)$ and $N[u] \cap A = [f_a(u), l_a(u)] \cap A$.

Similarly, there exist $f_b(u), l_b(u) \in B$ such that $f_b(u) \leq l_b(u)$ and $N[u] \cap B = [f_b(u), l_b(u)] \cap B$.

Adjacency



Adjacency

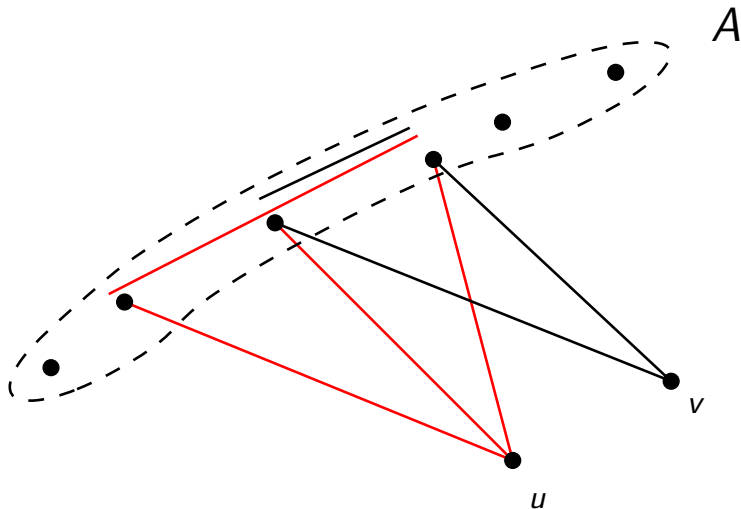


Overlapping

Lemma

Let $u < v$ be two non-adjacent vertices. Then, $f_a(u) \leq f_a(v)$, $l_a(u) \leq l_a(v)$, $f_b(u) \leq f_b(v)$, and $l_b(u) \leq l_b(v)$.

Overlapping



Shortest path

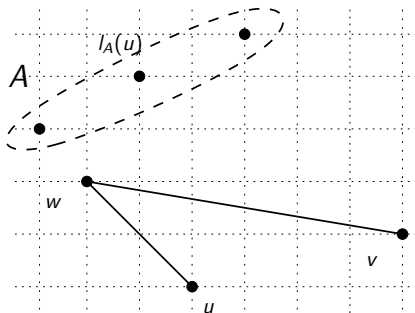
Lemma

Let $u < v$ be two non-adjacent vertices. Then there exists a shortest path $u, w_1, \dots, w_{k-1}, v$ such that $w_1 \in \{l_a(u), l_b(u)\}$, and, if $d_G(u, v) \geq 3$, $w_{k-1} \in \{f_a(v), f_b(v)\}$.

Shortest path

Lemma

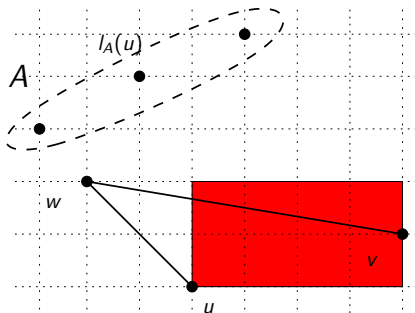
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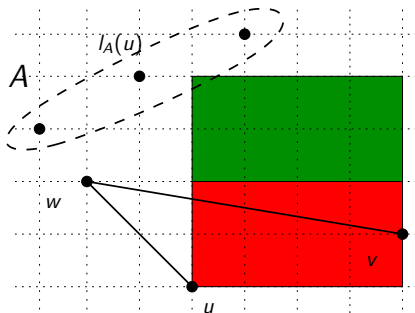
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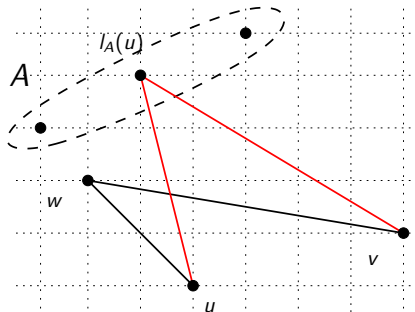
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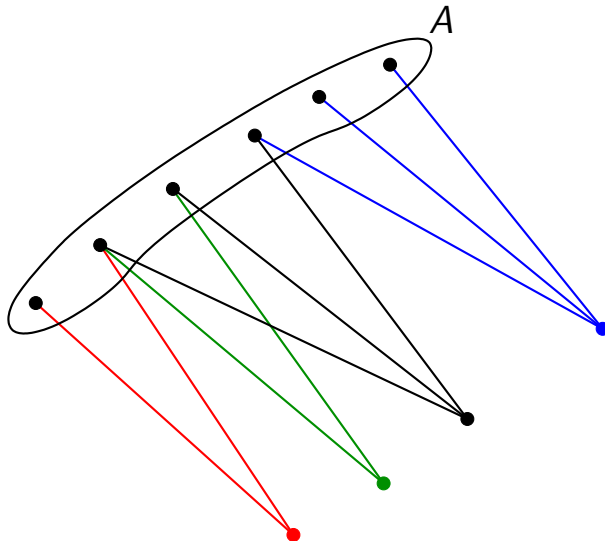


Interval graphs

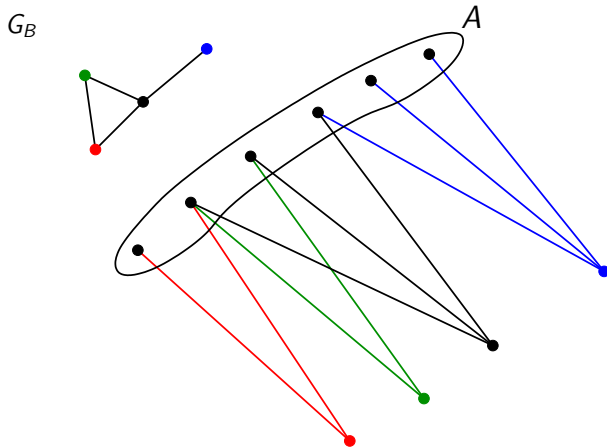
For each vertex u , we denote $I_B(u) = [f_a(u), l_a(u)]$ and $I_A(u) = [f_b(u), l_b(u)]$.

We denote by G_A the intersection graph defined by the set of intervals $\{I_A(u) \mid u \in A\}$. And G_B the intersection graph defined by the set of intervals $\{I_B(u) \mid u \in B\}$.

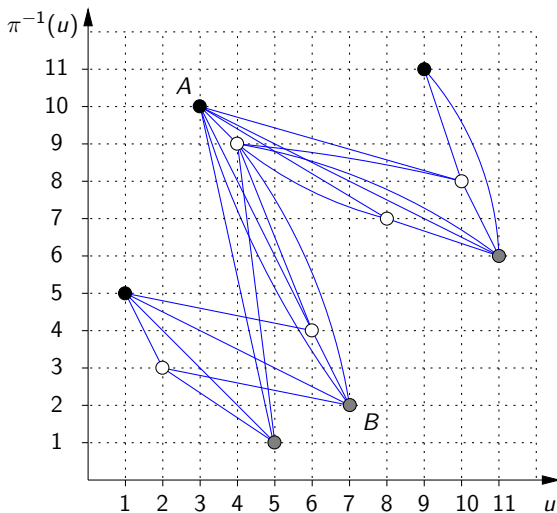
Interval graphs



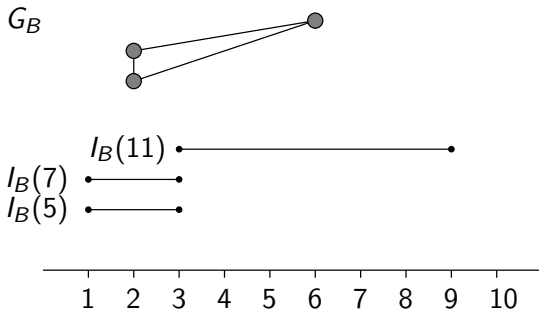
Interval graphs



Interval graphs



Interval graphs



Distance

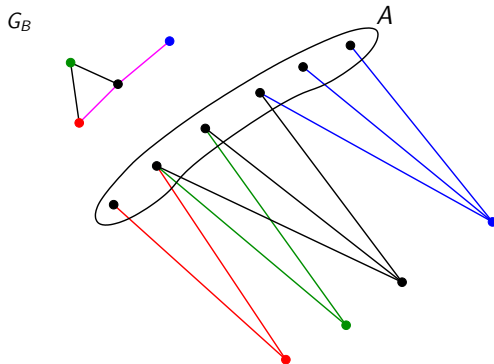
Lemma

For all $u, v \in A$, $d_G(u, v) = 2d_{G_A}(l_A(u), l_A(v))$. Similarly, for all $u, v \in B$, $d_G(u, v) = 2d_{G_B}(l_B(u), l_B(v))$.

Distance

Lemma

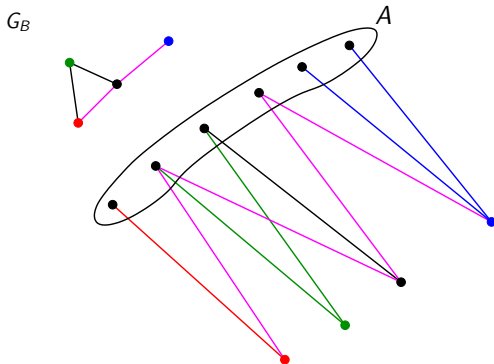
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Let u, v two vertices such that $u < v$. Then,

- 1 if $\pi^{-1}(u) > \pi^{-1}(v)$ then $d_G(u, v) = 1$;

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- ③ otherwise, if $f_a(v) \leq l_a(l_b(u))$ or $f_b(v) \leq l_b(l_a(u))$ then $d_G(u, v) = 3$;
- ④ otherwise, $d_G(u, v)$ is the minimum between:
 - $2 + 2d_{G_B}(l_B(l_b(u)), l_B(f_b(v)))$
 - $2 + 2d_{G_A}(l_A(l_a(u)), l_A(f_a(v)))$
 - $3 + 2d_{G_A}(l_A(l_a(l_b(u))), l_A(f_a(v)))$
 - $3 + 2d_{G_B}(l_B(l_b(l_a(u))), l_B(f_b(v)))$

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- $4 \lceil \log n \rceil$ for $l_a(l_b(u))$ and $l_b(l_a(u))$

Conclusion

Lemma

Labels are of $9 \lceil \log n \rceil + 6$ bits at most, and it takes constant time to decode the distance from the labels.

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Theorem

Permutation graphs have a shortest-path routing scheme with constant time protocol, and with $O(\log n)$ bit addresses, and, for every vertex u , the routing table of u is of size $O(\deg(u) \log n)$ bits. If the graph is bipartite then the size of the local routing tables can be reduced to $O(\log n)$ bit per vertex.

Conclusion

