Distance labeling of permutation graphs

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Distance labeling

$$\text{Dist}(L(x), L(y)) = \text{dist}_G(x, y) = 3$$
Existing results

Distance labeling schemes:

- proper interval graphs: $2 \log n$ [GP03]
- permutation graphs: $\log^2 n$ [KKP00]

Routing schemes:

- $\Delta \log n$ with an additive factor 1 [DL04]
A permutation graph $G[\pi] = (V, E)$ is defined as follows:

- $V = \{1, 2, \ldots, n\}$
- $\{i, j\} \in E \iff (i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$
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\[ \pi = 4 \ 2 \ 5 \ 1 \ 3 \]
Permutation graphs

A permutation graph is the intersection graph of segments between two parallel lines.

\[ \pi = 4\ 2\ 5\ 1\ 3 \]
Non-separating linear extension

For all $u, v, w \in V$ such that:

- $u < v < w$
- $\{u, v\}, \{v, w\} \in E$
Non-separating linear extension

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\[ \begin{array}{c}
\text{u} & \text{v} & \text{w} \\
\text{v} & \text{u} \\
\end{array} \]
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For all \( u, v, w \in V \) such that:

- \( u < v < w \)
- \( \{u, v\}, \{v, w\} \in E \)

\[ \Rightarrow \{u, w\} \in E \]
Non-separating linear extension

Lemma

For all $u, v, w \in V$ such that $u \leq v \leq w$,

- if $\{u, w\} \in E$, then $\{u, v\} \in E$ or $\{v, w\} \in E$;
- if $\{u, w\} \not\in E$, then $\{u, v\} \not\in E$ or $\{v, w\} \not\in E$. 

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Each vertex $i \in V$ is associated with the point of coordinates $(i, \pi^{-1}(i))$.

$$\{i, j\} \in E \iff (i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$$
Permutation graph draw in the plane

\[ \pi^{-1}(u) \]

\[ \pi = 5, 7, 2, 6, 1, 11, 8, 10, 4, 3, 9 \]
Neighbors

\[ \pi^{-1}(u) \]

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Particular sets

\[ A = \{ u \in V \mid N^-(u) = \emptyset \} \]

\[ B = \{ u \in V \mid N^+(u) = \emptyset \}. \]
Particular sets

\[ \pi^{-1}(u) \]

1 2 3 4 5 6 7 8 9 10 11

A

B

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Distance labeling of permutation graphs
Adjacency

**Lemma**

For every $u \in V$, there exist $f_a(u), l_a(u) \in A$ such that $f_a(u) \leq l_a(u)$ and $N[u] \cap A = [f_a(u), l_a(u)] \cap A$.

Similarly, there exist $f_b(u), l_b(u) \in B$ such that $f_b(u) \leq l_b(u)$ and $N[u] \cap B = [f_b(u), l_b(u)] \cap B$. 
Adjacency

\[ A \]

\[ l_A(u) \]

\[ f_A(u) \]

\[ u \]
Adjacency

\[ f_A(u) \]

\[ l_A(u) \]
Overlapping

Lemma

Let $u < v$ be two non-adjacent vertices. Then, $f_a(u) \leq f_a(v)$, $l_a(u) \leq l_a(v)$, $f_b(u) \leq f_b(v)$, and $l_b(u) \leq l_b(v)$. 
**Shortest path**

**Lemma**

Let \( u < v \) be two non-adjacent vertices. Then there exists a shortest path \( u, w_1, \ldots, w_{k-1}, v \) such that \( w_1 \in \{l_a(u), l_b(u)\} \), and, if \( d_G(u, v) \geq 3 \), \( w_{k-1} \in \{f_a(v), f_b(v)\} \).
Lemma

Let $u < v$ be two non-adjacent vertices. Then there exists a shortest path $u, w_1, \ldots, w_{k-1}, v$ such that $w_1 \in \{l_a(u), l_b(u)\}$, and, if $d_G(u, v) \geq 3$, $w_{k-1} \in \{f_a(v), f_b(v)\}$.
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Shortest path

Lemma

Let $u < v$ be two non-adjacent vertices. Then there exists a shortest path $u, w_1, \ldots, w_{k-1}, v$ such that $w_1 \in \{l_a(u), l_b(u)\}$, and, if $d_G(u, v) \geq 3$, $w_{k-1} \in \{f_a(v), f_b(v)\}$. 
Interval graphs

For each vertex $u$, we denote $I_B(u) = [f_a(u), l_a(u)]$ and $I_A(u) = [f_b(u), l_b(u)]$.

We denote by $G_A$ the intersection graph defined by the set of intervals $\{I_A(u) \mid u \in A\}$. And $G_B$ the intersection graph defined by the set of intervals $\{I_B(u) \mid u \in B\}$. 
Interval graphs
Interval graphs
Interval graphs
Interval graphs

$G_B$

$I_B(11)$

$I_B(7)$

$I_B(5)$
Distance

Lemma

For all $u, v \in A$, $d_G(u, v) = 2d_{G_A}(l_A(u), l_A(v))$. Similarly, for all $u, v \in B$, $d_G(u, v) = 2d_{G_B}(l_B(u), l_B(v))$. 
Lemma

For all $u, v \in A$, $d_G(u, v) = 2d_{G_A}(I_A(u), I_A(v))$. Similarly, for all $u, v \in B$, $d_G(u, v) = 2d_{G_B}(I_B(u), I_B(v))$. 
Lemma

For all \( u, v \in A \), \( d_G(u, v) = 2d_{G_A}(I_A(u), I_A(v)) \). Similarly, for all \( u, v \in B \), \( d_G(u, v) = 2d_{G_B}(I_B(u), I_B(v)) \).
Distance Theorem

Theorem

Let \( u, v \) two vertices such that \( u < v \). Then,

1. if \( \pi^{-1}(u) > \pi^{-1}(v) \) then \( d_G(u, v) = 1 \);
Distance Theorem

Theorem

Let $u, v$ two vertices such that $u < v$. Then,

1. if $\pi^{-1}(u) > \pi^{-1}(v)$ then $d_G(u, v) = 1$;
2. otherwise, if $f_a(v) \leq l_a(u)$ or $f_b(v) \leq l_b(u)$, then $d_G(u, v) = 2$;
Distance Theorem

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Let \( u, v \) two vertices such that \( u < v \). Then,

1. if \( \pi^{-1}(u) > \pi^{-1}(v) \) then \( d_G(u, v) = 1 \);
2. otherwise, if \( f_a(v) \leq l_a(u) \) or \( f_b(v) \leq l_b(u) \), then \( d_G(u, v) = 2 \);
3. otherwise, if \( f_a(v) \leq l_a(l_b(u)) \) or \( f_b(v) \leq l_b(l_a(u)) \) then \( d_G(u, v) = 3 \);
Distance Theorem

Theorem

Let $u, v$ two vertices such that $u < v$. Then,

1. if $\pi^{-1}(u) > \pi^{-1}(v)$ then $d_G(u, v) = 1$;
2. otherwise, if $f_a(v) \leq l_a(u)$ or $f_b(v) \leq l_b(u)$, then $d_G(u, v) = 2$;
3. otherwise, if $f_a(v) \leq l_a(l_b(u))$ or $f_b(v) \leq l_b(l_a(u))$ then $d_G(u, v) = 3$;
4. otherwise, $d_G(u, v)$ is the minimum between:
   - $2 + 2d_{GB}(l_B(l_b(u)), l_B(f_b(v)))$
   - $2 + 2d_{GA}(l_A(l_a(u)), l_A(f_a(v)))$
   - $3 + 2d_{GA}(l_A(l_a(l_b(u))), l_A(f_a(v)))$
   - $3 + 2d_{GB}(l_B(l_b(l_a(u))), l_B(f_b(v)))$
Label size

For each vertex $u$: 
Label size

For each vertex $u$:

- $2 \lceil \log n \rceil$ for the coordinates of $u$ in the plane
Label size

For each vertex $u$:

- $2 \lceil \log n \rceil$ for the coordinates of $u$ in the plane
- $4 \lceil \log n \rceil$ for $l_a(u)$ and $l_b(u)$
Label size

For each vertex $u$:

- $2 \lceil \log n \rceil$ for the coordinates of $u$ in the plane
- $4 \lceil \log n \rceil$ for $l_a(u)$ and $l_b(u)$
- $4 \lceil \log n \rceil$ for $f_a(u)$ and $f_b(u)$
Label size

For each vertex $u$:

- $2 \lceil \log n \rceil$ for the coordinates of $u$ in the plane
- $4 \lceil \log n \rceil$ for $l_a(u)$ and $l_b(u)$
- $4 \lceil \log n \rceil$ for $f_a(u)$ and $f_b(u)$
- $4 \lceil \log n \rceil$ for $l_a(l_b(u))$ and $l_b(l_a(u))$
Lemma

Labels are of $9 \lceil \log n \rceil + 6$ bits at most, and it takes constant time to decode the distance from the labels.
Conclusion

Lemma

Labels are of $9 \lceil \log n \rceil + 6$ bits at most, and it takes constant time to decode the distance from the labels.

Theorem

Permutation graphs have a shortest-path routing scheme with constant time protocol, and with $O(\log n)$ bit addresses, and, for every vertex $u$, the routing table of $u$ is of size $O(\deg(u) \log n)$ bits. If the graph is bipartite then the size of the local routing tables can be reduced to $O(\log n)$ bit per vertex.
Conclusion