

The relationship between minimum degree and maximum independent sets in planar graphs*

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Abstract

We explore the relationship between the minimum degree, $\delta(G)$, and the size of the maximum independent set, $\alpha(G)$, in a planar graph G with n vertices. For $\delta(G) \in \{0, 1, 2\}$, we prove that $\alpha(G) \leq n - \delta(G)$. This bound is tight in the sense that there exist planar graphs for which $\alpha(G) = n - \delta(G)$. When $\delta(G) \in \{3, 4, 5\}$ and G is a maximal planar graph, namely a triangulation, we prove that $\alpha(G) \leq (2n - 4)/\delta(G)$. This bound is also tight since for each $\delta \in \{3, 4, 5\}$ we show a family of maximal planar graphs of minimum degree δ such that $\alpha(G) = (2n - 4)/\delta$.

1 Introduction

The size of independent sets in planar graphs is an important parameter that is studied in graph theory and has algorithmic applications. For example, Kirkpatrick [20] and Dobkin and Kirkpatrick [16] used repeated removal of independent sets from a triangulation to design a data structure for efficient planar point location and a data structure for efficient distance computation between convex polytopes, respectively. Biedl and Wilkinson [7] studied the size of bounded degree independent sets in triangulations. The celebrated 4-color theorem [3, 26] immediately implies that every planar graph has an independent set of size at least $n/4$ where n is the number of vertices in the graph. Surprisingly, this bound is the best achievable as there are planar graphs that do not have a larger independent set; for example consider disjoint copies of the complete graph with 4 vertices. Some weaker lower bounds are also known [1, 13] that avoid the complexity of the 4-color theorem (as per suggestion of Erdős [5]).

From the algorithmic point of view, the problem of finding maximum independent set in planar graphs is NP-hard even if restricted to planar graphs of maximum degree 3 [18, 25] or planar triangle-free graphs [23]. The focus has then been turned into finding large independent sets through approximation algorithms

[2, 4, 10, 11, 17, 23], parallel algorithms [12, 14, 19], or in some minor-free planar graphs [17, 22, 24].

The independent set problem has also been studied on planar geometric graphs such as relative-neighborhood graphs [21], Delaunay triangulations [8], and half-Theta-six graphs [6].

In this paper, we explore the relationship between the minimum degree of a vertex and the size of the maximum independent set in planar graphs. We show that for certain classes of planar graphs, there are large independent sets, much larger than $n/4$ and that the minimum degree plays a role in these graphs. Intuitively, a planar graph with many low degree vertices should have large independent sets. We give tight bounds that corroborate this intuition.

2 Notation and Preliminaries

Given a set S , we denote its cardinality by $|S|$. We refer the reader to Bondy and Murty [9] or Diestel [15] for graph theoretic notation and some well-known graph-theoretic results we use such as Euler’s formula for planar graphs. Let $G = (V(G), E(G))$ be a graph where $V(G)$ is the vertex set and $E(G)$ is the edge set. We denote the degree of a vertex v as $\deg(v)$. We adopt the convention that $n = |V(G)|$ and $m = |E(G)|$. Given $V' \subseteq V(G)$, the graph *induced* on V' , denoted as $G[V']$, is the graph with vertex set V' that contains all edges of G that have both endpoints in V' . A subset $V' \subseteq V(G)$ is an independent set provided that $G[V']$ has no edges. Let $\alpha(G)$ represent the size of the maximum independent set in G . Let $\delta(G)$ be the minimum degree of a vertex in G , i.e. $\delta(G) = \min\{\deg(v) : v \in V(G)\}$. When G is clear from the context, to reduce notational clutter, we refer to these parameters as α and δ , respectively. A graph G is *planar* provided that it has a planar embedding, i.e. it can be drawn in the plane such that its edges do not cross each other. Throughout this paper the term “planar graph” refers to both the abstract graph and a planar embedding of the graph (with straight-line edges), interchangeably. We denote the set of faces of a planar graph G by $F(G)$. The *degree* of a face of a planar graph is the number of edges on the boundary of the face. We denote the degree of a face f as $\deg(f)$.

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3 Warm up

In this section, we consider planar graphs G such that $\delta(G) \in \{0, 1, 2\}$. The results in this section are rather simple and trivial, however, they give an intuition of when and where the non-planarity can break the trend in the relationship between $\delta(G)$ and $\alpha(G)$.

Observation 1 *For every n -vertex planar graph G with $\delta(G) = 0$, $\alpha(G) \leq n$. For every $n \geq 1$, there exists an n -vertex planar graph G with $\delta(G) = 0$ and $\alpha(G) = n$.*

Proof. The upper bound follows since $|V(G)| = n$. A planar graph G on n vertices with no edges has $\delta(G) = 0$ and $\alpha(G) = n$. \square

Lemma 1 *For every n -vertex planar graph G with $\delta(G) = 1$, $\alpha(G) \leq n - 1$. For every $n \geq 2$, there exists an n -vertex planar graph G with $\delta(G) = 1$ and $\alpha(G) = n - 1$.*

Proof. For the upper bound, let x be a vertex in a maximum independent set of G . Since $\deg(x) \geq 1$, x has at least one neighbor that cannot be in the independent set; this gives the upper bound of $n - 1$. For the lower bound, let G be a tree on $n \geq 2$ vertices that has one vertex v adjacent to all other vertices. This graph is sometimes called a *star*. The degree of v is $n - 1$ and the vertices adjacent to v form an independent set of size $n - 1$. \square

Lemma 2 *For every n -vertex planar graph G with $\delta(G) = 2$, $\alpha(G) \leq n - 2$. For every $n \geq 3$, there exists an n -vertex planar graph G with $\delta(G) = 2$ and $\alpha(G) = n - 2$.*

Proof. For the upper bound, let x be a vertex in a maximum independent set of G . Since $\deg(x) \geq 2$, x has at least 2 neighbors that cannot be in the independent set; this gives the upper bound of $n - 2$. For the lower bound, let G be $K_{2,n-2}$ which is the complete bipartite graph with bipartitions of size 2 and $n - 2$. This graph is planar, has minimum degree 2, contains at least 3 vertices, and has an independent set of size $n - 2$. \square

Notice that the above process of having 1 or 2 vertices dominating the independent set in a planar graph breaks down when $\delta(G) \geq 3$. Since $K_{3,3}$ is not planar [27], it is not possible to construct a planar graph with 3 vertices x, y, z and an independent set of at least 3 vertices of degree-3 where each vertex is joined to x, y, z . The following theorem summarizes the results of this section.

Theorem 3 *For every n -vertex planar graph G with $\delta(G) \in \{0, 1, 2\}$, $\alpha(G) \leq n - \delta(G)$. For any $n \geq 3$ and any $d \in \{0, 1, 2\}$, there exists an n -vertex planar graph G with $\delta(G) = d$ and $\alpha(G) = n - d$.*

The minimum size of a maximum independent set over all planar graphs G on n vertices is $n/4$. The upper bound follows from the 4-color theorem. The lower bound follows from a disjoint collection of complete graphs on 4 vertices, i.e. K_4 . As such, we turn our attention to the relationship between $\delta(G)$ and $\alpha(G)$ in maximal planar graphs, i.e. triangulations.

4 Upper Bounds

In this section, we prove some upper bounds on the size of the maximum independent set in a triangulation G as a function of the minimum degree $\delta(G)$. Since G is planar, $\delta(G) \leq 5$. Since G is a triangulation, $\delta(G) \geq 3$. Therefore, $\delta(G) \in \{3, 4, 5\}$.

Lemma 4 *Let G be an n -vertex planar graph and let F_x be the number of faces of degree at least x , with $x \geq 3$, in G . Then $F_x \leq (2n - 4)/(x - 2)$.*

Proof. Let G' be the graph obtained from G by triangulating every face of degree at least x . By construction, G' has n vertices and is planar. The number of triangles in G' is at least $(x - 2)F_x$ since every face of degree at least x gives rise to at least $x - 2$ triangular faces. Thus $(x - 2)F_x \leq |F(G')|$. Since G' is planar, $|F(G')| \leq 2n - 4$; this upper bound holds for any planar graph G' regardless of its connectivity, see e.g. [9]. Therefore $F_x \leq (2n - 4)/(x - 2)$. \square

Theorem 5 *Let G be any n -vertex maximal planar graph with $n \geq 3$. Then $\alpha(G) \leq (2n - 4)/\delta(G)$.*

Proof. Let I be a maximum independent set in G , so with $\alpha(G) = |I|$. Let $V' = V - I$. Let $H = G[V']$. Since G is maximal (i.e. a triangulation) and the vertices in I are independent, the boundary of every face f of H is a simple polygon containing at most one vertex v of I . For any such f and v , the vertex v is connected to every vertex of f in G because G is maximal, and thus $\deg(f) = \deg(v)$. As the minimum vertex degree is $\delta(G)$, only the faces of H with degree at least $\delta(G)$ can contain vertices of I . Therefore $|I| \leq F_\delta$, where F_δ is the number of faces in H with degree at least $\delta(G)$. By applying Lemma 4 on faces of degree at least $\delta(G)$ in H , we have that $F_\delta \leq (2|V'| - 4)/(\delta(G) - 2)$. Since $V' = V - I$, we can rewrite this inequality in terms of $n = |V|$,

$$\begin{aligned} |I| &\leq F_\delta \\ &\leq (2|V'| - 4)/(\delta(G) - 2) \\ &= (2n - 2|I| - 4)/(\delta(G) - 2). \end{aligned}$$

Rearranging the previous inequality gives

$$|I| = \alpha(G) \leq (2n - 4)/\delta(G).$$

\square

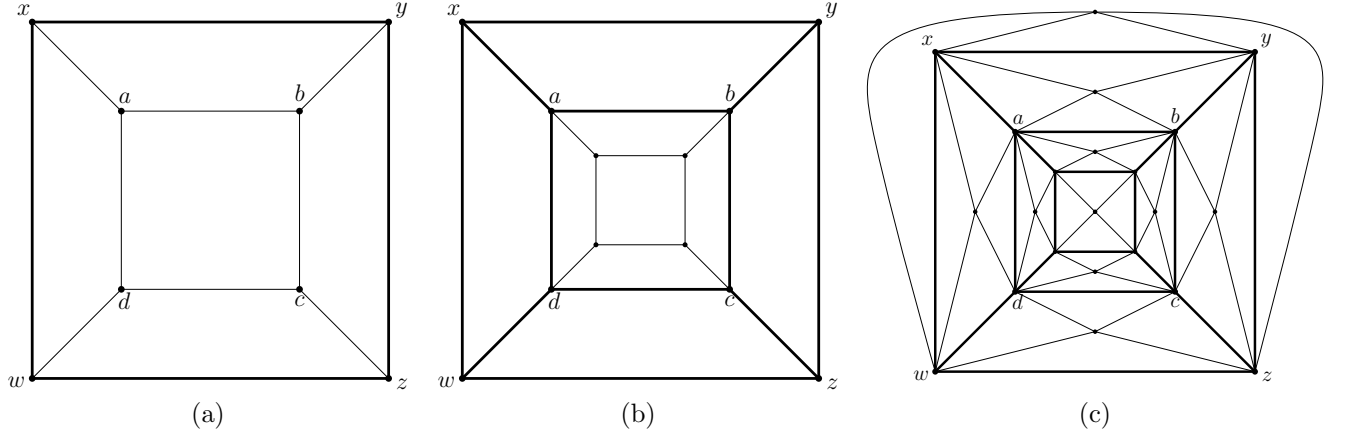


Figure 1: (a) An example of H_1 (bold edges represent H_0). (b) An example of H_2 (bold edges represent H_1). (c) Triangulation obtained from H_2 (bold edges represent H_2).

5 Lower Bounds

In this section, we show that the upper bound shown in Theorem 5 is tight for each $\delta \in \{3, 4, 5\}$; this is shown by constructing an infinite family of triangulations that attain the claimed bound for α . We begin by introducing some families of graphs that will form the underlying *skeleton* for the construction of these triangulations.

We first introduce the skeleton used for building our triangulations of minimum degree 3. For $i \geq 0$, let T_i be any triangulation on $i+3$ vertices. We note that a simple application of Euler's formula for planar graphs allows us to conclude that $|V(T_i)| = i+3$, $|E(T_i)| = 3i+3$ and $|F(T_i)| = 2i+2$.

Next, we introduce the skeleton used for building our triangulations with minimum degree 4. We recursively define a family of planar graphs where every face is a quadrilateral. Let H_0 be a 4-cycle. For $i \geq 1$, to construct H_i , take a copy of H_{i-1} . Add a 4-cycle a, b, c, d inside the innermost quadrilateral face x, y, z, w of H_{i-1} . Add the edges ax, by, cz and dw . The graph H_i is planar, every face is a 4-cycle by construction and it has 4 more vertices than H_{i-1} . We note that $|V(H_i)| = 4i+4$, $|E(H_i)| = 8i+4$ and $|F(H_i)| = 4i+2$. See Figure 1(a) for an example of H_1 , where we bolded the edges of H_0 . An example of H_2 is illustrated in Figure 1(b).

The skeleton used to build triangulations with minimum degree 5 is constructed as follows. Let D be the 15 vertex graph shown in Figure 2(a). This graph is planar and every face except the outer-face is a 5-cycle. We use the graph D in our recursive construction, much the same way we used the 4-cycles in the previous construction. The base graph is a 5-cycle denoted S_0 . The invariant we maintain is that every face of S_i is a 5-cycle. For $i \geq 1$, to construct S_i , take a copy of S_{i-1} which by the invariant has that every face is a 5-cycle. Add D inside the inner most 5-cycle and connect the edges as shown in Figure 2(b). By construction, the graph S_i is

planar and every face is a 5-cycle. Thus, the invariant is maintained. S_i has 15 more vertices than S_{i-1} since we added D to S_{i-1} . We note that $|V(S_i)| = 15i+5$, $|E(S_i)| = 25i+5$ and $|F(S_i)| = 10i+2$. See Figure 2 for an example of S_1 , where we bolded the edges of S_0 . An example of S_2 is illustrated in Figure 2(c).

Based on these skeletons, we prove the existence of infinite families of triangulations that achieve the bounds proven in Theorem 5.

Lemma 6 *For all $i \geq 0$, there exists a triangulation T on $n = 3i + 5$ vertices, with $\delta(T) = 3$ and $\alpha(T) \geq (2n - 4)/\delta(T)$.*

Proof. For $i \geq 0$, we build a triangulation by starting with a copy of the skeleton T_i , which is an arbitrary triangulation with $i+3$ vertices. Thus $|V(T_i)| = i+3$, and recall that $|F(T_i)| = 2i+2$. Place a vertex v in each face of T_i (including the outerface) and add 3 edges from v to the vertices on the face containing v in T_i . This results in a triangulation T with minimum degree 3. Moreover, T has an independent set of size at least $|F(T_i)| = 2i+2$ because none of the vertices added to T_i are adjacent. In other words, the vertices added to T_i form an independent set in T . The number n of vertices in T is $|V(T_i)| + |F(T_i)| = 3i+5$, since we add $2i+2$ vertices to T_i in the construction of T . We can rewrite $i = (n-5)/3$ which gives the desired bound since $2i+2 = (2n-4)/3$. \square

Lemma 7 *For all $i \geq 0$, there exists a triangulation T on $n = 8i + 6$ vertices, with $\delta(T) = 4$ and $\alpha(T) \geq (2n - 4)/\delta(T)$.*

Proof. For $i \geq 0$, we build a triangulation by starting with a copy of the skeleton H_i . Recall that $|V(H_i)| = 4i+4$ and $|F(H_i)| = 4i+2$. Place a vertex v in each face of T_i and add the 4 edges from v to the vertices

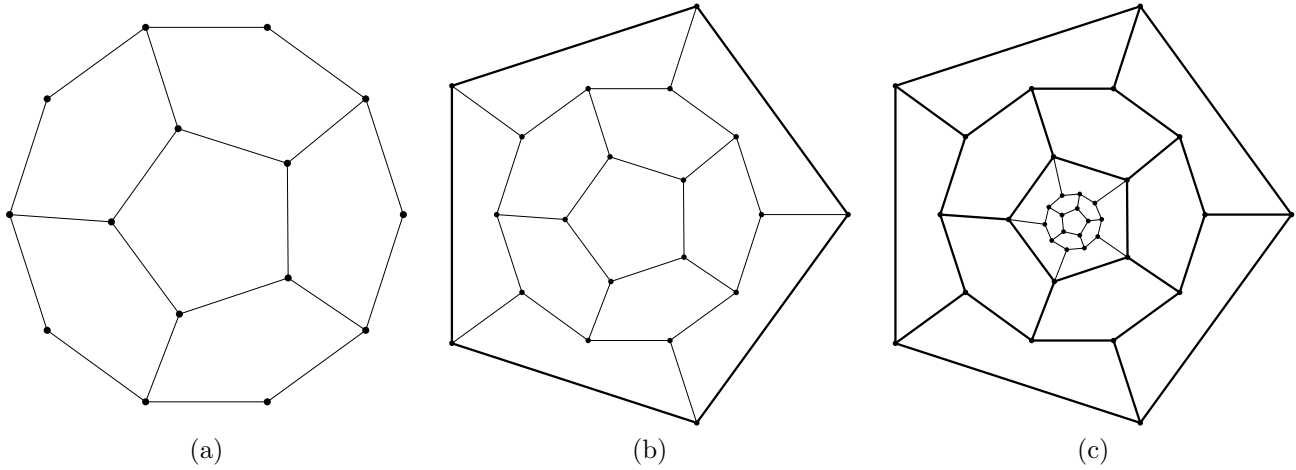


Figure 2: (a) The graph D that is added in the recursive construction of S_i . (b) An example of S_1 (bold edges represent S_0). (c) An example of S_2 (bold edges represent S_1).

on the face containing v in H_i . This results in a triangulation T with minimum degree 4. Moreover, T has an independent set of size at least $|F(H_i)| = 4i + 2$ because none of the vertices added to H_i are adjacent by construction. The number n of vertices in T is $|V(H_i)| + |F(H_i)| = 8i + 6$, since we add $4i + 2$ vertices to H_i in the construction of T . We can rewrite $i = (n - 6)/8$ which gives the desired bound since $4i + 2 = (2n - 4)/4$. \square

See Figure 1(c) for an example of the triangulation we obtain using H_2 as the underlying skeleton.

Lemma 8 *For all $i \geq 0$, there exists a triangulation T on $n = 25i + 7$ vertices, with $\delta(T) = 5$ and $\alpha(T) \geq (2n - 4)/\delta(T)$.*

Proof. For $i \geq 0$, we build a triangulation T by starting with a copy of S_i . Recall that $|V(S_i)| = 15i + 5$ and $|F(S_i)| = 10i + 2$. Place a vertex v in each face of S_i and add the 5 edges from v to the vertices on the face containing v in S_i . This results in a triangulation T that has an independent set of size at least $|F(S_i)| = 10i + 2$ since none of the vertices added to S_i are adjacent by construction. The reason that in the construction of S_i for $i \geq 1$, we added the graph D as opposed to a 5-cycle is to ensure that the minimum degree is 5. Note that $\delta(S_i) = 3$ for $i \geq 1$. Thus, when the vertices are added to each face of S_i , we have that all the vertices of S_i have degree at least 6 and each of the vertices of the independent set has degree 5. Thus, $\delta(T) = 5$. The number n of vertices in T is $|V(S_i)| + |F(S_i)| = 25i + 7$, since we add $10i + 2$ vertices to S_i in the construction of T . We can rewrite $i = (n - 7)/25$ which gives the desired bound since $10i + 2 = (2n - 4)/5$. \square

Combining Lemma 6, Lemma 7, Lemma 8, and Theorem 5, the following theorem summarizes our result.

Theorem 9 *For any $d \in \{3, 4, 5\}$, there exists an infinite family of triangulations \mathcal{G} such that for any $G \in \mathcal{G}$, we have that $\delta(G) = d$ and $\alpha(G) = (2|V(G)| - 4)/d$.*

6 Conclusions

We studied the relationship between the minimum degree, δ , and the size of the maximum independent set, α , in planar graphs. We proved tight upper bounds for α —with respect to δ —in general planar graphs with $\delta \in \{0, 1, 2\}$ and in maximal planar graphs with $\delta \in \{3, 4, 5\}$. It would be interesting to find tight upper bounds for α in general planar graphs with $\delta \in \{3, 4, 5\}$.

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