On Space-Stretch Trade-Offs: Lower Bounds

Ittai Abraham\textsuperscript{1} \hspace{1em} Cyril Gavoille\textsuperscript{2} \hspace{1em} Dahlia Malkhi\textsuperscript{1,3}

\textsuperscript{1}Hebrew University of Jerusalem, Israel
\textsuperscript{2}University of Bordeaux, France
\textsuperscript{3}Microsoft Research

SPAA 2006
The Compact Routing Problem

Input: a network $G$ (a weighted connected graph)

Output: a routing scheme for $G$

A routing scheme is a distributed algorithm that allows any source node to route messages to any destination node, given the destination’s network identifier.
The Compact Routing Problem

**Input:** a network $G$ (a weighted connected graph)

**Output:** a **routing scheme** for $G$

A routing scheme is a distributed algorithm that allows **any** source node to route messages to **any** destination node, given the destination’s network identifier

**Goal:** to minimize the size of the routing tables
Example: Grid with X,Y-coordinates

Routing algorithm: X,Y-routing
Example: Grid with X,Y-coordinates

Routing algorithm: X,Y-routing
Complexity Measures: Space & Stretch

\textbf{Space} = \text{size of the largest local routing tables}

\textbf{Stretch} = \text{ratio between length of the route and distance} \\
|\text{route}(x, y)| \leq \text{stretch} \cdot \text{dist}(x, y)

In the example: space = \(O(\log n)\) bits

In the example: stretch = 1 (shortest path)

Question: for a given family of graphs, find the best space-stretch trade-off
Complexity Measures: Space & Stretch

**Space** = size of the largest local routing tables

(more precisely, size of the smallest local routing algorithm including all constants and data-structures)

In the example: \( \text{space} = O(\log n) \) bits

**Stretch** = ratio between length of the route and distance

\[ |\text{route}(x, y)| \leq \text{stretch} \cdot \text{dist}(x, y) \]

In the example: stretch = 1 (shortest path)

Question: for a given family of graphs, find the best space-stretch trade-off
Complexity Measures: Space & Stretch

**Space** = size of the largest local routing tables

(more precisely, size of the smallest local routing algorithm including all constants and data-structures)

In the example: space = $O(\log n)$ bits

**Stretch** = ratio between length of the route and distance

$$|\text{route}(x, y)| \leq \text{stretch} \cdot \text{dist}(x, y)$$

In the example: stretch = 1 (shortest path)
Complexity Measures: Space & Stretch

**Space** = size of the largest local routing tables

(more precisely, size of the smallest local routing algorithm including all constants and data-structures)

In the example: space = $O(\log n)$ bits

**Stretch** = ratio between length of the route and distance

$$|\text{route}(x, y)| \leq \text{stretch} \cdot \text{dist}(x, y)$$

In the example: stretch = 1 (shortest path)

**Question:** for a given family of graphs, find the best space-stretch trade-off
Two variants: Name-independent vs. Labeled

The destination enters the network with its name, which is determined by either the designer of the routing scheme (labeled), or an adversary (name-independent).

**Labeled:** the designer is free to name the nodes according to the topology and the edge weights of the graph

**Name-independent:** the input is a graph with fixed node names
An overview: Labeled model

Labels are of polylogarithmic size
\( \tilde{O}(f(n)) = f(n) \cdot \text{polylog}(n) \)

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>1</td>
<td>( \tilde{O}(n) )</td>
</tr>
</tbody>
</table>
An overview: Labeled model

Labels are of polylogarithmic size
\( \tilde{O}(f(n)) = f(n) \cdot \text{polylog}(n) \)

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>1</td>
<td>( \tilde{O}(n) )</td>
</tr>
<tr>
<td>4k - 5</td>
<td>( \tilde{O}(n^{1/k}) )</td>
<td>[Thorup, Zwick]</td>
</tr>
</tbody>
</table>
An overview: Labeled model

Labels are of polylogarithmic size
\[ \tilde{O}(f(n)) = f(n) \cdot \text{polylog}(n) \]

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>1</td>
<td>( \tilde{O}(n) )</td>
</tr>
<tr>
<td></td>
<td>4( k - 5 )</td>
<td>( \tilde{O}(n^{1/k}) )</td>
</tr>
<tr>
<td>tree</td>
<td>1</td>
<td>( \tilde{O}(1) )</td>
</tr>
</tbody>
</table>
An overview: Labeled model

Labels are of polylogarithmic size
\[ \tilde{O}(f(n)) = f(n) \cdot \text{polylog}(n) \]

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>1</td>
<td>( \tilde{O}(n) )</td>
</tr>
<tr>
<td></td>
<td>4( k - 5 )</td>
<td>( \tilde{O}(n^{1/k}) )</td>
</tr>
<tr>
<td>tree</td>
<td>1</td>
<td>( \tilde{O}(1) )</td>
</tr>
<tr>
<td>doubling-( \alpha ) dim.</td>
<td>( 1 + \varepsilon )</td>
<td>( \log \Delta )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \tilde{O}(1) )</td>
</tr>
</tbody>
</table>
An overview: Labeled model

Labels are of polylogarithmic size
\[ \tilde{O}(f(n)) = f(n) \cdot \text{polylog}(n) \]

<table>
<thead>
<tr>
<th></th>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>1</td>
<td>\tilde{O}(n)</td>
<td>[\text{folk}]</td>
</tr>
<tr>
<td></td>
<td>4k - 5</td>
<td>\tilde{O}(n^{1/k})</td>
<td>[\text{Thorup,Zwick}]</td>
</tr>
<tr>
<td>tree</td>
<td>1</td>
<td>\tilde{O}(1)</td>
<td>[\text{TZ/Fraigniaud,G.}]</td>
</tr>
<tr>
<td>doubling-(\alpha) dim.</td>
<td>1 + (\varepsilon)</td>
<td>\log \Delta</td>
<td>[\text{Talwar/Slivkins}]</td>
</tr>
<tr>
<td>planar</td>
<td>1 + (\varepsilon)</td>
<td>\tilde{O}(1)</td>
<td>[\text{Chan et al./Abraham et al.}]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>[\text{Thorup}]</td>
</tr>
</tbody>
</table>
An overview: Labeled model

Labels are of polylogarithmic size
$\tilde{O}(f(n)) = f(n) \cdot \text{polylog}(n)$

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>1</td>
<td>$\tilde{O}(n)$</td>
<td>[folk]</td>
</tr>
<tr>
<td></td>
<td>$4k - 5$</td>
<td>$\tilde{O}(n^{1/k})$</td>
<td>[Thorup,Zwick]</td>
</tr>
<tr>
<td>tree</td>
<td>1</td>
<td>$\tilde{O}(1)$</td>
<td>[TZ/Fraigniaud,G.]</td>
</tr>
<tr>
<td>doubling-$\alpha$ dim.</td>
<td>$1 + \varepsilon$</td>
<td>$\log \Delta$</td>
<td>[Talwar/Slivkins]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\tilde{O}(1)$</td>
<td>[Chan et al./Abraham et al.]</td>
</tr>
<tr>
<td>planar</td>
<td>$1 + \varepsilon$</td>
<td>$\tilde{O}(1)$</td>
<td>[Thorup]</td>
</tr>
<tr>
<td>$H$-minor-free</td>
<td>$1 + \varepsilon$</td>
<td>$\tilde{O}(1)$</td>
<td>[Abraham,G.]</td>
</tr>
</tbody>
</table>
## An overview: Name-independent model

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>bounded growth</td>
<td>$1 + \varepsilon$</td>
<td>$\tilde{O}(1)$</td>
</tr>
</tbody>
</table>

[Abraham et al.]
## An overview: Name-independent model

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
<th>Refs</th>
</tr>
</thead>
<tbody>
<tr>
<td>bounded growth</td>
<td>$1 + \varepsilon$</td>
<td>$\tilde{O}(1)$</td>
<td>[Abraham et al.]</td>
</tr>
<tr>
<td>doubling-$\alpha$ dim.</td>
<td>$9 + \varepsilon$</td>
<td>$\tilde{O}(1)$</td>
<td>[Konjevod et al./Abraham et al.]</td>
</tr>
</tbody>
</table>
An overview: Name-independent model

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>bounded growth</td>
<td>$1 + \varepsilon$</td>
<td>$\tilde{O}(1)$</td>
<td>[Abraham et al.]</td>
</tr>
<tr>
<td>doubling-(\alpha) dim.</td>
<td>$9 + \varepsilon$</td>
<td>$\tilde{O}(1)$ [Konjevod et al./Abraham et al.]</td>
<td></td>
</tr>
<tr>
<td>(H)-minor-free</td>
<td>$O(1)$</td>
<td>$\tilde{O}(1)$</td>
<td>[Abraham et al.]</td>
</tr>
<tr>
<td>(unweighted)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
An overview: Name-independent model

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>bounded growth</td>
<td>$1 + \varepsilon$</td>
<td>$\tilde{O}(1)$</td>
</tr>
<tr>
<td>doubling-$\alpha$ dim.</td>
<td>$9 + \varepsilon$</td>
<td>$\tilde{O}(1)$</td>
</tr>
<tr>
<td>$H$-minor-free (unweighted)</td>
<td>$O(1)$</td>
<td>$\tilde{O}(1)$</td>
</tr>
<tr>
<td>trees</td>
<td>$2^k - 1$</td>
<td>$\tilde{O}(n^{1/k})$</td>
</tr>
</tbody>
</table>
An overview: Name-independent model

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>bounded growth</td>
<td>$1 + \varepsilon$</td>
<td>$\tilde{O}(1)$ [Abraham et al.]</td>
</tr>
<tr>
<td>doubling-$\alpha$ dim.</td>
<td>$9 + \varepsilon$</td>
<td>$\tilde{O}(1)$ [Konjevod et al./Abraham et al.]</td>
</tr>
<tr>
<td>$H$-minor-free (unweighted)</td>
<td>$O(1)$</td>
<td>$\tilde{O}(1)$ [Abraham et al.]</td>
</tr>
<tr>
<td>trees</td>
<td>$2^k - 1$</td>
<td>$\tilde{O}(n^{1/k})$ [Laing]</td>
</tr>
<tr>
<td>arbitrary</td>
<td>$O(k^2 2^k)$</td>
<td>$\tilde{O}(n^{1/k})$ [Arias et al./Awerbuch,Peleg]</td>
</tr>
<tr>
<td></td>
<td>$O(k)$</td>
<td>$\tilde{O}(n^{1/k})$ [next talk]</td>
</tr>
</tbody>
</table>
Lower bounds for name-independent

Rem: lower bound for labeled $\Rightarrow$ lower bound for name-indep
**Lower bounds for name-independent network stretch space/node (bits)**

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>$&lt; 1.4$</td>
<td>$\Omega(n \log n)$ [G.,Pérennès.]</td>
</tr>
<tr>
<td>$&lt; 3$</td>
<td>$\Omega(n)$     [G.,Gengler]</td>
<td></td>
</tr>
<tr>
<td>(only $k = 1, 2, 3, 5$)</td>
<td>$&lt; 2k + 1$</td>
<td>$\Omega(n^{1/k})$ [Thorup,Zwick]</td>
</tr>
</tbody>
</table>
Lower bounds for name-independent network stretch space/node (bits)

<table>
<thead>
<tr>
<th>network</th>
<th>stretch</th>
<th>space/node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>$&lt; 1.4$</td>
<td>$\Omega(n \log n)$ [G., Pèrennès.]</td>
</tr>
<tr>
<td></td>
<td>$&lt; 3$</td>
<td>$\Omega(n)$       [G., Gengler]</td>
</tr>
<tr>
<td>(only $k = 1, 2, 3, 5$)</td>
<td>$&lt; 2k + 1$</td>
<td>$\Omega(n^{1/k})$ [Thorup, Zwick]</td>
</tr>
<tr>
<td>trees</td>
<td>$\leq 3$</td>
<td>$\Omega(\sqrt{n})$ [Laing, Rajaraman]</td>
</tr>
<tr>
<td></td>
<td>$\leq 9 - \varepsilon$</td>
<td>$\Omega(n^{(\varepsilon/60)^2})$ [Konjevod et al.]</td>
</tr>
</tbody>
</table>
### Lower bounds for name-independent networks

Rem: lower bound for labeled $\implies$ lower bound for name-indep

<table>
<thead>
<tr>
<th></th>
<th>Stretch</th>
<th>Space/Node (bits)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>arbitrary</strong></td>
<td>$&lt; 1.4$</td>
<td>$\Omega(n \log n)$ [G., Pérennès.]</td>
</tr>
<tr>
<td></td>
<td>$&lt; 3$</td>
<td>$\Omega(n)$       [G., Gengler]</td>
</tr>
<tr>
<td>(only $k = 1, 2, 3, 5$)</td>
<td>$&lt; 2k + 1$</td>
<td>$\Omega(n^{1/k})$ [Thorup, Zwick]</td>
</tr>
<tr>
<td><strong>trees</strong></td>
<td>$\leq 3$</td>
<td>$\Omega(\sqrt{n})$ [Laing, Rajaraman]</td>
</tr>
<tr>
<td></td>
<td>$\leq 9 - \varepsilon$</td>
<td>$\Omega(n^{(\varepsilon/60)^2})$ [Konjevod et al.]</td>
</tr>
<tr>
<td>for all $k \geq 1$</td>
<td>$&lt; 2k + 1$</td>
<td>$\Omega((n \log n)^{1/k})$ [this paper]</td>
</tr>
</tbody>
</table>
Theorem

1. Any name-indep. routing scheme using \( < (n \log n)^{1/k} \) bits/node has a \textbf{max stretch} \( \geq 2^k + 1 \) for some graph.

2. Any name-indep. routing scheme using \( < (n/k)^{1/k} \) bits/node has an \textbf{average stretch} \( \geq k/4 \) for some graph.
Theorem

1. Any name-indep. routing scheme using $<(n \log n)^{1/k}$ bits/node has a **max stretch** $\geq 2k + 1$ for some graph.

2. Any name-indep. routing scheme using $<(n/k)^{1/k}$ bits/node has an **average stretch** $\geq k/4$ for some graph.

Rem 1: All previous lower bounds for labeled case (Peleg, Upfal / G., Pérennès / G., Gengler / Kranakis, Krizanc / Thorup, Zwick) are based on the construction of **dense large girth** graphs.

If stretch $< 2k + 1$, then $u$ is forced to "know" the edge $(u, v)$. 

Diagram:
- $u$ and $v$ are nodes in a graph.
- The edge $(u, v)$ is indicated.
- The stretch $2k + 2$ is shown between $u$ and $v$. 

if stretch $< 2k + 1$, then $u$ is forced to "know" the edge $(u, v)$.
Theorem

1. Any name-indep. routing scheme using \( < (n \log n)^{1/k} \) bits/node has a **max stretch** \( \geq 2k + 1 \) for some graph.

2. Any name-indep. routing scheme using \( < (n/k)^{1/k} \) bits/node has an **average stretch** \( \geq k/4 \) for some graph.

Erdös Conjecture: \( \exists \) graph of girth \( 2k + 2 \) with \( \Omega(n^{1+1/k}) \) edges (proved only for \( k = 1, 2, 3, 5 \)). So, the extra \( (\log n)^{1/k} \) term **cannot** be obtained with a girth approach.
Theorem

1. Any name-indep. routing scheme using $< (n \log n)^{1/k}$ bits/node has a max stretch $\geq 2k + 1$ for some graph.
2. Any name-indep. routing scheme using $< (n/k)^{1/k}$ bits/node has an average stretch $\geq k/4$ for some graph.

Rem 2: It makes a clear separation between labeled and name-independent routing.

In the labeled model, $O(\text{polylog}(n))$ space and $O(1)$ average stretch exists for every graph! [Abraham, Bartal, Chan, Gupta, Kleinberg et al. (FOCS05)]

In the name-indep model, if space is $O(\text{polylog}(n))$, then the average stretch must be $\Omega(\log n / \log \log n)$ for some graphs.
Proof: The model

(only point 1)

\[ R(x_i, h_i, q_i) = (h_{i+1}, p_{i+1}) \]

\( R(x_i, \cdot, \cdot) \) describes the routing algorithm in \( x_i \)
Proof: The model

(only point 1)

\[ R(x_i, h_i, q_i) = (h_{i+1}, p_{i+1}) \]

\( R(x_i, \cdot, \cdot) \) describes the routing algorithm in \( x_i \)

**Definition (Kolmogorov Complexity)**

Given an object \( P \), let \( \mathcal{K}(P) \) denote the length of the smallest program that prints \( P \) and halts.
Proof: The counter-example

Lemma \( \exists L \subset \{1, \ldots, n\} \) with \(|L| = n/2\) such that every \( P \subset L \) satisfies \(|P| \leq K(P) + 2 \log n\).

Edge weight is 1 or \( k \): \( w(e) = 1 \) iff \( \text{port}(e) \in L \).

Node naming: light (=red) edges lead to name \( \leq n/2 \).
Proof: The counter-example

Lemma

\( \exists L \subset \{1, \ldots, n\} \text{ with } |L| = n/2 \text{ such that every } P \subset L \text{ satisfies } |P| \leq \mathcal{K}(P) + 2 \log n. \)
Proof: The counter-example

\[ L = \{2, 3, 5, 7\} \]

Lemma

\[ \exists L \subset \{1, \ldots, n\} \text{ with } |L| = n/2 \text{ such that every } P \subset L \text{ satisfies } |P| \leq \mathcal{K}(P) + 2 \log n. \]

Edge weight is 1 or \( k \): \( w(e) = 1 \) iff \( \text{port}(e) \in L \).
Proof: The counter-example

\[ L = \{2, 3, 5, 7\} \]

Lemma

\[ \exists L \subset \{1, \ldots, n\} \text{ with } |L| = n/2 \text{ such that every } P \subset L \text{ satisfies } |P| \leq K(P) + 2 \log n. \]

Edge weight is 1 or \( k \): \( w(e) = 1 \) iff \( \text{port}(e) \in L \).
Node naming: light (=red) edges lead to name \( \leq n/2 \)
Proof: The counter-example

\[ L = \{2, 3, 5, 7\} \]

\[ \exists L \subset \{1, \ldots, n\} \text{ with } |L| = n/2 \text{ such that every } P \subset L \text{ satisfies } |P| \leq K(P) + 2 \log n. \]

Edge weight is 1 or \( k \): \( w(e) = 1 \) iff \( \text{port}(e) \in L \).
Node naming: light (red) edges lead to name \( \leq n/2 \)
Assume any $R$ is given (no limits on the headers), and uses $\leq M$ bits at every node.

**Idea:** To analyze all the walks from the root to all light destinations (names $\leq n/2$).
Proof: Analysis 1/3

Assume any $R$ is given (no limits on the headers), and uses $\leq M$ bits at every node

\[ L = \{2, 3, 5, 7\} \]

Idea: To analyze all the walks from the root to all light destinations (names $\leq n/2$).

Basic fact: If stretch $< 2k + 1$, then no heavy edge is used
Proof: Analysis 2/3

Assume now the stretch of $R$ is $< 2k + 1$

$L = \{2, 3, 5, 7\}$

$P_i = \{\text{all ports in alive walks after } i\text{th routing decision of } r\}$
Proof: Analysis 2/3

Assume now the stretch of $R$ is $< 2k + 1$

$L = \{2, 3, 5, 7\}$

$P_i = \{\text{all ports in alive walks after } i\text{th routing decision of } r\}$

- $P_i \subset L$ for all $i$
Assume now the stretch of $R$ is $< 2k + 1$

$P_i = \{\text{all ports in alive walks after } i\text{th routing decision of } r\}$

- $P_i \subseteq L$ for all $i$
- $\mathcal{K}(P_1) \leq M$ (why?)
Proof: Analysis 2/3

Assume now the stretch of $R$ is $< 2k + 1$

$L = \{2, 3, 5, 7\}$

$P_i = \{\text{all ports in alive walks after } i\text{th routing decision of } r\}$

- $P_i \subset L$ for all $i$
- $\mathcal{K}(P_1) \leq M \quad \text{(why?)}$
- $\mathcal{K}(P_{i+1}) \leq M \cdot (|P_i| + 1) \quad \text{(why?)}$
Proof: Analysis 2/3

Assume now the stretch of $R$ is $< 2k + 1$

$L = \{2, 3, 5, 7\}$

$P_i = \{\text{all ports in alive walks after } i\text{'th routing decision of } r\}$

- $P_i \subset L$ for all $i$
- $\mathcal{K}(P_1) \leq M$ (why?)
- $\mathcal{K}(P_{i+1}) \leq M \cdot (|P_i| + 1)$ (why?)

So, $\mathcal{K}(P_{i+1}) \leq M \cdot (\mathcal{K}(P_i) + 2 \log n + 1)$, and therefore

$$|P_{i+1}| \leq \mathcal{K}(P_{i+1}) + 2 \log n + 1 \leq (1 + o(1)) \cdot M^{i+1}$$
Proof: Analysis 3/3

\[ |P_{i+1}| \leq (1 + o(1)) \cdot M^{i+1} \]

\[ W_t = \{ \text{light destinations reached after the } t\text{th routing decision of } r \} \]
Proof: Analysis 3/3

\[ |P_{i+1}| \leq (1 + o(1)) \cdot M^{i+1} \]

\( W_t = \{ \text{light destinations reached after the } t\text{th routing decision of } r \} \)

- \( |W_t| \leq |P_1| + |P_2| + \cdots + |P_t| \)
Proof: Analysis 3/3

\[ |P_{i+1}| \leq (1 + o(1)) \cdot M^{i+1} \]

\( W_t = \{ \text{light destinations reached after the } t\text{th routing decision of } r \} \)

- \( |W_t| \leq |P_1| + |P_2| + \cdots + |P_t| \)
- If \( |W_t| < n/2 \), then stretch \( \geq 2t + 1 \).
  
  So, stretch \( < 2k + 1 \) implies \( |W_k| \geq n/2 \).
Proof: Analysis 3/3

\[ |P_{i+1}| \leq (1 + o(1)) \cdot M^{i+1} \]

\( W_t = \{ \text{light destinations reached after the } t \text{th routing decision of } r \} \)

- \( |W_t| \leq |P_1| + |P_2| + \cdots + |P_t| \)
- If \( |W_t| < n/2 \), then stretch \( \geq 2t + 1 \).
  
  So, stretch \( < 2k + 1 \) implies \( |W_k| \geq n/2 \).

Combining: \( n/2 \leq \sum_{i=1}^{k} (1 + o(1)) \cdot M^i = O(M^k) \), i.e.,

\[ M = \Omega(n^{1/k}) \]
Proof: Analysis 3/3

\[ |P_{i+1}| \leq (1 + o(1)) \cdot M^{i+1} \]

\[ W_t = \{ \text{light destinations reached after the } t\text{th routing decision of } r \} \]

- \[ |W_t| \leq |P_1| + |P_2| + \cdots + |P_t| \]
- If \( |W_t| < n/2 \), then stretch \( \geq 2t + 1 \).
  
  So, stretch \( < 2k + 1 \) implies \( |W_k| \geq n/2 \).

Combining: \( n/2 \leq \sum_{i=1}^{k} (1 + o(1)) \cdot M^i = O(M^k) \), i.e.,
\[ M = \Omega(n^{1/k}). \]

Actually, a finer argument yields \( M = \Omega((n \log n)^{1/k}) \).

QED
Thank you!