

Object Location Using Path Separators

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To design **distributed** data-structures for networks supporting **location** queries like distance or routing

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⇒ **small-world problem**

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What knowledge is needed to solve a task?

Overview for distance labeling

Networks = weighted graph with n nodes and aspect ratio Δ

Trade-offs between stretch and label size

network	stretch	label size (bits)
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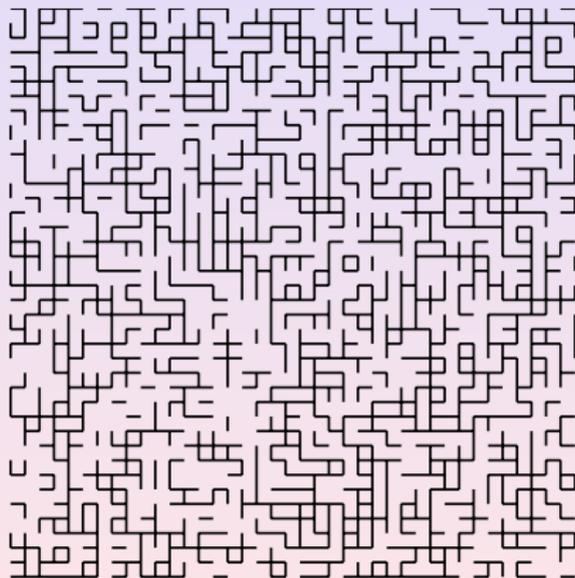
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bounded doubling dimension	$1 + \varepsilon$	$\log \Delta$
(Euclidian, bounded growth, ...)		$\log n \log(\varepsilon^{-1} \log \Delta)$

Similar bounds for compact routing ...

Distance labeling in planar networks

Shortest path metrics of planar graphs are difficult to capture

- Planars are \neq Euclidian networks (TSP, ℓ_p embedding, ...)
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$1 + \varepsilon$	$\varepsilon^{-1} \log n \log \Delta$	Thorup [JACM '04]

In this paper ...

Contribution

Weighted graphs excluding a **fixed minor** have an stretch- $(1 + \varepsilon)$ distance labeling scheme with $O(\varepsilon^{-1} \log n \log(\varepsilon^{-1} \log \Delta))$ -bit labels.

Rem: scheme polynomially constructible, constant depending on the excluded minor. Similar bounds for routing.

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 $\subset K_{3,3}$ -minor-free

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Definition

H is a **minor** of G if H can be obtained from G by edge contraction and taking subgraph (node and edge deletion).

Minor-free graphs

A lot of network topologies can be characterized in terms of graphs excluding some fixed minor

- trees $\subset K_3$ -minor-free
- outerplanar $\subset K_{2,3}$ -minor-free
- series-parallel $\subset K_4$ -minor-free
- planar $\subset K_5$ -minor-free
- genus- g $\subset K_{O(\sqrt{g})}$ -minor-free
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The Graph Minor Theorem [Robertson & Seymour]

Every graph family closed under minor taking can be characterized by a finite set of finite forbidden minors.

Small-world problems

Polylog(n) number of hops (in expectation) known for: grids (Kleinberg), d -dimensional meshes (Fraigniaud et al.), bounded growth (Duchon et al.), bounded doubling dimension (Slivkins), trees & bounded treewidth (Fraigniaud), ... Open for planars

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Every weighted graph excluding a **fixed minor** can be augmented with one long range link per node so that greedy routing performs in $O(\log^2 n \log^2 \Delta)$ expected number of hops.

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Our approach: path separator technique

Definition (Main)

A weighted graph G with n nodes is *k -path separable* if there exists a subgraph S , called k -path separator, such that:

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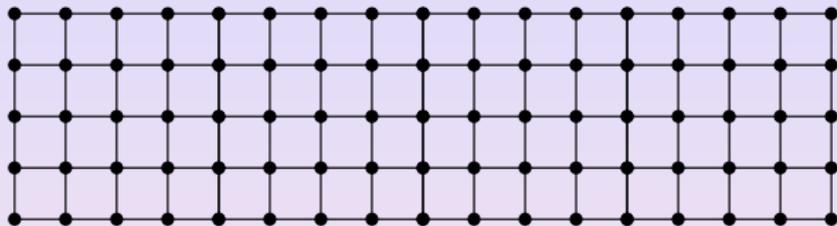
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If Q is a path forming P_i , then:

- Q is not necessarily of bounded size
- Q is not necessarily a shortest path in G

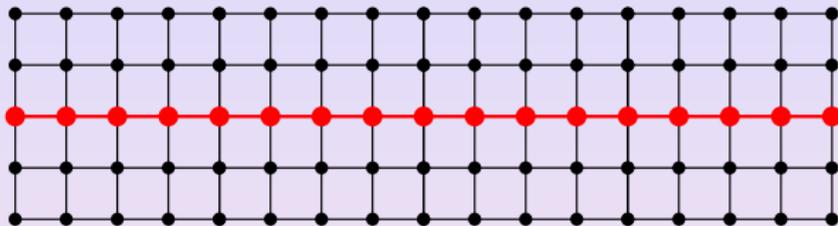
Some basic examples

- Unweighted meshes are 1-path separable



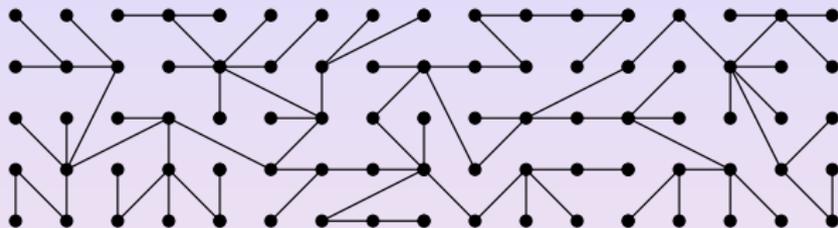
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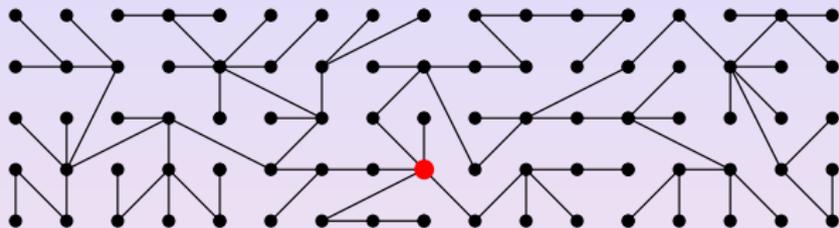
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Lemma (Thorup [JACM '04])

Every n -node planar graph G has a shortest-path tree T with at most 3 leaves such that each component of $G \setminus T$ has $\leq n/2$ nodes.

\Rightarrow planars are 3-path separable

Conjecture (Thorup)

Every n -node H -minor-free graph has a shortest-path tree T with at most $\ell = \ell(H)$ leaves such that each component of $G \setminus T$ has $\leq n/2$ nodes.

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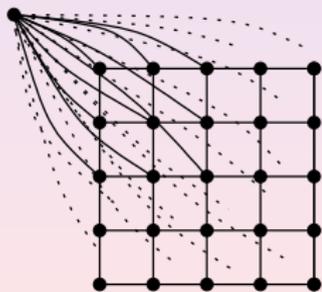
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- genus $\Omega(n)$
- tree-width $\Omega(\sqrt{n})$
- no K_6 minor
- $\Omega(\sqrt{n})$ shortest paths to halve
- ... but is 2-path separable

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Theorem (Main)

Every H -minor-free graph is k -path separable for $k = k(H)$.

A k -path separator can be found in $n^{O(k)}$ time

Consequences of the Main Theorem

Theorem (Object Location)

Let G be a weighted k -path separable graph of aspect ratio Δ

- 1 stretch- $(1 + \varepsilon)$ distance labeling with $O(k\varepsilon^{-1} \log n \log(\varepsilon^{-1} \log \Delta))$ -bit labels
- 2 stretch- $(1 + \varepsilon)$ labeled routing scheme with $O(k\varepsilon^{-1} \log^3 n / \log \log n)$ -bit headers and routing tables
- 3 One can augment G with 1 directed edge per node such that greedy routing performs in $O(k^2 \log^2 n \log^2 \Delta)$ expected number of hops
- 4 And others: reachability, distance oracles in digraphs, ...

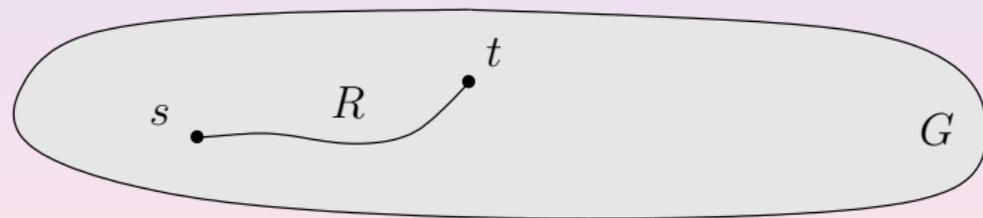
Proving the Object Location Theorem

(\approx extension of Thorup's data-structures)

$\forall s, t$ -shortest path R in G there exist:

- a subgraph G' in the separator decomposition of G ;
- a k -path separator S' of G' ; and
- a path Q that composes S' such that

Q and R intersect and **both** are shortest paths in G' .



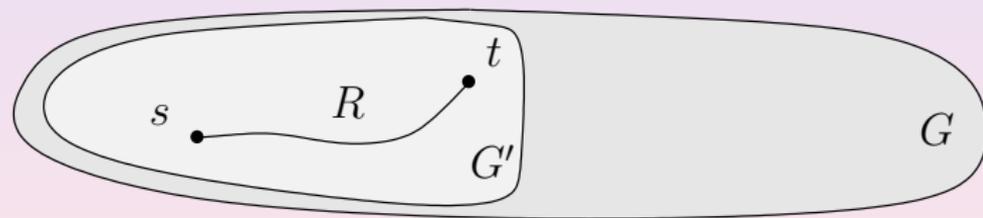
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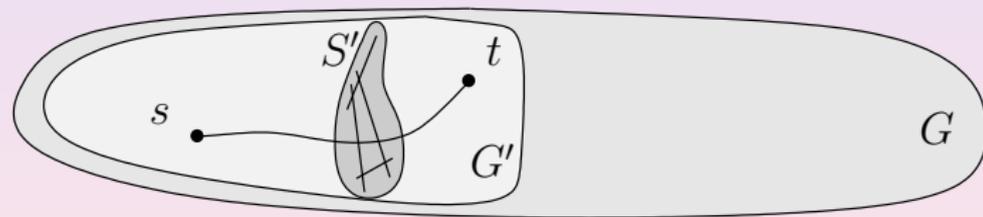
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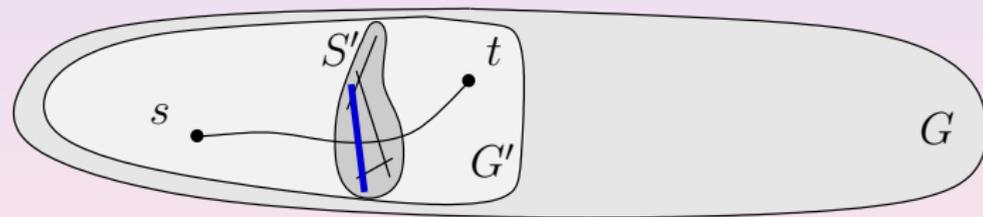
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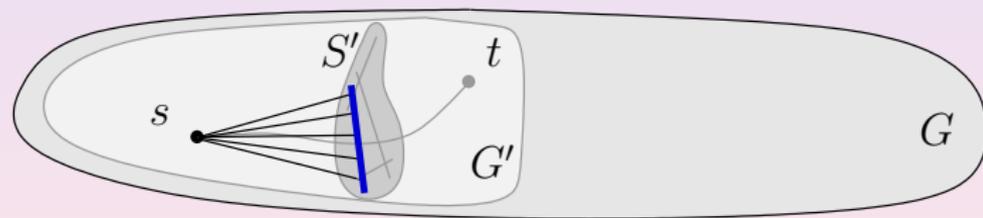
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Node s can select, **independently** of t , few “landmarks” on Q so that one of these landmarks is close to $R \cap Q$

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Roughly speaking,

Theorem (Graph Minor-16, 2003)

Every graph excluding a fixed minor has a tree-decomposition in subgraphs that are h -almost embeddable on a surface of bounded Euler genus.

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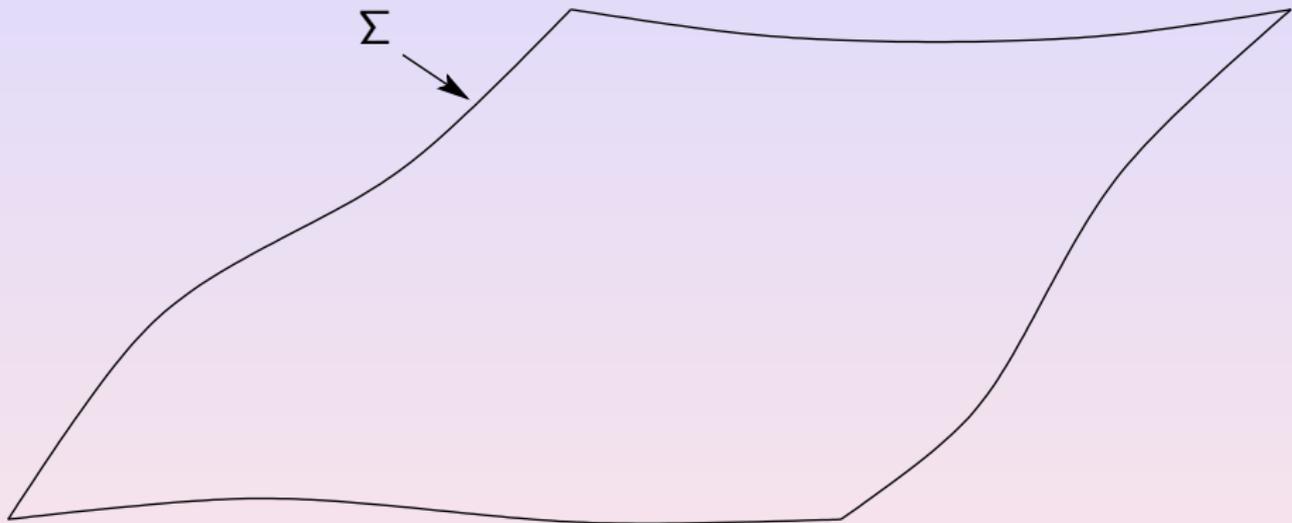
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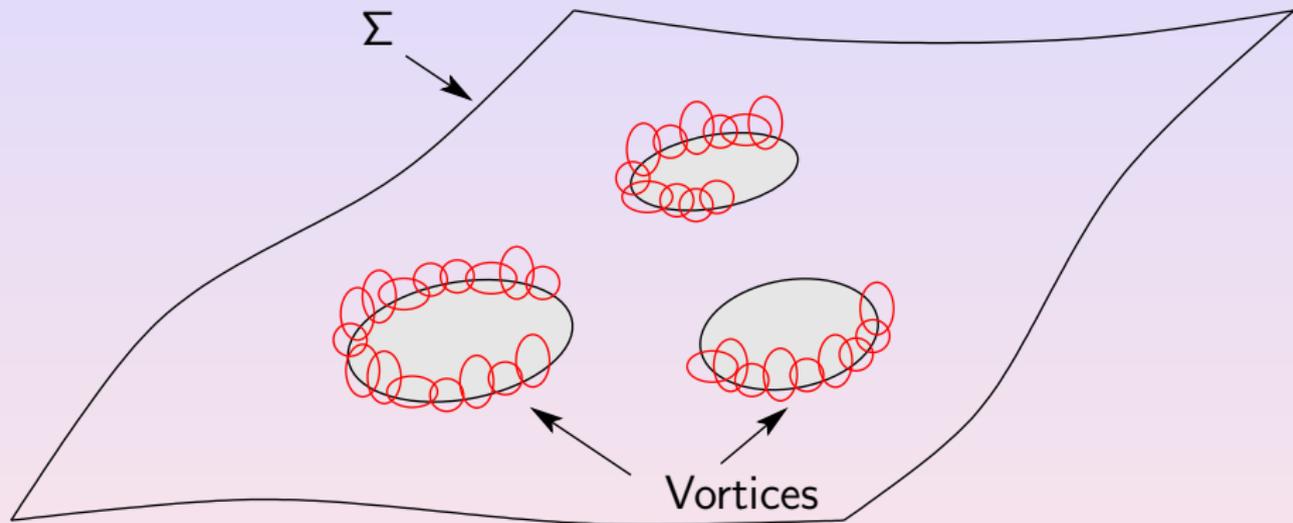
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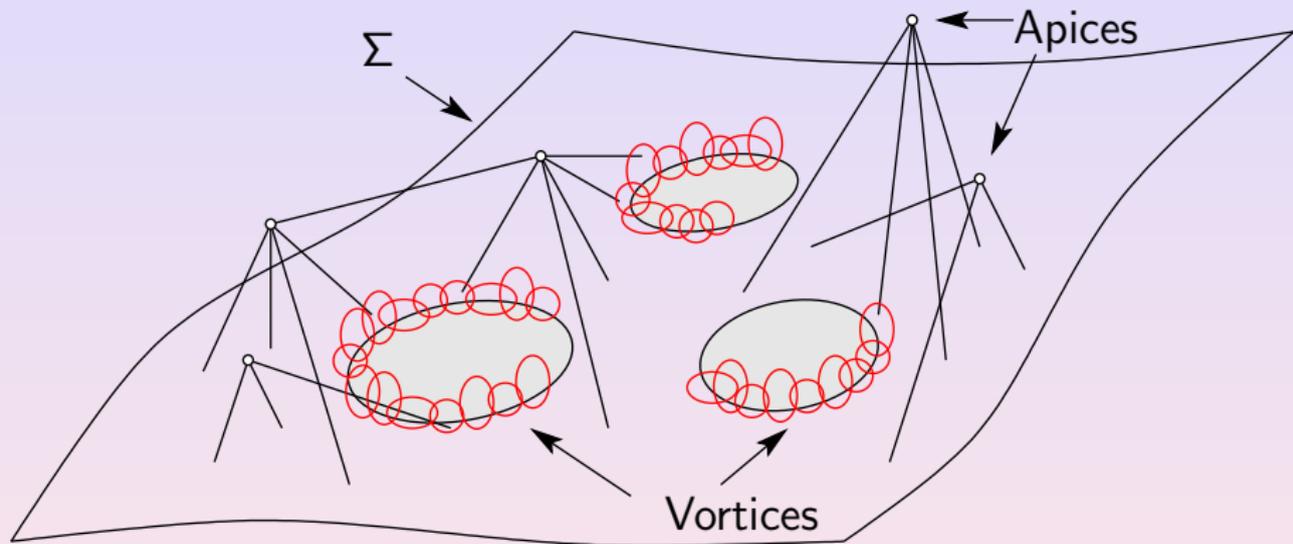
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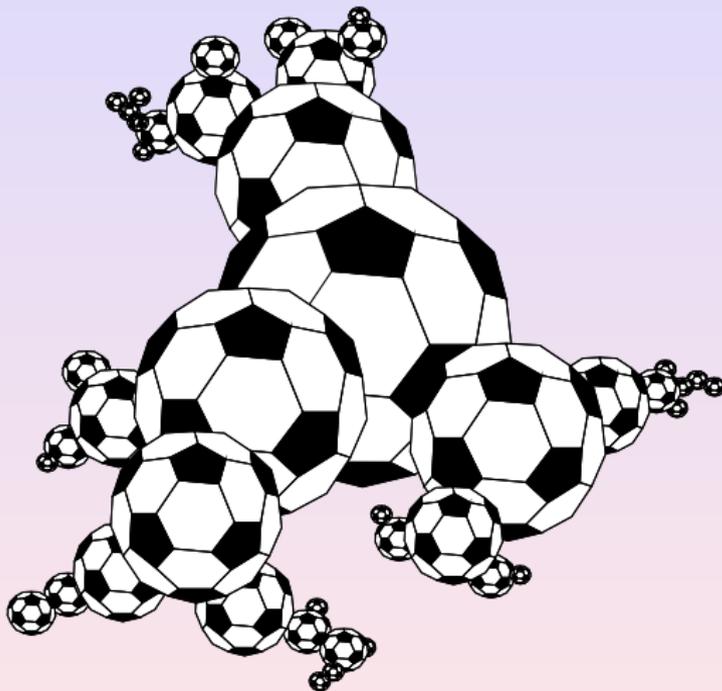
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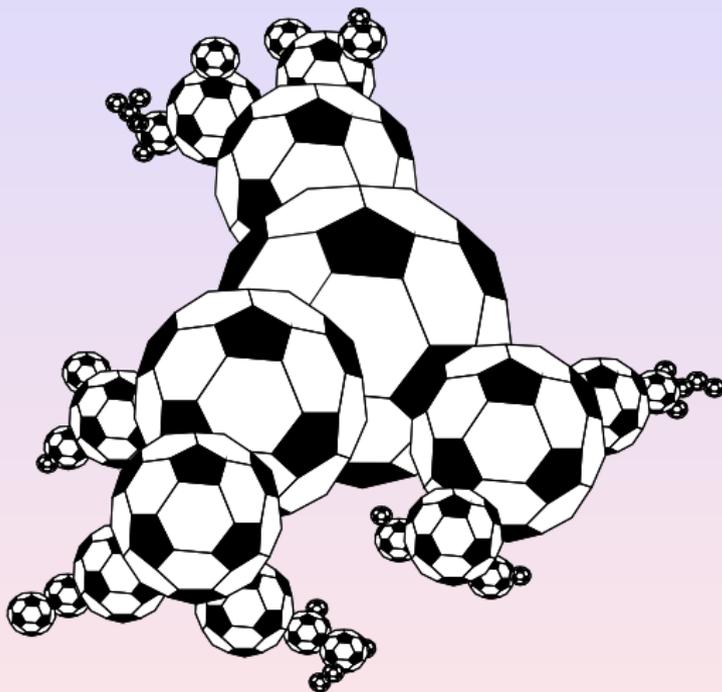
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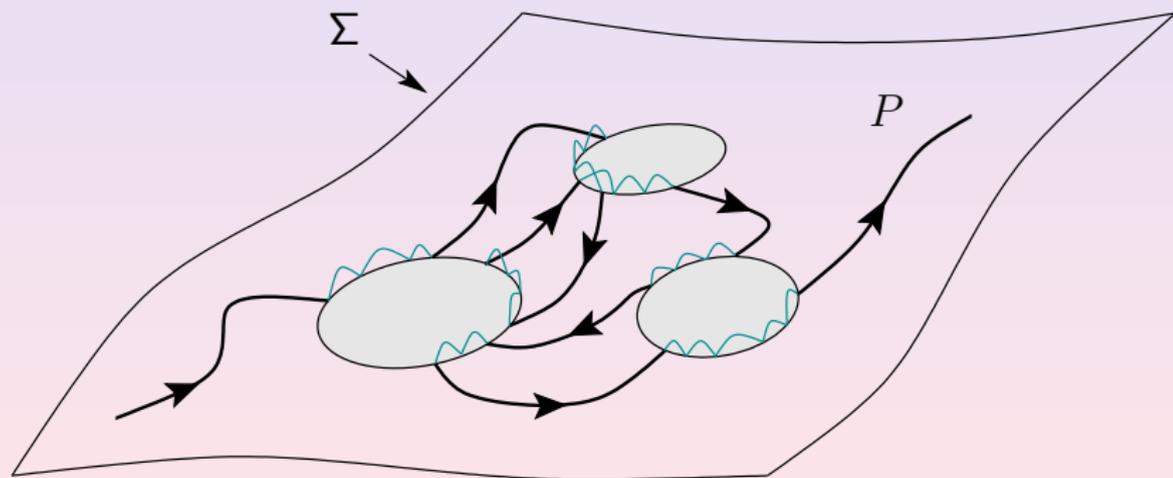
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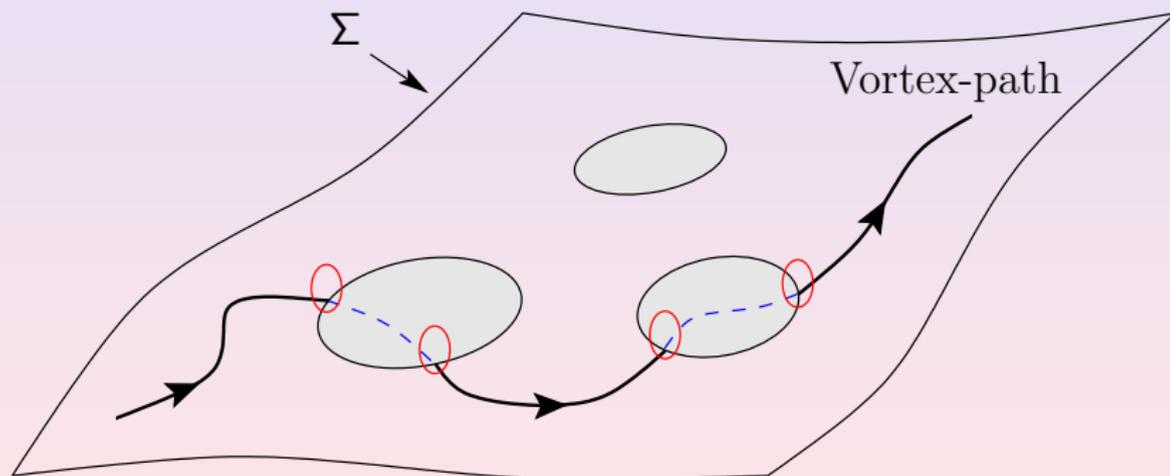
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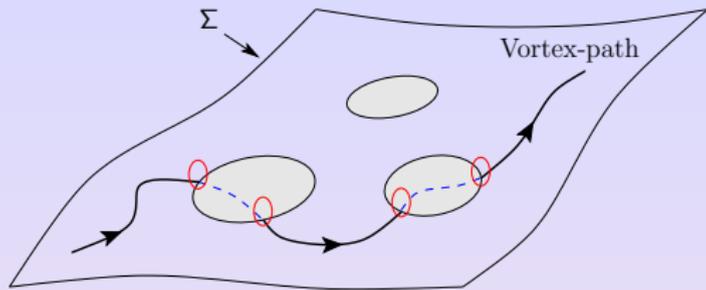


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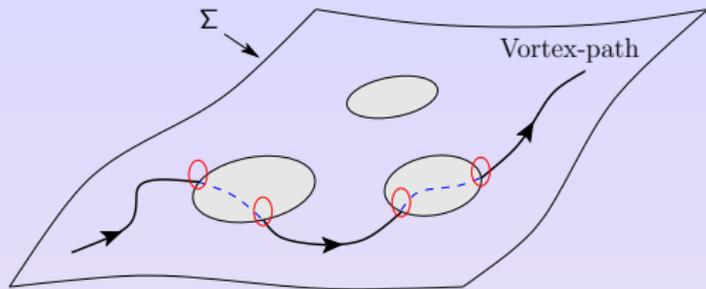
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Lemma

If the center subgraph is "nearly-planar" (= no apices and $\Sigma = \mathbb{R}^2$), there are three vortex-paths whose segments are shortest paths, and whose deletions leave components of size $\leq n/2$.

Q.E.D.

Thanks!