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Dominating Set

We recall the two distributed algorithms discussed in the practical sessions that compute a dominating set S of size close (within a constant factor) to the optimal for a given graph that is either a tree or a forest (TREE), or a planar outer graph (OUTER).

- Algorithm TREE. A vertex is selected in the set S if its degree is at least two.
- Algorithm OUTER. Each vertex calculates its degree and sends it to its neighbors. Then, a vertex is selected in the set S if its degree is at least 4, or if all its neighbors have a degree smaller than 4.

Question 1. Describe the OUTER algorithm in the LOCAL model using the language and conventions from the course, specifying the number of rounds. [NB : We are not asking for Java code using JBotSim! You can use NEWROUND, ID, SEND, SENDALL, RECEIVE, ...]

Question 2. Same question for the TREE algorithm, correcting the issue with connected components having one or two vertices. Specify the number of rounds.

Let Q_n be the planar outer graph obtained from a path P_n with n vertices by adding all edges between vertices at distance two in P_n , see the example below.



Question 3. Show that for all n , P_n has a dominating set with at most $\lceil n/3 \rceil$ vertices.

Question 4. Show that for all n , Q_n has a dominating set with at most $\lceil n/5 \rceil$ vertices.

Communication Complexity

Question 5. In the b -CONGEST model, explain what the problem of « detecting an H » in G corresponds to, for a fixed graph H . [NB : We are not asking for its solution.]

Question 6. Explain what a communication protocol that computes a Boolean function f is. What are the main rules ? What is the goal ?

Question 7. Which distributed algorithmic problem seen in the course did communication complexity help us solve ? State the result.

We consider the Boolean function $\text{inf}_n(x, y) \in \{0, 1\}$ which, given two binary strings x, y , each of n bits, returns 1 if and only if « $x \leq y$ », that is, if the integer represented in binary by x is less than or equal to the integer represented by y . For instance, $\text{inf}_2(01, 11) = 1$ because « $01 \leq 11$ ». Indeed, $01_{\text{binary}} = 1_{\text{decimal}} \leq 3_{\text{decimal}} = 11_{\text{binary}}$.

Question 8. Provide the matrix of the function \inf_n , for $n = 2$.

We recall that a discriminating set for a Boolean function f and a value $v \in \{0, 1\}$ is a set F_v of inputs (x, y) for f such that, for all $(x, y) \neq (x', y') \in F_v$, $f(x, y) = f(x', y') = v$ and $\bar{v} \in \{f(x, y'), f(x', y)\}$, where $\bar{v} = 1 - v$ is the complementary value of v . This implies that we cannot have $f(x, y') = f(x', y) = v$.

Question 9. Provide the largest possible discriminating set F_1 for \inf_2 and its cardinality. Do the same for a discriminating set F_0 .

Question 10. Prove that any discriminating set F_v , for a given function f and value v , cannot have two entries located in the same row or column of the matrix of f .

Question 11. Generalize question 9 to any n , i.e., construct the largest possible discriminating sets F_1 and F_0 for \inf_n . Justify the constructions and their optimality. [Hint : Question 10 can be used.]

Question 12. Provide a lower bound, as a function of n , on the minimum number of scripts for any communication protocol computing \inf_n . Justify. Deduce the communication complexity of \inf_n .

We now consider communication protocols that are prefix-free and do not include empty strings. Let f_0 be the Boolean function defined on binary strings of 2 bits, whose matrix is as follows :

| x \ y | | | | |
|-------|----|----|----|----|
| | 00 | 01 | 10 | 11 |
| 00 | 1 | 1 | 0 | 0 |
| 01 | 1 | 1 | 0 | 0 |
| 10 | 0 | 0 | 1 | 1 |
| 11 | 0 | 0 | 1 | 1 |

Question 13. Show that any communication protocol for f_0 requires at least 4 scripts.

Question 14. Provide a communication protocol for f_0 with cost 2. [Its principle is sufficient.]

We say that a Boolean function f on n -bit strings is k -similar if the matrix of f contains only k distinct rows (out of 2^n possible rows). For instance, the function f_0 above is 2-similar.

Question 15. Propose a non-distributed algorithm that, given the matrix of f , computes the smallest k such that f is k -similar. Provide its complexity as a function of n . [Its principle is sufficient.]

Question 16. Propose a communication protocol for a function f that is k -similar. Specify, as a function of k (and possibly n), the number of scripts in your protocol and its cost. [Hint : Refer to the results of questions 13 and 14.]

Question 17. For any integers k and n , $1 \leq k \leq n$, find a Boolean function on n -bit strings that is k -similar and has communication complexity at least $\log_2(k) + 1$. [The description of its matrix is sufficient to describe the function, however you need to argue for its communication complexity.]

END.