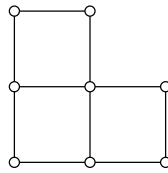




A *Maximal Independent Set* (MIS for short) for a graph  $G$  is an independent set  $M$  which is maximal for inclusion. More precisely,  $M$  is a subset of nodes of  $G$  such that no two nodes of  $M$  are neighbors (independent), and every node not in  $M$  is adjacent to at least one node of  $M$  (maximal).

**Question 1.** For the graph depicted below, give a 3-coloring such that, for each color  $i$ , the set of nodes of color  $i$  is not a MIS.



In the remaining we consider the LOCAL model.

**Question 2.** Recall the characteristics of the LOCAL model.

Recall that  $G$  is 1-oriented (or has a 1-orientation) if all its edges are oriented into arcs such that each of its nodes has at most one outgoing arc.

In the distributed setting, whenever we say that  $G$  is 1-oriented (or has a 1-orientation), we mean that every node  $u$  with outgoing arc  $u \rightarrow v$  has a local variable  $\text{PARENT}(u) = v$ . We set  $\text{PARENT}(w) = \perp$  if a node  $w$  has no outgoing arc. Similarly, whenever we say that  $G$  has a MIS  $M$ , it means that every node  $u$  has a boolean variable  $M(u)$  that is TRUE iff  $u \in M$ .

**Question 3.** Consider a cycle that is 1-oriented and with a MIS  $M$ . Give a constant time distributed algorithm that computes, in a variable COLOR, a 3-coloring for such a graph. Express the number of rounds. Justify.

We would like now to answer to the same question not only for cycles, but for more general graphs. Unfortunately, that's not so easy.

**Question 4.** Construct a 1-oriented graph with a MIS for which the algorithm you gave at the previous question fails.

**Question 5.** Based on your previous construction, describe a 1-oriented graph with  $n$  nodes and a MIS for which the number of rounds to compute a 3-coloring is unbounded. Give a lower bound in term of  $n$  on this number of rounds. (Hint: Reduce to the coloring lower bound proved in the course for oriented cycles.)

A simple algorithm to compute a MIS  $M$  consists to apply (possibly in parallel) the following rule: if node  $u$  and none of its neighbors is in  $M$ , then add  $u$  to  $M$ , under the condition that none of the neighbors of  $u$  applies the same rule simultaneously.

**Question 6.** Assume that a graph  $G$  with  $n$  nodes has a  $k$ -coloring via a local variable  $\text{COLOR}(u)$  for each node  $u$ . Give a distributed algorithm computing a MIS for  $G$ . Express its number of rounds.

**Question 7.** Explain briefly how to distributively compute a MIS for a cycle or a path with  $n$  nodes in  $O(\log^* n)$  rounds. Unlike previous questions, note that here the cycle has no orientation.

We now consider the problem of efficiently computing a “small” dominating set. The goal is to show that in  $O(\log^* n)$  rounds one can compute a small dominating set for every connected graph with a 1-orientation.

Recall that a dominating set of a graph  $G$  is a subset  $D$  of nodes of  $G$  such that every node not in  $D$  is adjacent to at least one node of  $D$ . Observe that a MIS is a particular dominating set. We say that  $X \subseteq V(G)$  is “small” if there is an absolute constant  $\lambda < 1$  such that  $|X| < \lambda n + 1$  where  $n = |V(G)|$ . Recall that  $|X|$  denotes the cardinality of the set  $X$ .

**Question 8.** *Construct a graph with  $n$  nodes and  $m \leq n$  edges with no small dominating set.*

**Question 9.** *Show that any MIS of a connected graph of maximum degree at most two is a small dominating set.*

Let  $G$  be a connected graph having a 1-orientation. We consider the following two subsets of nodes of  $G$ :  $B := \{u : \deg(u) > 2\}$  (the “big” degree nodes); and  $S := V(G) \setminus B$  (the “small” degree nodes). The general idea to find a small dominating set is to apply a different strategy for nodes in  $B$  and in  $S$ .

For nodes in  $B$ , this is very simple: just add all of them to the final wanted dominating set  $D$ . (You will need to argue that  $B$  is necessarily small.) For nodes in  $S$ , compute efficiently a MIS  $M$  for  $G[S]$ , the subgraph of  $G$  induced by the nodes of  $S$ . Then, we would like to argue that  $M$  is a small dominating set and conclude by setting  $D := B \cup M$ .

Unfortunately, this does not work because  $G[S]$  is not necessarily connected. So, we cannot positively conclude that  $M$  is a small dominating set of  $G[S]$ . Observe also that the union of two small disjoint sets is not necessarily small. Fortunately, since we have  $B \subseteq D$ , we can analyze more carefully the situation of the nodes in  $S$ , and apply the following local rule: a node  $u \in S$  is added to  $D$  only if  $u \in M$  and  $u$  has at least a neighbor in  $S$ .

**Question 10.** *Summarize the above discussion, and give a distributed algorithm that, given a connected 1-oriented graph  $G$  with  $n > 1$  nodes, that computes a small dominating set  $D$  for  $G$  in  $O(\log^* n)$  rounds. Express the value  $\lambda$  that bounds the size of  $D$ .*