
All your answers must be fully justified. The language for your answers can be either French or English, but not both! Document: one double-sided A4 sheet allowed.

Course Questions

Question 1. Recall the main characteristics of the LOCAL, CONGEST, and b-CONGEST models.

Question 2. Explain what the distributed algorithmic problem of detecting a fixed graph H in the graph G corresponds to. [Note : You are not asked for its solution, but for the specifics in terms of inputs and outputs.]

Question 3. Describe the principle of an algorithm in the LOCAL model for detecting a C_6 (a cycle of length six) in a graph G . Specify the number of rounds.

Question 4. Deduce an algorithm for detecting C_6 in the CONGEST model. Specify its time complexity (number of rounds) as a function of the maximum degree Δ of the graph G .

Communication Complexity

Question 5. Explain what a communication protocol that computes a Boolean function f is. What are its main rules? What is the objective?

Question 6. Explain the principle of the "trivial" protocol for computing f . What is its cost?

Consider the function $\text{aleq}_n(x, y) \in \{0, 1\}$ (for *ALmost EQual*) which, given two binary words x, y , each of n bits, returns 1 if and only if $|x - y| \leq 1$, i.e., if the integers represented in binary by the words x and y differ by at most one. For example, $\text{aleq}_2(11, 10) = 1$ because $|3 - 2| \leq 1$. On the other hand, $\text{aleq}_2(01, 11) = 0$ because $|1 - 3| \not\leq 1$. Obviously, $\text{aleq}_n(x, x) = 1$ for all $x \in \{0, 1\}^n$.

Question 7. Give the matrix of the function aleq_n for $n = 2$.

Recall that a discriminating set for a Boolean function f and a value $v \in \{0, 1\}$ is a set F_v of inputs (x, y) for f such that, for all $(x, y) \neq (x', y') \in F_v$, $f(x, y) = f(x', y') = v$ and $\bar{v} \in \{f(x, y'), f(x', y)\}$, where $\bar{v} = 1 - v$ is the complementary value of v .

Question 8. Give the largest possible discriminating set F_1 for aleq_2 and its cardinality. Do the same for a discriminating set F_0 .

Question 9. Explain the structure of the matrix of aleq_n for $n \geq 3$.

Question 10. Recall the definition of $\text{CC}(f)$, the communication complexity of a Boolean function f .

Question 11. Generalize Question 8 to all $n \geq 3$, i.e., construct the largest possible discriminating sets F_1 and F_0 for aleq_n and give their cardinalities.

Question 12. Give a lower bound, as a function of n , on the number of scripts of any communication protocol computing aleq_n . Justify. Deduce the exact value of $\text{CC}(\text{aleq}_n)$.

Consider the equality function $\text{eq}_n(x, y) \in \{0, 1\}$ which is 1 if and only if the n -bit binary words x and y are equal. It can be admitted that $\text{CC}(\text{eq}_n) = n + 1$.

Question 13. Give the details of a communication protocol without empty words of cost $n + 2$ computing eq_n in 4 rounds, such that Alice sends half of the bits of x to Bob and Bob sends half of the bits of y to Alice. You may assume n is even.

Consider a particular transmission process, denoted $T_{1,2}$, which consists of sending small messages of variable length. More precisely, the participant (say Alice) who wants to send a binary word $S = s_0s_1s_2 \dots s_{t-1}$ of t bits to the other (Bob) splits it into messages of 1 or 2 bits. When it is her turn, and the bits $s_0 \dots s_{i-1}$ have already been transmitted, Alice sends in her message the bit $m = s_{i+1}$ if $s_i = 0$, or sends the two bits $m = s_{i+1}s_{i+2}$ if $s_i = 1$. Note that in all cases, s_i is not directly transmitted. However, Bob, who receives such a small message m from Alice, can deduce the bits $s_i s_{i+1}$ or $s_i s_{i+1} s_{i+2}$ depending on the size $|m| \in \{1, 2\}$. If $i \in \{n-2, n-1\}$, the bits s_{i+1} and s_{i+2} may not exist. In this case, Alice adds 1 or 2 arbitrary bits, which does not cause a problem for Bob who will not take these bits into account if he knows t .

Question 14. Inspired by the protocol from Question 13, combined with the transmission process $T_{1,2}$, give a communication protocol without empty words computing eq_n . Give its cost. [Hint : First assume that Alice and Bob finish transmitting their part at the same time. Then adapt the protocol to work in all cases.]

Question 15. Noting that the communication protocol for eq_n from Question 14 has a cost $\leq n$, explain the apparent contradiction with the fact that $CC(eq_n) = n + 1$.

END.