Distributed Algorithms

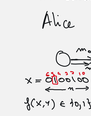
Dotted a C_4 or a C_3

CONGEST model
- synchronous rounds
- limit on msg size $O(\log n)$ bits

Previously: $O(\sqrt{n})$ rounds.
Can we do better? $\Rightarrow$ NO.

Communication Complexity

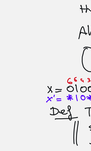
Tool to prove lower bound in the CONGEST.



In order to "compute" $f(x,y)$ Alice & Bob apply some "communication protocole".
It has to follow the following rules:

1. Alice starts, prepare a first $m_0 \to$ Bob
2. Rounds after rounds Bob $\to$ Alice $\to$ Bob $\to \dots$
3. The last message must $f(x,y)$.

Π = " Alice & Bob send their positions of bits set to 1, in turn, in ascending order, and the first of Alice & Bob that conclude sends $f(x,y)$. "



Def. The ST of a com. protocole Π is the sum of the message length exchanged, in the worst-case inputs (x,y).

Exercice Compute/estimate the cost of this protocole Π for $x, y \in \{0,1\}^n$?

Answer. Alice & Bob send positions written in binary, $i \in \{0,\dots,n-1\} \Rightarrow \lceil \log_2(i) \rceil$ bits $\leq \lceil \log_2 n \rceil$ bits
The positions that are transmitted $(i_1, i_2, i_3, \dots, i_k, f(x,y))$
$k \leq n$
$\text{cost}(\Pi) \leq k \cdot \lceil \log_2 n \rceil + 1 \leq O(n \log_2 n)$ bits.

Rem. There is a protocole $(\forall f)$ with cost $\leq n+1$ bits.
Trivial protocole Π_0 = "Alice sends x to Bob, then Bob computes $f(x,y)$ and sends to Alice"

What we would to say: $\forall \Pi$ (of a given f) the $\text{cost}(\Pi) \geq M(n)$.

Def. We would to define the Communication Complexity of f, as the cost of a minimum communication protocole that computes f.

Given a protocole Π, and $x, y \in \{0,1\}^n$

$$\text{Script}_\Pi(x,y) = (\underbrace{m_0}_{A}, \underbrace{m_1}_{B}, \dots, \underbrace{m_\ell}_{f(x,y)})$$

We concentrate our attention on function

SET-DISJOINTNESS

$$\text{Disj}_n(x,y) = \neg\left(\bigvee_{i=0}^{n} x[i] \wedge y[i]\right) = \begin{cases} 1 \Leftrightarrow x, y \text{ are disjoints} \\ \text{"no 1's in common"} \\ 0 \Leftrightarrow x, y \text{ common} \end{cases}$$

$A, B \subseteq \{0,\dots,n-1\}$ $\quad A = \{2,4,5\}$ $\quad B = \{1,4,5,6\}$ intersect.
$x = 0101100$ $\quad y = 1110001$
$A \cap B = \{5\} \Rightarrow \text{Disj}_7(x,y) = 0$

What is the com. complexity (min. number of bits to exchange) of Disj_n?

The matrix of f : $\{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ ⇔ the "truth table"

$f = \text{Disj}_{n=2}$	00	01	10	11
00	1	1	1	1
01	1	0	1	0
10	1	1	0	0
11	1	0	0	0

(Alice & Bob one owner of M of f)

7 scripts $(x,\bar{x})$ $(x',\bar{x}')$
$\{(11,00),(10,01),(01,10),(00)\}$ as Disc. Set for $x=1$.
$\{(10,10),(01,11)\}$ for $x \neq 2$ scripts.

Rectangle property: Given f, Π
If $\text{script}_\Pi(x,y) = \text{script}_\Pi(x',y')$, then $\text{script}_\Pi(x',y') = \text{script}_\Pi(x,y) = \text{script}_\Pi(x,y)$

To determine a lower of the cost of $\Pi, \forall \Pi$, we try to minimize the number of scripts.

Ex Assume that f uses 2 scripts for some Π,

$S_1 = (\underbrace{\text{0100100}}_{x}, \underbrace{x_1, x_2, \dots}_{\bar{x}}, \underbrace{m_1}_{A})$
$S_2 = (\underbrace{\text{0101100}}_{x'}, \underbrace{x_1', \dots}_{\bar{x}'}, \underbrace{m_1}_{A})$

$\} \Rightarrow \text{cost}(\Pi) = \max(|S_1|, |S_2|)$

Then, there is another protocole Π' for f, that uses only 2 bits $\leq \text{cost}(\Pi)$.
- S_1, S_2 can be computed by Alice & Bob.
- If the first difference is at Xth position (m_1 or m_2) Alice can send m_1 or m_2, but she can directly send m_1 or m_2.
- If the "set of x is odd", Bob decides if S_1 or S_2 and sends the answer to Alice, and first Alice starts with "0" message.
  At most 2 bits protocole.

{ Number of scripts in any protocole } is related to the com. complexity of f.

Again, intuition is: if for all protocoles for f, we need 2^k scripts, then Alice & Bob needs to transmit k bits.

Idea of the proof: (H) $\text{script}_\Pi(x,y) = \text{script}_\Pi(x',y')$

$\text{script}_\Pi(x,y)[0] = m_0 = \text{script}_\Pi(x,y')$ $\forall y'$
because the 1st message of Alice does not depend on y, y'.

$\text{script}_\Pi(x,y')[0] = \text{script}_\Pi(x',y')$ $\forall x$
because the 1st msg of Alice does not depend on Bob's input.

$\Rightarrow$ (H) $\text{script}_\Pi(x,y')_{[0]} = \text{script}_\Pi(x,y')_{[0]} = \text{script}_\Pi(x,y)_{[0]}$

By induction on ℓ, $\text{script}_\Pi(x,y)[0 \dots i] \dots$ ∎

Discriminant Set

This is a set of entries in M $\{\dots (x,y) \cdot (x',y') \dots\}$ that pairwise contradicts the rectangle property

$\forall x \in \{0,1\}^n$, $F_x \subseteq \{0,1\}^n \times \{0,1\}^n$

F_x is a discriminant set for x and f if:

$$\begin{cases} \forall (y,\bar{y}), (y',\bar{y}') \in F_x \\ \quad \bullet\ f(x,y) = f(x',y') = 1^n \\ \quad \bullet\ \bar{y} \in \bar{y}' \text{ or } f(x,y') \neq f(x',y) \end{cases}$$

$\Rightarrow$ need to use different scripts for all $(x,y) \in F_x$.

Proposition If F_0, F_1 are discriminant sets (= fooling set) for f ($f_{|F_0} = 0, 1$), then the number of scripts for EVERY com. protocoles that computes f is AT LEAST $|F_0| + |F_1|$.

Proposition Every protocole for Disj_n has at least $2^n + n$ different scripts.

Proof. $F_x = \{(x,\bar{x}) : x \in \{0,1\}^n\}$ is a disc. set for $x=1$.
x and $\bar{x}$ are disjoint $\Leftrightarrow \forall i (x[i] = \bar{x}[i] = 1)$.

$F_0 = \{(2^i, 2^i) : i \in \{0,\dots,n-1\}\}$
$\text{Disj}_n(2^i, 2^j) = 1$ if $i \neq j$

∎