

UE: APPLIED ALGORITHMS - 4TIN915U

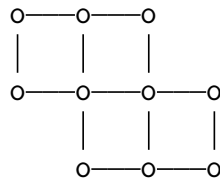
(test – duration 1h30)

- *Answers can be done in English or in French* –
 - *Documents (course and personal notes) allowed* –
(the official lecture notes will be available)
-

Vertex Cover

Question 1. *Explain what is a “minimum vertex cover” for a given graph.*

Question 2. *Find a minimum vertex cover for the graph G_0 depicted below. (You can draw the graph with your solution).*



Question 3. *For G_0 , find a matching of maximum size, i.e., a set of independent edges of maximum cardinality. (You can draw the graph with your solution).*

Question 4. *For a general graph G , what is the relation between the size of a matching and the size of a vertex cover?*

We consider the following algorithm (seen in the lecture), that applies to any connected graph G . See Figure 1 for an example.

Algorithme $\text{Approx_VC}_2(G)$

1. Compute for G a DFS spanning tree T .
 2. Return the set S of inner vertices of T , i.e., all its vertices but its leaves.
-

We have seen that Approx_VC_2 is a 2-approximation for the minimum vertex cover, assuming the graph is connected.

Question 5. *Explain what does it mean that Approx_VC_2 is a 2-approximation?*

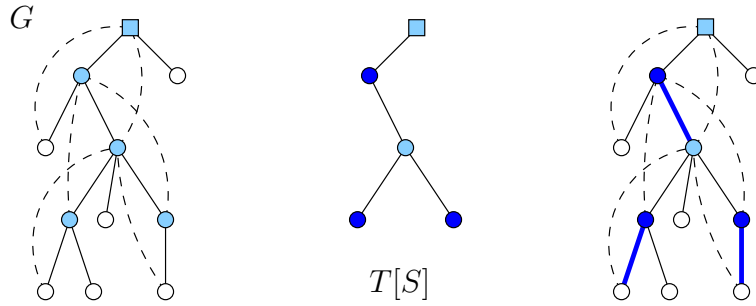


Figure 1: A vertex cover S (colored vertices) for G , computed from a DFS spanning tree T . Edges of $G \setminus T$ are dotted. A 2-coloring for $T[S]$ shows that G (in fact T) contained a matching of size at least $|S|/2$ (bold edges in the left picture).

Question 6. Show that the tree T in Figure 1 contains a matching of size 5.

Question 7. What is the approximation factor of Approx_VC2 for the graph G , given the tree T , as in Figure 1? Justify.

Question 8. Find a critical instance for Approx_VC2 , noting that the input graph must be connected. Justify.

In a rooted tree, a *pre-leaf* is a vertex that has a child which is a leaf. For instance, the tree T in Figure 1 has 5 pre-leaves, and it contains a matching of the same size.

Question 9. Do you think that, in general, the number of pre-leaves is a good lower bound in order to upper bound the approximate factor of Approx_VC2 ? i.e., discuss of the relation between minimum vertex cover for general graph G and the number of pre-leaves in any DFS spanning tree T .

We propose the following modification for Approx_VC2 . At Step 2, return all vertices of T with degree > 1 in T (instead of inner vertices).

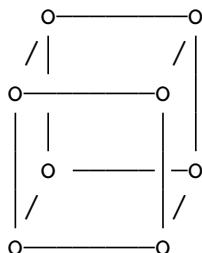
Question 10. Does this modification lead to a 2-approximation for minimum vertex cover? Justify.

Sphere Cover

Consider a set P of points in the 3D Euclidean space, that is $P \subset \mathbb{R}^3$. The *circularity* of P is the largest number c such that there exists a plane $H \subset \mathbb{R}^3$ with a circle $C \subset H$ containing c points of P .

For instance, if P is the set of the 8 points corresponding to the vertices of a cube (see the figure below), then the circularity of P is 4. Indeed, (1) each face defines a plane H

with a circle C containing 4 vertices (there are planes other than these 6 faces containing 4 vertices as well); and (2) no planes of \mathbb{R}^3 can contain a circle with more than 4 vertices of the cube.



We will assume to be given two routines running in constant time:

- $\text{circle}(u, v, w)$ that, given 3 points $u, v, w \in \mathbb{R}^3$, returns a circle C containing u, v, w .
- $\text{check}(p, C)$ that says whether a point p belongs to a circle C or not.

In the case where u, v, w are aligned, we can assume that $\text{circle}(u, v, w)$ returns a special circle C with infinite radius (in fact a line), and that $\text{check}(p, C)$, in this case, says whether p is aligned with u, v, w or not.

Question 11. Give an algorithm that computes the circularity of any point set $P \subset \mathbb{R}^3$. Describe its principle and give its complexity expressed as a function of $n = |P|$.

A *sphere cover* for the point set $P \subset \mathbb{R}^3$ is a collection \mathcal{C} of spheres such that every point of P belongs to the surface of some sphere of \mathcal{C} . We consider the decision problem SPHERE COVER defined as follows:

Instance: A point set $P \subset \mathbb{R}^3$ and $k \in \mathbb{N}$.
Question: Does P has a sphere cover of size k ?

Recall that it takes a constant time to check whether a points $p \in \mathbb{R}^3$ belongs to the surface of some sphere described by its center u and radius r . Indeed, it is sufficient to check that the Euclidean distance between u and p equals r .

Question 12. Give a reduction rule showing that SPHERE COVER admits a kernel whenever the problem is parametrized by k and the circularity c of P . (A complete justification is required here! You have to show that your reduction is valid and is polynomial (give its complexity), and you have to express the size of the kernel as function of the parameters.) [Hint: Observe that the circularity of any subset of P is at most c , and that four points uniquely defined a sphere that can be computed (center and radius) in constant time.]

FIN.