

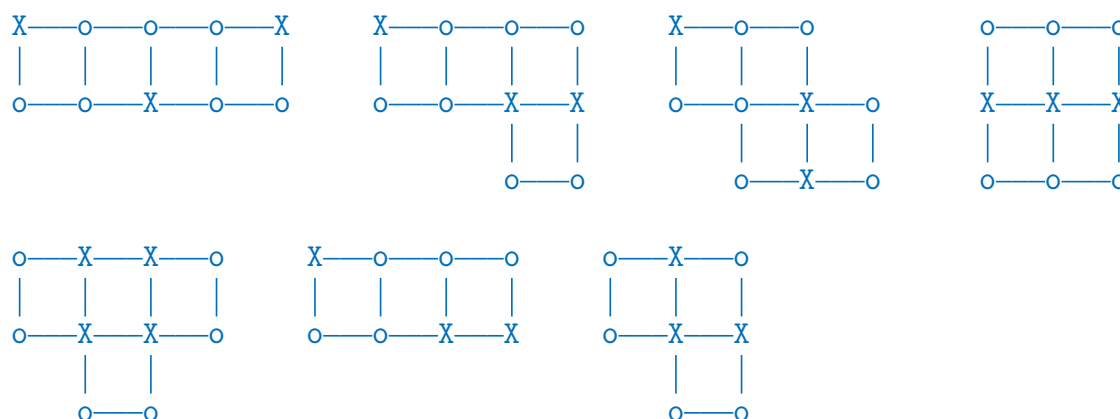
Answers can be done in English or in French. Documents (course and personal notes) allowed. The official lecture notes will be available.

Tetris

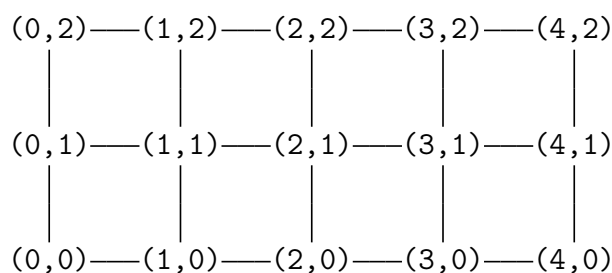
A *tetris-graph* is a 2-connected subgraph of a 2D grid containing 3 or 4 squares. Recall that a k -connected graph is a connected graph that needs the removal of at least k vertices to be disconnected. Two cycles sharing a single vertex is a connected graph but not 2-connected. However, the 5×3 -grid drawn below is a 2-connected graph.

Question 1. Draw all the possible tetris-graphs with a dominating set of minimum size for each of them. We do not ask to prove optimality, just to draw an optimal solution.

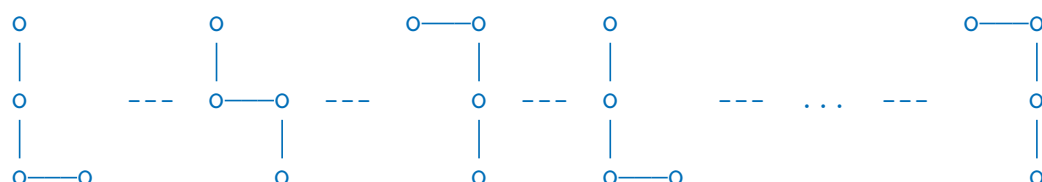
SOLUTION. [10 pts | $\sum 10.0$]



Question 2. Give a tree-decomposition for the 5×3 -grid as drawn below with the smallest possible width. Give the tree, the bags and their contents. [Hint : Check that all the properties hold. Try a tree-decomposition that is a path.]



SOLUTION. [3 pts | $\sum 13.0$] La décomposition est un chemin, et chaque sac est un chemin de 4 sommets de la grille utilisant deux colonnes consécutives :



Plus précisément, cela donne :

$\{02, 01, 00, 10\}$ --- $\{02, 01, 11, 10\}$ --- $\{02, 12, 11, 10\}$ ---
 $\{12, 11, 10, 20\}$ --- $\{12, 11, 21, 20\}$ --- $\{12, 22, 21, 20\}$ ---
 $\{22, 21, 20, 30\}$ --- $\{22, 21, 31, 30\}$ --- $\{22, 32, 31, 30\}$ ---
 $\{32, 31, 30, 40\}$ --- $\{32, 31, 41, 40\}$ --- $\{32, 42, 41, 40\}$

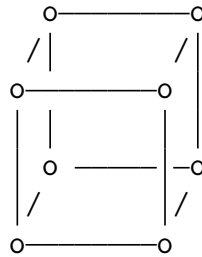
Question 3. Conclude that the tree-width of tetris-graphs is no more than 3.

SOLUTION. [3 pts | $\sum 16.0$] Tout tetris-graph est un sous-graphe de la grille 3×5 . Or la tree-width d'un sous-graphe de G ne peut pas dépasser celle de G (c'est même clos par mineur).

Covering Points by Spheres

Consider a set P of points in the 3D Euclidean space, that is $P \subset \mathbb{R}^3$. The *circularity* of P is the largest number c such that there exists a plan $H \subset \mathbb{R}^3$ with a circle $C \subset H$ containing c points of P .

For instance, if P is the set of the 8 points corresponding to the vertices of a cube (see the figure below), then the circularity of P is 4. Indeed, (1) each face defines a plan H with a circle C containing 4 vertices (there are other plans than these 6 faces containing 4 vertices as well); and (2) no other plan of $\mathbb{R}^3$ can contain a circle with more than 4 vertices of the cube.



We will assume to be given two routines running in constant time :

- $\text{circle}(u, v, w)$ that, given 3 points $u, v, w \in \mathbb{R}^3$, returns a circle C containing u, v, w .
- $\text{check}(p, C)$ that says whether a point p belongs to a circle C or not.

In the case where u, v, w are aligned, we can assume that $\text{circle}(u, v, w)$ returns a special circle C with infinite radius (in fact a line), and that $\text{check}(p, C)$, in this case, says whether p is aligned with u, v, w or not.

Question 4. Give an algorithm that computes the circularity of any point set $P \subset \mathbb{R}^3$. Describe its principle and give its complexity as a function of $n = |P|$.

SOLUTION. [3 pts | $\sum 19.0$] Pour chaque triplet u, v, w de P , calculer $C = \text{circle}(u, v, w)$. Puis vérifier combien de points de P appartiennent à C , avec $\text{check}(\cdot, C)$. Cela prend un temps $O(n)$ pour chaque triplet, soit $O(n^4)$.

A *sphere cover* for the point set $P \subset \mathbb{R}^3$ is a collection $\mathcal{C}$ of spheres such that every point of P belongs to the *surface* of some sphere of $\mathcal{C}$.

We consider the decision problem SPHERE COVER defined as follows :

Instance : A point set $P \subset \mathbb{R}^3$ and $k \in \mathbb{N}$.

Question : Does P has a sphere cover of size k ?

We will also consider the optimization variant, called MINIMUM SPHERE COVER, that asks to find the minimum k such that P has a sphere cover of size k .

Recall that it takes a constant time to check whether a points $p \in \mathbb{R}^3$ belongs to the surface of some sphere described by its center u and radius r . Indeed, it is sufficient to check that the Euclidean distance between u and p equals r .

Question 5. Give a reduction rule showing that SPHERE COVER admits a kernel whenever the problem is parametrized by k and the circularity c of P . (A complete justification is required here! You have to show that your reduction is valid and is polynomial (give its complexity), and you have to express the size of the kernel as function of the parameters.) [Hint : Observe that the circularity of any subset of P is at most c .]

SOLUTION. [3 pts |Σ22.0] Prendre une sphère contenant plus de ck points où c est la circularité de P . La remarque est que deux sphères s'intersectent en un cercle, cercle qui ne peut contenir qu'au plus c points de P par définition de c . Soit S un ensemble de $> ck$ points appartenant à une certaine sphère. Si on ne prend pas la sphère couvrant S alors tout autre sphère ne pourra contenir qu'au plus c points. Il en faudra donc $> k$ sphères pour couvrir S . Il y a que $\binom{n}{4} = O(n^4)$ sphères possibles à tester (pour définir une sphère quatre points étant nécessaires et suffisants), ce qui est polynomial et prend un temps en $O(n^5)$ pour tester toutes les sphères et points de P . Et avec k sphères couvrant chacune au plus ck points, on ne couvre que ck^2 points au maximum. Donc le noyau est de taille ck^2 .

Question 6. Explain to which general problem seen in the course MINIMUM SPHERE COVER is a particular case. In particular, what are the sets B , $\mathcal{F}$, and their size $|B|$, $|\mathcal{F}|$?

SOLUTION. [3 pts |Σ25.0] Il s'agit de SET COVER pour $(B, \mathcal{F})$ avec $B = P$ et $\mathcal{F}$ est l'ensemble des sphères définies (de manière unique) par quatre points de P . On a donc $|B| = |P| = n$ et $|\mathcal{F}| \leq \binom{|P|}{4} \leq n^4$.

Question 7. Explain how to approximate MINIMUM SPHERE COVER in polynomial time. What is the approximation factor? Try to be accurate.

SOLUTION. [3 pts |Σ28.0] L'algorithme est GreedyCover($B, \mathcal{F}$) qui produit une solution avec un facteur d'approximation $H(k) < 1 + \ln |B| = 1 + \ln n$.

Hitting Set

In this exercise, we consider again points in $\mathbb{R}^3$. We denote by G_n all the points corresponding to the vertices of a regular 3D grid $n \times n \times n$. Note that G_n has n^3 points, and that G_2 is the point set of the cube as depicted in the previous exercise.

We assume that each point $u \in G_n$ is associated with a ball B_u centered at u and with some real radius $r_u > 0$. Depending on r_u , each ball B_u contains some subset of points of G_n . For instance, in the cube example, assuming unit length edges and $r_u = 1$ for all $u \in G_2$, each ball B_u contains exactly 4 points of G_2 : the center u and its 3 neighbors.

Let $\mathcal{B}_n = \{B_u : u \in G_n\}$ be the collection of balls for the points of G_n . We are looking for a *hitting set* for $\mathcal{B}_n$ that is in G_n , that is a subset $S \subseteq G_n$ of minimum size that intersects each ball of G_n . More precisely, for each ball $B_u \in \mathcal{B}_n$ there must exist a point $s \in S \cap B_u$.

Question 8. Give a hitting set S of minimum size for $\mathcal{B}_2$ for the cube with unit length edges and unit radii.

SOLUTION. [3 pts |Σ31.0] Il y a une solution avec 2 points à distance 3 diagonalement opposés.

Question 9. In the light of Question , explain how to approximate the MINIMUM HITTING SET problem for $\mathcal{B}_n$ in polynomial time, and give the approximation factor. Try to be accurate. [Hint : The size of $\mathcal{B}_n$ is not n .]

SOLUTION. [3 pts |Σ34.0] Cela revient à SET COVER. Il faut choisir pour B l'ensemble des boules, donc $B = \mathcal{B}_n$, et pour $\mathcal{F}$ l'ensemble des boules contenant un point u donné, soit $\mathcal{F} = \left\{ \bigcup_{v \in B_u} \{B_v\} : u \in G_n \right\}$. On a $|B| = n^3$ et $|\mathcal{F}| = n^3$. Il suit que le facteur d'approximation est au plus $H(n^3) < 1 + 3 \ln n$.

END.