# Edge Partition of Planar Graphs into Two Outerplanar Graphs

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# ABSTRACT

An outerplanar graph is a planar graph that can be embedded in the plane without crossing edges, in such a way that all the vertices are on the outer boundary. In this paper, we prove a conjecture of Chartrand, Geller, and Hedetniemi that any planar graph G = (V, E) has a bipartition of its edge set  $E = A \cup B$  such that the graphs induced by these subsets, G[A] and G[B], are outerplanar.

#### **Categories and Subject Descriptors**

G.2.2 [DISCRETE MATHEMATICS]: Graph Theory

#### **General Terms**

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# Keywords

Edge partition, outerplanar graphs, planar graphs

# **1. INTRODUCTION**

Much work has been done in partitioning the edge sets of graphs such that each subset induces a subgraph of a certain form. See for example the concepts of chromatic index [15], arboricity [13], thickness [12], or the track number of a graph [11]. In this vein, Chartrand, Geller and Hedetniemi ([2] and Problem 6.3 in [9]) made the famous [m, n]-conjecture. They defined the graphs with property  $P_m$  as the graphs containing no subdivision of  $K_{m+1}$  or  $K_{\lceil m/2 \rceil + 1, \lfloor m/2 \rfloor + 1}$ . Observe that the graphs with property  $P_4$  (resp.  $P_3$ ) are the planar graphs (resp. outerplanar graphs). The [m, n]-conjecture was that any graph with property  $P_m$  has an edge partition into m - n + 1 graphs with property  $P_n$ , for  $m \ge n \ge 2$ . In [8], Gutin, Kostochka and Toft proved that the conjecture

is false when  $m > cn^2$ , for some constant c. In this paper, we prove that the conjecture holds for m = 4 and n = 3, that is, that planar graphs have an edge partition into two outerplanar graphs. There have been various results toward this special case of the conjecture. Several proofs have been claimed, but later proved to be incorrect. Colbourn and El-Mallah [5] gave a first partial result showing that every planar graph has an edge partition into two partial 3-trees. Then Kedlaya [10] and Ding et al. [4] proved that a partition into two partial 2-trees exists. Another result in [4] is that planar graphs have an edge partition into two outerplanar graphs and a vee-forest (each connected component is a  $K_{1,1}$  or a  $K_{1,2}$ ).

A simple case of planar graphs that can be divided into two outerplanar graphs are the hamiltonian planar graphs. The first outerplanar graph is constructed with the edges of an hamiltonian cycle together with the edges in the interior of this cycle, and the second one with the edges of this hamiltonian cycle together with the edges in the exterior of this cycle. There is a lot of flexibility in this type of partition since the edges of the hamiltonian cycle can be in both subgraphs. In [16], Whitney proved that 4-connected triangulations are hamiltonian. In [14], Tutte generalized this result to 4-connected planar graphs. So we know that the conjecture holds for 4-connected planar graphs. We note that with this type of partition the two graphs obtained are outerplanarly embedded. This means that, given an embedding of the planar graph, the embedding it induces for each subgraph is such that there is a face containing all the vertices on its boundary. But there is not always a bipartition into outerplanarly embedded subgraphs. An interesting result of Kedlaya [10] is that embedded planar graphs cannot always by partitioned into two outerplanar graphs, one of them being outerplanarly embedded. This implies for example that there are planar graphs with no edge partition into an outerplanar graph and a forest (forests being always outerplanarly embedded).

A triangulation is an embedded maximal planar graph. Since every planar graph is a subgraph of a triangulation and since every subgraph of an outerplanar graph is outerplanar, we can restrict our work to triangulations. Given a triangulation T, let  $T^*$  be the triangulation created by adding to T one vertex in each bounded face of T, this vertex being connected to all the vertices of that face. If a triangulation  $T^*$  has an edge partition into two outerplanar graphs, then the partition it induces in T also divides it into two outerplanar graphs. We say that this partition

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Figure 1: An Outerplanar Sprout

of T is extendable. If all the triangulations (e.g.  $T^*$ ) have an edge bipartition into outerplanar graphs, then all the triangulations T have an extendable one. In this paper, we construct extendable bipartitions because they allow simple inductions. It is possible to extend to  $T^*$  an extendable outerplanar bipartitions of T, but it is not simple. Consider a triangulation T with an extendable outerplanar bipartition  $A \cup B$ . Let z be the vertex of  $T^*$  we added in the face uvw of T. At least two of the edges zu, zv and zw have to be in the same subset. If we put zu and zv in A, we may destroy the outerplanarity of  $T^*[A]$ . There are various possible combinations, and we know that at least one is correct. We need more information on the outerplanar graphs induced by Aand B; It is why we introduce the concept of outerplanar sprout in the next section. In Section 3 and 4, we present a theorem that implies the conjecture. In the conclusions, we discuss on the consequences of our result for other graph problems, and we raise two open questions.

#### 2. OUTERPLANAR SPROUTS

In a triangulation T, we denote by  $V_T$ ,  $E_T$ , and  $F_T$  the vertex set, the edge set, and the inner face set of T. The edges and the faces of T are respectively denoted as ab and abc, according to their adjacent vertices. Given an outeplanar graph H, we define its growing capacity function,  $\Gamma_H : E_H \to \{0, 1, 2\}$ , of an edge e, as the number of times we pass through e going around the outer boundary of H, in an outerplanar embedding of H. The bridges of H have growing capacity two, the other edges of the outer boundary have growing capacity one, and the rest of the edges of H have growing capacity zero. Observe that  $\Gamma_H(uv)$  corresponds to the maximal number of new vertices we can connect to both u and v while maintaining the outerplanarity of H.

An outerplanar sprout  $S = (S, \sigma)$  is defined in a given triangulation T. The first element of this pair is an edge set  $S \subseteq E_T$  that induces an outerplanar graph T[S]. The second element is a function  $\sigma : E_T \longrightarrow \mathcal{P}_2(F_T)$ . If  $uv \in S$ , then we have  $\sigma(uv) \subseteq \{uvx, uvy\}$ , where uvx and uvy are the two faces incident to uv in T, and  $\Gamma_{T[S]}(uv) - |\sigma(uv)| \ge 0$ . If  $uv \notin S$ , then  $\sigma(uv) = \emptyset$ . With these outerplanar sprouts, it will be easier to extend a bipartition of T to  $T^*$ : Let  $E_T = A \cup B$  be an edge bipartition of T into two outerplanar graphs. If A is the edge set of an outerplanar sprout  $\mathcal{A} =$  $(A, \alpha)$  and if the face uvw is in  $\alpha(uv)$  then if we add a vertex z in uvw linked to u, v and w, we know that we can add the edges uz and vz to A (since,  $\Gamma_{T[A]}(uv) \ge |\alpha(uv)| \ge 1$ ), and the edge wz in B while keeping the graphs induced by A and B outerplanar. If  $f \in \alpha(e)$  we allow the outerplanar



Figure 2: The graph  $S_3$ 



Figure 3: Lemma 1

sprout  $\mathcal{A}$  to expand its 2-connected component containing the edge e inside the face f. So the function of  $\alpha$  is to indicate how some 2-connected components of  $\mathcal{A}$  can grow. When drawing an outerplanar sprout  $\mathcal{A} = (A, \alpha)$ , we will denote the fact that a face f belongs to  $\alpha(e)$ , for an edge e, by drawing an arrow from e to f (see Figure 1).

Next we extend some concepts of outerplanar graphs to outerplanar sprouts. For the rest of the section we consider two outerplanar sprouts  $\mathcal{A} = (A, \alpha)$  and  $\mathcal{B} = (B, \beta)$  in a triangulation T. The outerplanar sprout  $\mathcal{A}$  is a connected component of  $\mathcal{B}$  if T[A] is a connected component of T[B]and if  $\alpha(e) = \beta(e)$  over A. We say that A and B intersect in the subgraph  $G = (V_{T[A]} \cap V_{T[B]}, A \cap B)$  of T. The outerplanar sprout  $\mathcal{A}$  is spanning, if the vertex set of T[A] is  $V_T$ . The outerplanar sprout  $\mathcal{A}$  is *chordal*, if T[A] is chordal, this is without chordless cycle of length more than three. The outerplanar sprout  $\mathcal{A}$  is  $S_3$ -free (see Figure 2 for  $S_3$ ), if  $T[\mathcal{A}]$ is  $S_3$ -free, this is without subgraphs isomorphic to  $S_3$ . The growing capacity function  $\Gamma_{\mathcal{A}}: \mathcal{A} \to \{0, 1, 2\}$  of the outerplanar sprout  $\mathcal{A}$  is equal to  $\Gamma_{T[A]}(e) - |\alpha(e)|$ , for any edge  $e \in A$ .  $\Gamma_{\mathcal{A}}$  denotes the remaining growing capacity of the outerplanar graph T[A], considering the constraints imposed by  $\alpha$ . Observe that  $\Gamma_{\mathcal{A}}(uv)$  corresponds to the maximal number of vertices  $w \notin V_{T[A]}$  adjacent to both u and v such that adding the edges uw and vw to A and extending  $\alpha$  to them with  $|\alpha(uw)| \leq 1$  and  $|\alpha(vw)| \leq 1$ , the pair  $\mathcal{A} = (A, \alpha)$  remains an outerplanar sprout. We define the bridges of  $\mathcal{A}$  as the edges  $e \in A$  with growing capacity two. These are the bridges e of T[A] with  $\alpha(e) = \emptyset$ . We say that  $\mathcal{A}$  covers the edge e (resp. the face f) if  $e \in A$  (resp.  $f \in \alpha(e)$ , for some  $e \in A$ ).

We define the *union* of two outerplanar sprouts  $\mathcal{A} \cup \mathcal{B}$  as the pair  $(M, \mu)$  where  $M = A \cup B$  and  $\mu(e) = \alpha(e) \cup \beta(e)$ , for any edge e. Note that the union of two outerplanar sprouts may not be an outerplanar sprout, if T[M] is not an outerplanar graph, or if  $\Gamma_{T[M]}(e) - |\mu(e)| < 0$  for some edge  $e \in M$ . In the next lemmas, we present some cases in which the union produces an outerplanar sprout. These lemmas are used in the proofs of Theorem 1 and Theorem 2, it is why we just deal with some specific cases.



Figure 4: Lemma 2

LEMMA 1. If two chordal outerplanar sprouts,  $\mathcal{A}$  and  $\mathcal{B}$ , intersect in at most one vertex, their union  $\mathcal{M} = (\mathcal{M}, \mu) = \mathcal{A} \cup \mathcal{B}$  is also a chordal outerplanar sprout. Then  $\Gamma_{\mathcal{M}}(e) = \Gamma_{\mathcal{A}}(e)$ , for  $e \in A$ , and  $\Gamma_{\mathcal{M}}(e) = \Gamma_{\mathcal{B}}(e)$ , for  $e \in B$ . Furthermore, if two vertices are in distinct connected components of  $\mathcal{A}$  (resp.  $\mathcal{B}$ ) they remain in distinct connected components in  $\mathcal{M}$ .

PROOF. The graph T[M] is clearly outerplanar, and this union cannot produce any new chordless cycle. Since  $\Gamma_{T[M]}$ equals T[A] or T[B], the same occurs with  $\Gamma_{\mathcal{M}}$ ,  $\Gamma_{\mathcal{A}}$  and  $\Gamma_{\mathcal{B}}$ .  $\Box$ 

Let us observe some implications of this lemma needed in the proofs of Theorem 1 and Theorem 2. Since the growing capacity does not change, note that the bridges of  $\mathcal{A}$  and  $\mathcal{B}$ are still bridges in  $\mathcal{M}$ . This lemma confirms that an outerplanar sprout is the union of its connected components. Another simple example of this lemma is when  $\mathcal{B}$  is just a bridge ( $B = \{uv\}$  and  $\beta(uv) = \emptyset$ ) such that  $u \in V_{T[A]}$  and  $v \notin V_{T[A]}$ . This implies that, if an edge  $e \in E_T$  has just one end in  $V_{T[A]}$  and if we add this edge to A and let  $\alpha(e) = \emptyset$ , then  $\mathcal{A}$  remains an outerplanar sprout.

LEMMA 2. If two chordal outerplanar sprouts,  $\mathcal{A}$  and  $\mathcal{B}$ , intersect in a path  $P = (v_1, ..., v_k)$  such that the edges  $v_i v_{i+1}$ are bridges of  $\mathcal{B}$ , their union  $\mathcal{M} = (\mathcal{M}, \mu) = \mathcal{A} \cup \mathcal{B}$  is also a chordal outerplanar sprout. Then  $\Gamma_{\mathcal{M}}(e) = \Gamma_{\mathcal{A}}(e)$ , for  $e \in A$ , and  $\Gamma_{\mathcal{M}}(e) = \Gamma_{\mathcal{B}}(e)$ , for  $e \in B \setminus A$ . Furthermore, if two vertices are in distinct connected components of  $\mathcal{A}$  (resp.  $\mathcal{B}$ ) they remain in distinct connected components in  $\mathcal{M}$ .

PROOF. Remove the edges of the path from  $\mathcal{B}$ , and then just apply Lemma 1 for each connected component of  $\mathcal{B}$ .

LEMMA 3. If two chordal outerplanar sprouts,  $\mathcal{A}$  and  $\mathcal{B}$ , intersect in an edge  $e_1$  such that  $\Gamma_{\mathcal{A}}(e_1) \geq 1$  and  $\Gamma_{\mathcal{B}}(e_1) \geq 1$ , their union  $\mathcal{M} = (\mathcal{M}, \mu) = \mathcal{A} \cup \mathcal{B}$  is also a chordal outerplanar sprout. Then  $\Gamma_{\mathcal{M}}(e) = \Gamma_{\mathcal{A}}(e)$ , for  $e \in \mathcal{A} \setminus \{e_1\}$ , and  $\Gamma_{\mathcal{M}}(e) = \Gamma_{\mathcal{B}}(e)$ , for  $e \in \mathcal{B} \setminus \{e_1\}$ . Furthermore, if two vertices are in distinct connected components of  $\mathcal{A}$  (resp.  $\mathcal{B}$ ) they remain in distinct connected components in  $\mathcal{M}$ .

PROOF. If  $e_1$  is a bridge in  $\mathcal{A}$  or  $\mathcal{B}$ , then just apply Lemma 2. Otherwise, if  $e_1$  is a bridge in T[A] (resp. T[B]) with  $\alpha(e_1) = \{f\}$ , then remove  $e_1$  from A, add f to  $\beta(e_1)$  (since  $\Gamma_{\mathcal{B}}(e_1) \geq 1$ ) and then add each connected component of  $\mathcal{A}$  using Lemma 1.



Figure 5: Lemma 3



Figure 6: Lemma 4

Otherwise, if  $e_1$  is, in both T[A] and T[B], an edge of the outer boundary of a 2-connected component, with  $\alpha(e_1) = \beta(e_1) = \emptyset$ . In this case outerplanarly draw T[A] in the plane, in such a way that T[A] is in a half-plane delimited by a line prolonging  $e_1$ . Then, outerplanarly draw T[B] in the other half-plane. It is clear that the graph obtained is outerplanar and that  $\Gamma_{\mathcal{M}}$  differs just for  $e_1$ .  $\Box$ 

Let us observe some implications of this lemma needed in the proofs of Theorem 1 and Theorem 2. Let uv be an edge of A with  $\Gamma_{\mathcal{A}}(uv) \geq 1$ . If  $\mathcal{B}$  is such that  $B = \{uv\}$ and  $\beta(uv) = \{uvw\}$  the lemma just shows us that, since  $\Gamma_{\mathcal{A}}(uv) \geq 1$ , we can add the face uvw to  $\alpha(uv)$ . Another simple example of this lemma is when  $\mathcal{B}$  forms a face  $(B = \{uv, vw, uw\}, \beta(uv) = \beta(uw) = \emptyset$  and  $\beta(vw) = \{uvw\}$ . This implies that, if a face  $uvw \in F_T$  has an incident edge  $uv \in E_{T[A]}$  such that  $\Gamma_{\mathcal{A}}(uv) \geq 1$  and if we add the edges uw and vw to A and let  $\alpha(uw) = \emptyset$  and  $\alpha(vw) = \{uvw\}$ , then  $\mathcal{A}$  remains an outerplanar sprout.

LEMMA 4. Consider a chordal outerplanar sprout  $\mathcal{A}$  and a face  $uvw \in F_T$ , such that the edges uw and vw are in  $\mathcal{A}$  with  $\Gamma_{\mathcal{A}}(uw) = 2$  and  $\Gamma_{\mathcal{A}}(vw) \geq 1$ . If we add this face uvw to  $\alpha(uw)$  and the edge uv to  $\mathcal{A}$  with  $\alpha(uv) = \emptyset$ , then  $\mathcal{A}$ remains a chordal outerplanar sprout. The growing capacity  $\Gamma_{\mathcal{A}}(e)$  remains unchanged for every edge  $e \in \mathcal{A} \setminus \{uw, vw, uv\}$ ; for the edge uv,  $\Gamma_{\mathcal{A}}(uv) = 1$ . Furthermore, if two vertices were in distinct connected components of  $\mathcal{A}$  they remain in distinct connected components.

Proof. See Figure 6.  $\Box$ 



Figure 7: The triangulations T,  $T_{ext}$  and  $T_{int}$ 

## **3. THE GENERAL CASE**

The following theorem is the main result of this paper (see the sktech on the right of Figure 7).

THEOREM 1. Given a triangulation T and any external vertex triplet (a, b, c) of T, there are two spanning chordal outerplanar sprouts  $\mathcal{A} = (A, \alpha)$  and  $\mathcal{B} = (B, \beta)$  that cover all the edges and all the inner faces of T such that :

- A is connected.
- B has two connected components, one with b and one with c.
- The edge ab is in A and  $\Gamma_{\mathcal{A}}(ab) \geq 1$ .
- The edge bc is a bridge of  $\mathcal{A}$  (i.e.  $\Gamma_{\mathcal{A}}(bc) = 2$ ).
- The edge ac is a bridge of  $\mathcal{B}$  (i.e.  $\Gamma_{\mathcal{B}}(ca) = 2$ ).

All triangulations are 3-connected. The 4-connected triangulations are the triangulations in which each cycle of length three delimits a face. A separating 3-cycle is a cycle of length three that does not delimit a face. If a triangulation T has a separating 3-cycle we can split T into two smaller triangulations, the first one,  $T_{int}$ , with the edges of the cycle and the edges inside the cycle, and the other one,  $T_{ext}$ , with the edges of the cycle and the edges outside the cycle.

PROOF. We now proceed by induction on the number of vertices in T. If T is a 4-connected triangulations, the proof follows from Theorem 2 presented in the next section. So we assume that T is not 4-connected and let C be one of its separating 3-cycles. Let  $T_{int}$  and  $T_{ext}$  be the two triangulations respectively in the interior and in the exterior of C. They both have less vertices than T so that the induction hypothesis holds. Let  $\mathcal{A}_{ext} = (A_e, \alpha_e)$  and  $\mathcal{B}_{ext} = (B_e, \beta_e)$ be the two outerplanar sprouts in  $T_{ext}$  for the triplet (a, b, c). Now consider the face delimited by C in  $T_{ext}$ . The induction hypothesis ensures that this face is covered by  $\mathcal{A}_{ext}$  or  $\mathcal{B}_{ext}$ . Without loss of generality let  $\mathcal{A}_{ext}$  be a sprout covering this face. We denote by a', b' and c' the vertices of C, so that  $a'b' \in A_e$  and  $a'b'c' \in \alpha_e(a'b')$  (see Figure 7). Note that, since bc and ac are bridges, we have that  $a'b' \neq bc$  and  $a'b' \neq ac$ . By the induction hypothesis, let  $\mathcal{A}_{int} = (A_i, \alpha_i)$ and  $\mathcal{B}_{int} = (B_i, \beta_i)$  be the two outerplanar sprouts in  $T_{int}$ for the triplet (a', b', c').

We want to construct  $\mathcal{A}$  and  $\mathcal{B}$  by respectively considering the union of  $\mathcal{A}_{ext}$  with  $\mathcal{A}_{int}$  and the union of  $\mathcal{B}_{ext}$  with  $\mathcal{B}_{int}$ ; but we have to slightly modify  $\mathcal{A}_{ext}$ ,  $\mathcal{A}_{int}$  and  $\mathcal{B}_{int}$ before. First, since there is no more face delimited by C to be covered in T, we remove this face from  $\alpha_e(a'b')$ . Since there is one element less in  $\alpha_e(a'b')$ , the growing capacity  $\Gamma_{\mathcal{A}_{ext}}(a'b')$  increases by one. Secondly, the edges b'c' and c'a' being already covered by  $\mathcal{A}_{ext}$  or  $\mathcal{B}_{ext}$  we remove them from  $\mathcal{A}_{int}$  and  $\mathcal{B}_{int}$ . Since b'c' was a bridge of  $\mathcal{A}_{int}$ , by removing it from  $\mathcal{A}_{int}$ , we have not uncovered any face of  $T_{int}$ , and now b' and c' are in distinct connected components of  $\mathcal{A}_{int}$ . Similarly, since a'c' was a bridge of  $\mathcal{B}_{int}$ , by removing it from  $\mathcal{B}_{int}$ , we have not uncovered any face of  $T_{int}$ . Furthermore, since b' was already in a distinct connected component from a' and c', now a', b' and c' are in three distinct connected components of  $\mathcal{B}_{int}$ . Now we can let  $\mathcal{A} = \mathcal{A}_{ext} \cup \mathcal{A}_{int}$  and  $\mathcal{B} = \mathcal{B}_{ext} \cup \mathcal{B}_{int}$ .

Now we prove that  $\mathcal{A}$  and  $\mathcal{B}$  are well-defined spanning chordal outerplanar sprouts. For  $\mathcal{B}$ , let the outerplanar sprouts  $\mathcal{B}_{int}^1$ ,  $\mathcal{B}_{int}^2$ , and  $\mathcal{B}_{int}^3$  be the connected components of  $\mathcal{B}_{int}$ . We have that  $\mathcal{B} = \mathcal{B}_{ext} \cup \mathcal{B}_{int}^1 \cup \mathcal{B}_{int}^2 \cup \mathcal{B}_{int}^3$ . Since a', b' and c' are in distinct components  $\mathcal{B}_{int}^i$ , for each of these unions, the outerplanar sprouts have at most one vertex in common. So, by Lemma 1, we have that  $\mathcal{B}$  is a well-defined spanning chordal outerplanar sprout where  $\Gamma_{\mathcal{B}}(ca) = 2$  and with b and c in distinct connected components. Furthermore, since  $\mathcal{B}_{ext}$  is spanning (it covers a', b', and c'), none of the  $\mathcal{B}_{int}^i$  creates a new connected component. So there are still two connected components, one with b and one with c. For  $\mathcal{A}$ , let the outerplanar sprouts  $\mathcal{A}_{int}^1$  and  $\mathcal{A}_{int}^2$  be the connected components of  $\mathcal{A}_{int}$ , such that a'b' is in  $\mathcal{A}_{int}^1$ . We have that  $\mathcal{A} = \mathcal{A}_{ext} \cup \mathcal{A}_{int}^1 \cup \mathcal{A}_{int}^2$ . Since  $\Gamma_{\mathcal{A}_{ext}}(a'b')$  has decreased by one, we have that  $\Gamma_{\mathcal{A}_{ext}}(a'b') = 2$ , if a'b' = ab, or that  $\Gamma_{\mathcal{A}_{ext}}(a'b') \geq 1$ , if  $a'b' \neq ab$ . We know that the union of  $\mathcal{A}_{ext}$  with  $\mathcal{A}_{int}^1$  is correct, by Lemma 2 if a'b' = ab, or by Lemma 3 otherwise. For the other union, since the outerplanar sprouts intersect in at most one vertex, c', we can apply Lemma 1. We then obtain that  $\mathcal{A}$  is a well-defined chordal outerplanar sprout with  $\Gamma_{\mathcal{A}}(ab) \geq 1$  and  $\Gamma_{\mathcal{A}}(ca) = 2$ . Since  $\mathcal{A}_{ext}$  and  $\mathcal{A}_{int}$  were spanning, and since all the  $\mathcal{A}_{int}^{i}$  intersect  $\mathcal{A}_{ext}, \mathcal{A}$  is spanning and connected.

Since all the edges and all the inner faces were covered before the unions, they are all covered by  $\mathcal{A}$  or  $\mathcal{B}$ .

Note that if the face delimited by C in  $T_{ext}$  was covered by  $\mathcal{B}_{ext}$ , instead of  $\mathcal{A}_{ext}$ , the only difference would be that  $\mathcal{B} = \mathcal{B}_{ext} \cup \mathcal{A}_{int}$  and  $\mathcal{A} = \mathcal{A}_{ext} \cup \mathcal{B}_{int}$ .



Figure 8: Case 1, there is a chord  $a_i b_j$ 

# 4. PARTIAL 4-CONNECTED TRIANGULATIONS

Given a cycle C in a triangulation T, we say that the graph T' induced by the edges of C and the edges inside C is a partial triangulation. We say that T' is a partial 4-connected triangulation if T is a 4-connected triangulation. In this case, T' may not be 4-connected but it has no separating 3-cycles. Consider a partial 4-connected triangulation T' bounded by the cycle C, and let C be partitioned into three paths of length at least one denoted  $(a_1a_2...a_p), (b_1b_2...b_q)$  and  $(c_1c_2...c_r)$  with  $a_p = b_1, b_q = c_1, c_r = a_1$ . We say that T' is  $(a_1a_2...a_p)-(b_1b_2...b_q)-(c_1c_2...c_r)$ -bounded, or that  $(a_1a_2...a_p)-(b_1b_2...b_q)-(c_1c_2...c_r)$  is the 3-bound of T', if there are no chords  $a_ia_j, b_ib_j$  or  $c_ic_j$ .

THEOREM 2. Any  $(a_1...a_p)$ - $(b_1...b_q)$ - $(c_1...c_r)$ -bounded partial 4-connected triangulation T has two spanning chordal outerplanar sprouts  $\mathcal{A} = (A, \alpha)$  and  $\mathcal{B} = (B, \beta)$  that cover all the edges and all the inner faces of T and such that :

- A is connected.
- $\mathcal{B}$  has two connected components, one with  $b_1$  and one with  $b_p$ .
- The edge  $a_1a_2$  is in A and  $\Gamma_{\mathcal{A}}(a_1a_2) \geq 1$ .
- The edges  $a_i a_{i+1}$  are bridges of  $\mathcal{B}$ , for  $i \geq 2$  (i.e.  $\Gamma_{\mathcal{B}}(a_i a_{i+1}) = 2$ ).
- The edges  $b_i b_{i+1}$  are bridges of  $\mathcal{A}$  (i.e.  $\Gamma_{\mathcal{A}}(b_i b_{i+1}) = 2$ ).
- The edges  $c_i c_{i+1}$  are bridges of  $\mathcal{B}$  (i.e.  $\Gamma_{\mathcal{B}}(c_i c_{i+1}) = 2$ ).
- $\mathcal{A}$  and  $\mathcal{B}$  are outerplanarly embedded.
- $\mathcal{A}$  and  $\mathcal{B}$  are  $S_3$ -free.

This theorem implies Theorem 1 for 4-connected triangulations. Given a 4-connected triangulation T and a triplet (a, b, c) of its vertices on the outer boundary, we can apply Theorem 2 to T and the 3-bound (ab)-(bc)-(ca) and obtain two sprouts  $\mathcal{A}$  and  $\mathcal{B}$  that satisfy all the conditions of Theorem 1.

PROOF. Note that the two last items are consequences of the rest of the theorem. If  $\mathcal{A}$  was not outerplanarly embedded, it would have a cycle and a vertex inside.  $\mathcal{A}$  being

chordal, this vertex would be inside a cycle of length three, contradicting the definition of partial 4-connected triangulation (Without separating 3-cycle). If  $\mathcal{A}$  had a subgraph  $S_3$  outerplanarly embedded, it would be impossible to cover the face in the center. So, we focus on the other properties.

Now we proceed by induction on the number of vertices in T. The base case is the  $K_3$  with its  $(a_1, a_2)-(b_1, b_2)-(c_1, c_2)$ bound. For this graph, we set  $A = \{a_1a_2, b_1b_2\}, B = \{c_1c_2\}, \alpha(a_1a_2) = \{a_1b_1c_1\}, \alpha(b_1, b_2) = \emptyset$  and  $\beta(c_1c_2) = \emptyset$ . Now to prove the induction, we consider different cases according to the presence of a chord  $a_ib_j$  or  $a_ic_j$  in T.

Case 1: There is a chord  $a_i b_j$ , for  $i \ge 2$  (Figure 8). First note that  $i \ne p$  and  $j \ne 1$ ; otherwise, we would have a chord  $b_1 b_j$  or a chord  $a_i a_p$ . If there are various such edges, we consider the one that maximizes j, so there is no edge  $a_i b_k$  with k > j. We split T into two smaller partial 4connected triangulations,  $T_l$  and  $T_r$ , which are respectively to the left and to the right of  $a_i b_j$ . Each of these partial triangulation has a 3-bound. For  $T_l$  (resp.  $T_r$ ), we consider  $(a_1 a_2 \dots a_i) - (a_i b_j \dots b_k) - (c_1 \dots c_k)$  (resp.  $(a_i b_j) - (b_j \dots b_1) - (a_p \dots a_i))$ . Indeed, there is no chord  $a_k a_l$ ,  $b_k b_l$ , or  $c_k c_l$  and no chord  $a_i b_k$ , for k > j. The partial triangulation  $T_l$  (resp.  $T_r$ ) having also less vertices than T, since  $b_1 \notin V_{T_l}$  (resp.  $a_1 \notin V_{T_r}$ ), we can apply the induction hypothesis for these 3-bounds and let  $\mathcal{A}_l$  and  $\mathcal{B}_l$  (resp.  $\mathcal{A}_r$  and  $\mathcal{B}_r$ ) be its two covering outerplanar sprouts.

Now let  $\mathcal{A} = \mathcal{A}_r \cup \mathcal{A}_l$  and  $\mathcal{B} = \mathcal{B}_l \cup \mathcal{B}_r$ . Since  $\mathcal{A}_l$  and  $\mathcal{A}_r$  have just the edge  $a_i b_j$  in common and since it is a bridge of  $\mathcal{A}_l$ ,  $\mathcal{A}$  is a well-defined spanning chordal outerplanar sprout, by Lemma 2. Lemma 2 also implies that  $a_1a_2 \in A$  with  $\Gamma_{\mathcal{A}}(a_1a_2) \geq 1$  and that all the edges  $b_ib_{i+1}$ are bridges of  $\mathcal{A}$ . Since  $\mathcal{B}_l$  and  $\mathcal{B}_r$  have just the vertices  $a_i$ and  $b_j$  in common and since these vertices are in distinct connected components of  $\mathcal{B}_r$ ,  $\mathcal{B}$  is a well-defined spanning chordal outerplanar sprout, by Lemma 1. This lemma implies that all the edges  $c_i c_{i+1}$  and all the edges  $a_i a_{i+1}$ , for  $i \geq 2$ , are bridges of  $\mathcal{B}$ . It also implies that the vertices  $a_i$ and  $b_p$  are in distinct connected components of  $\mathcal{B}$ , as they were in  $\mathcal{B}_l$ . So,  $a_i$  and  $b_1$  being in the same connected component of  $\mathcal{B}$ ,  $b_1$  and  $b_p$  are in distinct connected components of  $\mathcal{B}$ . Finally, since all the edges and all the inner faces of  $T_l$  and  $T_r$  were covered, all the edges and all the inner faces of T are now covered by  $\mathcal{A}$  or  $\mathcal{B}$ .



Figure 9: Case 2, there is a chord  $a_i c_j$ 



Figure 10: Case 3, there is a chord  $a_1b_2$ 

Case 2: There is a chord  $a_i c_j$ , for  $i \ge 2$  (Figure 9). This case is very similar to the previous one. Since the case with a chord  $a_i c_1$  has already been treated, let  $i \neq 1$ Since there is no chord  $a_1a_i$ , let  $j \neq r$ . If there are various such edges, we consider the one that maximizes i, so there is no edge  $c_i a_k$  with k > i. We split T into two smaller partial 4-connected triangulations,  $T_l$  and  $T_r$ , which are respectively to the left and to the right of  $a_i c_j$ . Each of these partial triangulations has a 3-bound. For  $T_l$  (resp.  $T_r$ ), we consider  $(a_1a_2...a_i)$ - $(a_ic_j)$ - $(c_j...c_k)$  (resp.  $(c_ja_i...a_p)$ - $(b_1...b_q)$ - $(c_1...c_i)$ . Indeed, there is no chord  $a_k a_l$ ,  $b_k b_l$  or  $c_k c_l$ and no chord  $c_i a_k$ , for k > i. The partial triangulation  $T_l$ (resp.  $T_r$ ) having also less vertices than T, since  $c_1 \notin V_{T_1}$ (resp.  $c_r \notin V_{T_r}$ ) we can apply the induction hypothesis for these 3-bounds and let  $\mathcal{A}_l$  and  $\mathcal{B}_l$  (resp.  $\mathcal{A}_r$  and  $\mathcal{B}_r$ ) be the two covering outerplanar sprouts.

Now let  $\mathcal{A} = \mathcal{A}_r \cup \mathcal{A}_l$  and  $\mathcal{B} = \mathcal{B}_r \cup \mathcal{B}_l$ . Since  $\mathcal{A}_l$  and  $\mathcal{A}_r$  have just the edge  $a_i c_j$  in common and since it is a bridge of  $\mathcal{A}_l$ , by Lemma 2, we have that  $\mathcal{A}$  is a well-defined spanning chordal outerplanar sprout. Lemma 2 also implies that  $a_1 a_2 \in \mathcal{A}$  with  $\Gamma_{\mathcal{A}}(a_1 a_2) \geq 1$  and that all the edges  $b_i b_{i+1}$  are bridges of  $\mathcal{A}$ . Since  $\mathcal{B}_l$  and  $\mathcal{B}_r$  have just the vertices  $a_i$  and  $c_j$  in common and since these vertices are in distinct connected components of  $\mathcal{B}_l$ , by Lemma 1, we have that  $\mathcal{B}$  is a well-defined spanning chordal outerplanar sprout. This lemma implies that all the edges  $c_i c_{i+1}$  and all the edges  $a_i a_{i+1}$ , for  $i \geq 2$ , are bridges of  $\mathcal{B}$ . It also implies that the vertices  $b_1$  and  $b_p$  are in distinct connected components of  $\mathcal{B}_r$ . Finally, since all the edges and all the inner faces of  $T_l$  and  $T_r$  were covered, all the edges and all the inner faces of T are now covered by  $\mathcal{A}$  or  $\mathcal{B}$ .



Figure 11: Case 4

Case 3: There is a chord  $a_1b_2$  and p = 2 (Figure 10). Observe that q > 2; otherwise, we would have a chord  $c_1c_r$ . We denote by T' the partial 4-connected triangulation bounded by  $(a_1b_2...b_qc_2...c_{r-1})$ . Considering the 3-bound  $(b_2a_1)-(c_r...c_1)-(b_q...b_2)$ , we apply the induction hypothesis and let  $\mathcal{A}'$  and  $\mathcal{B}'$  be its two outerplanar sprouts. We just have to add  $a_1b_1b_2$  to  $\alpha(b_2a_1)$ , using Lemma 3, and add the edges  $b_2b_1$  and  $a_1a_2$  to  $\mathcal{B}$  with  $\beta(b_2b_1) = \beta(a_1a_2) = \emptyset$ , using Lemma 1, to obtain the desired outerplanar sprouts.

Case 4: There is no such chord (Figure 11). Let  $d_1$  be a neighbour of  $b_1$  and  $b_2$ . This vertex is not in the outer boundary of T; otherwise, we would have a chord  $a_p x$ . Since there is no separating 3-cycle in T there is a face  $b_1b_2d_1$ . Let  $N_a$  be the set of neighbours of the vertices  $a_i$  for  $i \ge 2$ , excluding the vertices  $a_i$  with  $i \geq 2$  and the vertex  $b_2$ . This set contains the vertices  $d_1$  and  $a_1$  and induces a connected graph. Let  $(d_1 d_2 \dots d_s a_1)$  be the shortest path in this induced graph from  $d_1$  to  $a_1$ . Note that, by the construction, this path has no chord. Let T' be the partial 4-connected triangulation bounded by the cycle  $(b_2...b_qc_2...c_rd_s...d_1b_2)$ ). We observe that there could be in T' a chord linking  $d_1$  to  $b_i$ , for  $i \geq 3$ , or a chord linking  $b_2$  to x, x being  $c_r$  or a vertex  $d_j$ , for  $j \geq 2$ ; but we cannot have both. So T' is either 3-bounded by  $(b_2d_1...d_sc_r) - (c_r...c_1) - (b_q...b_2)$  or by  $(d_1b_2...b_p) - (c_1...c_r) - (c_1...c_r$  $(c_r d_s \dots d_1)$ . Since T' has less vertices than T  $(b_1 \notin V_{T'})$ we can use the induction hypothesis with one of these 3bounds. Unlike the notations of Theorem 2, we denote as  $\mathcal{B} = (B,\beta)$  the outerplanar sprout covering the edge  $d_1b_2$ and as  $\mathcal{A} = (\mathcal{A}, \alpha)$  the other one. With both 3-bounds, we obtain two outerplanar sprouts with similar characteristics.



Figure 12: Case A

Indeed, in both cases, the edge  $d_1b_2 \in B$  with  $\Gamma_{\mathcal{B}}(d_1b_2) \geq 1$ , all the edges  $c_ic_{i+1}$  are bridges of  $\mathcal{B}$ , all the edges  $b_ib_{i+1}$ , the edge  $d_id_{i+1}$ , and the edge  $d_sc_r$  are bridges of  $\mathcal{A}$ , and the vertices  $c_1$  and  $c_r$  are in distinct connected components of  $\mathcal{A}$ .

We now extend  $\mathcal{A}$  and  $\mathcal{B}$  to construct the two outerplanar sprouts covering T. We start with the face  $b_1b_2d_1$ . Using Lemma 3, we add this face to  $\beta(d_1b_2)$ . Then, since  $d_1$  and  $b_2$  are in distinct connected components of  $\mathcal{A}$ , using Lemma 1, we add the edges  $b_1d_1$  and  $b_1b_2$  to  $\mathcal{A}$  with  $\alpha(b_1d_1) = \alpha(b_1b_2) = \emptyset$ . Now the partial triangulation bounded by  $(b_1...b_qc_2...c_rd_s...d_1b_1)$  is covered by  $\mathcal{A}$  and  $\mathcal{B}$ , and the edges  $c_ic_{i+1}$  are bridges of  $\mathcal{B}$  while the other edges on the outer boundary are bridges of  $\mathcal{A}$ .

Consider an ordering of the edges  $d_x a_y$  with  $y \ge 2$  from the right to the left. The edge  $d_x a_y$  appears before  $d_z a_w$  if x < z or if x = z and y > w. We now present a process that will extend  $\mathcal{A}$  and  $\mathcal{B}$ . In each step of this process, a new edge  $d_x a_y$  is covered by  $\mathcal{A}$  following the ordering we defined. At a given step of this process, if we denote as  $d_x a_y$  the last edge added in  $\mathcal{A}$ , the spanning chordal outerplanar sprouts  $\mathcal{A}$  and  $\mathcal{B}$  are such that :

- All the edges and all the inner faces of the partial triangulation bounded by  $(d_x a_y ... a_p b_2 ... b_q c_2 ... c_r d_s ... d_x)$ are covered by  $\mathcal{A}$  or  $\mathcal{B}$ .
- A is connected
- $\mathcal{B}$  has two connected components, one with  $b_1$ , and one with  $b_p$ .
- The edge  $d_x a_y$  is in  $\mathcal{A}$  and  $\Gamma_{\mathcal{A}}(d_x a_y) \geq 1$ .
- The edges  $a_i a_{i+1}$  for  $i \geq y$  are bridges of  $\mathcal{B}$ .
- The edges  $b_i b_{i+1}$  are bridges of  $\mathcal{A}$ .
- The edges  $c_i c_{i+1}$  are bridges of  $\mathcal{B}$ .
- The edge  $c_r d_s$  and the edges  $d_{i+1} d_i$  for  $i \geq x$  are bridges of  $\mathcal{A}$ .

Note that these conditions are satisfied at the beginning of the process when  $d_1a_p$  is the only edge of the ordering covered by  $\mathcal{A}$ . For a given step, we denote by  $d_xa_y$  the last edge of the ordering covered by  $\mathcal{A}$  and by  $d_za_w$  the next one. We have to consider different cases according to the relation between  $d_za_w$  and  $d_xa_y$ .



Figure 13: Case B



Figure 14: Case C

Case A: z = x and w = y - 1 (see Figure 12). Since there are no separating 3-cycles, there is a face  $d_x a_y a_w$ . Using Lemma 3, we add  $d_x a_y a_w$  to  $\alpha(d_x a_y)$ . Then, using Lemma 1, we add the edge  $d_x a_w$  to A (resp  $a_w a_y$  to B) with  $\alpha(d_x a_w) = \emptyset$  (resp.  $\beta(a_w a_y) = \emptyset$ ). All the conditions are fulfilled and we can proceed to the next step.

Case B: z = x and w < y - 1 (see Figure 13). We denote by  $e_1 = a_y$ ,  $e_2$ , ...,  $e_t$  and  $a_w$  the neighbours of  $d_x$ , going from  $a_y$  to  $a_w$ . There are at least two vertices  $e_i$ ; otherwise, there would be a chord  $a_w a_y$ . Since  $\Gamma_{\mathcal{A}}(d_x e_1) \geq 1$ , according to Lemma 3 we add to  $\mathcal{A}$  the edges  $d_x e_2$  and  $e_1 e_2$  with  $\alpha(d_x e_2) = \emptyset$  and  $\alpha(e_1 e_2) = \{d_x e_1 e_2\}$ . We similarly add the edges  $d_x e_i$  and  $e_i e_{i-1}$  until reaching  $e_t$ . Then according to Lemma 1 we add the edge  $a_w e_t$  to  $\mathcal{B}$  with  $\beta(a_w e_t) = \{a_w e_t d_x\}$ .

Consider now the partial 4-connected triangulation T''bounded by  $(a_w...a_ye_2...e_ta_w)$ . There is no chord  $e_ie_j$ ; otherwise,  $(e_ie_jd_x)$  would be a separating 3-cycle. So T'' is 3-bounded by  $(e_ta_w)$ - $(a_w...a_y)$ - $(e_1...e_t)$ . We apply the induction hypothesis and obtain two outerplanar sprouts  $\mathcal{A}''$ and  $\mathcal{B}''$ . By Lemmas 1 and 2, let  $\mathcal{A} = \mathcal{A} \cup \mathcal{B}''$ , and by Lemmas 1 and 3, let  $\mathcal{B} = \mathcal{B} \cup \mathcal{A}''$ . All the conditions are satisfied and we can proceed to the next step.

Case C: z = x + 1 and w = y (see Figure 14). We consider the face  $a_y d_x d_z$  and by Lemma 4 we add the edge  $d_z a_y$  to  $\mathcal{A}$  with  $\alpha(d_z a_y) = \emptyset$  and put  $a_y d_x d_z$  in  $\alpha(d_x d_z)$ . All the conditions are satisfied and we can proceed to the next step.

Case D: z = x + 1 and w < y (see Figure 15). We denote by  $e_1 = a_y, e_2, ..., e_t$  and  $d_z$  the neighbours of  $d_x$  going from  $a_y$  to  $d_z$ . There are at least two vertices  $e_i$ ; other-



Figure 15: Case D



Figure 16: Case E

wise, we would have the previous case with w = y. Since  $\Gamma_{\mathcal{A}}(d_x a_y) \geq 1$ , according to Lemma 3 we add to  $\mathcal{A}$  the edges  $d_x e_2$  and  $e_2 a_y$  with  $\alpha(d_x e_2) = \emptyset$  and  $\alpha(e_2 a_y) = \{d_x e_2 a_y\}$ . We similarly add the edges  $d_x e_i$  and  $e_i e_{i-1}$  until reaching  $e_t$ . Then according to Lemma 4 we can add the edge  $d_z e_t$  to  $\mathcal{A}$  with  $\alpha(d_z e_t) = \emptyset$  and put the face  $d_x d_z e_t$  in  $\alpha(d_x d_z)$ .

We denote by  $f_1 = a_w$ ,  $f_2$ , ...,  $f_u = e_t$  the neighbours of  $d_z$  going from  $a_w$  to  $e_t$ . Using Lemma 1, we add the edge  $d_z f_1$  to  $\mathcal{A}$  with  $\alpha(d_z f_1) = \emptyset$ . Then according to Lemma 3 we add the edges  $d_z f_i$  and  $f_{i-1}f_i$  to  $\mathcal{A}$  with  $\alpha(d_z f_i) = \emptyset$  and  $\alpha(f_{i-1}f_i) = \{d_z f_{i-1}f_i\}$  until reaching  $f_{u-1}$ . Then, using Lemma 1, we put the edge  $f_{u-1}f_u$  in  $\mathcal{B}$  with  $\beta(f_{u-1}f_u) = \{f_{u-1}f_u d_z\}$ .

Consider now the partial 4-connected triangulation T''bounded by  $(a_w...a_ye_2...e_tf_{u-1}...f_1)$ . There is no chord  $e_ie_j$ (resp.  $f_if_j$ ); otherwise,  $(e_ie_jd_x)$  (resp.  $(f_if_jd_z)$ ) would be a separating 3-cycle. So T'' is 3-bounded by  $(f_u...f_1)$ - $(a_w...a_y)$ - $(e_1...e_t)$ . We apply the induction hypothesis and obtain two outerplanar sprouts  $\mathcal{A}''$  and  $\mathcal{B}''$ . Let  $\mathcal{A} = \mathcal{A} \cup \mathcal{B}''$ by applying Lemma 2 with the path  $(e_1...e_t)$  and then with the path  $(f_1...f_{u-1})$ . Using Lemmas 1 and 3, let  $\mathcal{B} = \mathcal{B} \cup \mathcal{A}''$ . All the conditions are fulfilled and we can proceed to the next step.

This process terminates with an edge  $d_s a_y$ , where  $y \ge 2$ . Now we have just two more cases to complete the proof.

Case E: y = 2 (see Figure 16). In this case, we simply use Lemma 4 to add the edge  $a_1a_2$  to  $\mathcal{A}$  with  $\alpha(a_1a_2) = \emptyset$ and put the face  $d_sa_1a_2$  in  $\alpha(a_1d_s)$ . Then we finally obtain the two outerplanar sprouts  $\mathcal{A}$  and  $\mathcal{B}$  that fulfill all the conditions of the theorem.



Figure 17: Case F

Case F: y > 2 (see Figure 17). We denote by  $e_1 = a_y$ ,  $e_2$ , ...,  $e_t = a_1$  the neighbours of  $d_s$  going from  $a_y$  to  $a_1$ . There are at least three vertices  $e_i$ ; otherwise, there would be a chord  $a_1a_y$ . Since  $\Gamma_{\mathcal{A}}(d_se_1) \geq 1$ , according to Lemma 3 we add to  $\mathcal{A}$  the edges  $d_se_2$  and  $e_2e_1$  with  $\alpha(d_xe_2) = \emptyset$ and  $\alpha(e_2e_1) = \{d_xe_2e_1\}$ . We similarly add the edges  $d_xe_i$ and  $e_ie_{i-1}$  until reaching  $e_{t-1}$ . Then according to Lemma 4 we add the edge  $e_{t-1}e_t$  to  $\mathcal{A}$  with  $\alpha(e_{t-1}e_t) = \emptyset$  and put the face  $e_{t-1}e_td_s$  in  $\alpha(e_td_s)$ .

Consider now the partial 4-connected triangulation T'' bounded by  $(a_1...a_y e_2...e_{t-1})$ . There is no chord  $e_i e_j$ ; otherwise,  $(e_i e_j d_s)$  would be a separating 3-cycle. So we consider the following 3-bound of T',  $(a_2a_1)-(e_t...e_1)-(a_y...a_2)$ . We apply the induction hypothesis and obtain two outerplanar sprouts  $\mathcal{A}''$  and  $\mathcal{B}''$ . Using Lemma 2, let  $\mathcal{A} = \mathcal{A} \cup \mathcal{A}''$ , and using Lemma 1, let  $\mathcal{B} = \mathcal{B} \cup \mathcal{B}''$ . Then we finally obtain the two outerplanar sprouts  $\mathcal{A}$  and  $\mathcal{B}$  that fulfill all the conditions of the theorem.  $\Box$ 

# 5. CONCLUSION

The proof being constructive, using appropriate data structures, we easily obtain a linear-time algorithm that, given a planar graph, computes a bipartition of its edges into two outerplanar graphs.

Observe that in a triangulation with n vertices, there is a gap between 6n - 6 (=  $2 \times (3n - 3)$ ), the number of edges or faces covered by two outerplanar sprouts, and 5n - 11 (= 3n - 6 + 2n - 5), the number of edges and inner faces. Nevertheless, it is impossible for any planar graph, to cover k times its edges with s outerplanar subgraphs, if  $\frac{s}{k} < 2$ . Indeed, in  $K_{2,2s+1}$  (with vertex sets  $\{a_1, a_2\}$  and  $\{b_1, ..., b_{2s+1}\}$ ), since outerplanar graphs are  $K_{2,3}$ -free, there is at least one path  $(a_1, b_i, a_2)$  that is included in none of the outerplanar subgraphs. So there are at least k subgraphs covering the edge  $a_1b_i$ , and k other subgraphs covering  $a_2b_i$ . So,  $s \geq 2k$ . This implies, since we have ratio 2, that our result is tight in this sense.

The track number t(G) of a graph G, is the minimal number of interval subgraphs of G needed to cover the edges of G. In [6], it is proved that planar graphs have track number at most four, and in [7], it is proved that this bound is tight. Since outerplanar graphs have track number two [11], our result provides a new proof of the upper bound four.

The thickness  $\theta(G)$  (resp. outerthickness  $\theta_o(G)$ ) of a graph G is the minimal number of planar (resp. outerplanar) subgraphs we need to cover all the edges of G (see [12] for a survey). We now have the relation  $\theta_o(G) \leq 2 \times \theta(G)$ , for any graph G. This result has also consequences in universal graphs: It decreases the number of vertices needed to construct a graph containing, as induced subgraphs, all planar graphs with a fixed number of vertices and a fixed maximum degree [3].

In [4], it is proved that if a graph G can be embedded in a surface S, then its edge set has a bipartition into two graphs with bounded tree-width. We can extend the definition of outerplanar graphs to other surfaces. Given a surface S, we say that a graph H is outer-S if it can be embedded in S in such a way that there is a face with all the vertices of H in its boundary. Can any graph embedded in S be edge-partitioned into two outer-S graphs ? It is true for 5connected toroidal triangulations. In [1], the authors proved that 5-connected toroidal triangulations have a contractible hamiltonian cycle. In this case the triangulation is divided into an outerplanar graph and an outertoroidal graph. Can any graph embedded in S be edge-partitioned into an outerplanar graph and an outer-S graph ?

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