

# A SYMMETRY PROPERTY OF SOME HARMONIC ALGEBRAIC CURVES

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ABSTRACT. The aim of this note is to give and to prove a simple and surprising symmetry property of some harmonic algebraic curves: when all the roots  $z_i$  of a complex polynomial  $P$  lie on the unit circle  $\mathcal{U}$ , the points of  $\mathcal{U}$  different from the  $z_i$ , and such that  $\text{Arg}(P(z)) = \theta$ , form a regular  $n$ -gon, where  $n$  is the degree of  $P$ .

Let  $\mathbf{z} = \{z_1, \dots, z_n\}$  be a multiset of  $n$  points in the complex plane  $\mathbb{C}$  and  $P$  the monic polynomial with root set  $\mathbf{z}$ :

$$P(z) = \prod_{i=1}^n (z - z_i).$$

For  $\theta$  a fixed real number of your choice, consider

$$C_\theta(P) = \{z \in \mathbb{C} : \text{Im}(e^{-i\theta} P(z)) = 0\}.$$

The set  $C_\theta(P)$  coincides up to  $\mathbf{z}$ , to the set  $\{z \in \mathbb{C} : \text{Arg}(P(z)) = \theta[\pi]\}$ . These curves arise in the Gauss approach to the Fundamental Theorem of Algebra (see e.g. Stillwell [3], and Martin & al. [1]). In their paper Martin & al. [1] and then Savitt [2] initiated the study of the combinatorial topology of the families  $C_\theta(P)$ . The idea are the following ones: the curves  $C_\theta(P)$  have  $2n$  asymptotes at angles  $(\pi k + \theta)/n$ , for  $k \in \{0, \dots, 2n - 1\}$ , and form in the generic case  $n$  non intersecting curves. This induces a matching:  $k$  and  $k'$  are matched if and only if the asymptotes  $(\pi k + \theta)/n$  and  $(\pi k' + \theta)/n$  lie on the same connected component in  $C_\theta(P)$ . The papers [1] and [2] aim at studying these matchings, and also the properties of the so-called necklaces, formed by the families of matchings obtained when  $\theta$  traverses the set  $[0, \pi]$ .

Let us now state and prove our result. The set  $\mathbf{z}$  is clearly included in  $C_\theta(P)$ . It turns out that when  $\mathbf{z}$  is included in the unit circle  $\mathcal{U} = \{z : |z| = 1\}$ , the set  $C_\theta(P) \cap \mathcal{U}$  presents a quite surprising symmetry – illustrated at Figure 1 – that can be stated as follows.

**Proposition 1.** *If  $\mathbf{z}$  is a subset of  $\mathcal{U}$ , then*

$$C_\theta(P) \cap \mathcal{U} = \mathbf{z} \cup G(\mathbf{z})$$

where  $G(\mathbf{z})$  is the regular  $n$ -gon on  $\mathcal{U}$ , with set of vertices  $\left\{ e^{i(\Omega + 2k\pi/n)}, k = 1, \dots, n \right\}$ , for

$$\Omega := \frac{2\theta - \sum_{j=1}^n \text{Arg}(z_j)}{n} - \pi$$

There exists a purely geometric proof of this Proposition using that the measure of a central angle is twice that of the inscribed angle intercepting the same arc; we provide below a more compact analytic proof.

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*Proof.* We will only consider  $z \notin \mathbf{z}$ . We have the equivalence:

$$z \in C_\theta(P) \setminus \mathbf{z} \iff z \notin \mathbf{z}, \sum_{i=1}^n \text{Arg}(z - z_i) = \theta \pmod{\pi},$$

where  $\text{Arg}(z) \in \mathbb{R}/2\pi\mathbb{Z}$  stands for (any chosen determination of) the argument of  $z \neq 0$ . Now for any  $\nu$  and  $\psi$  real numbers,

$$e^{i\nu} - e^{i\psi} = e^{i\frac{\nu+\psi}{2}} \left( e^{i\frac{\nu-\psi}{2}} - e^{i\frac{-\nu+\psi}{2}} \right) = 2i \sin((\nu - \psi)/2) e^{i\frac{\nu+\psi}{2}}.$$

Thus

$$\text{Arg}(e^{i\nu} - e^{i\psi}) = \frac{\nu + \psi}{2} + \frac{\pi}{2} + \pi \times \text{sgn}(\sin((\nu - \psi)/2)) \pmod{2\pi}$$

Hence,  $z \in C_\theta(P) \setminus \mathbf{z}$  is equivalent to:

$$z \notin \mathbf{z}, \sum_{j=1}^n \left( \frac{\text{Arg}(z) + \text{Arg}(z_j)}{2} + \frac{\pi}{2} \right) = \theta \pmod{\pi},$$

which leads to the conclusion at once.  $\square$

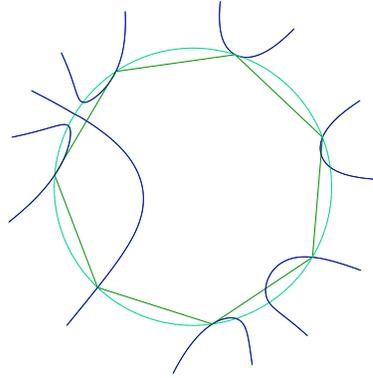


FIGURE 1. An example where  $n = 7$ ,  $\theta = 0$  and the roots  $z_i$  randomly chosen.

**Note.** If  $z_i$  is a root of multiplicity  $k$  of  $P$ , and if  $z_i$  belongs to  $G(\mathbf{z})$ , then in the neighborhood of  $z_i$ ,  $C_\theta(P)$  has  $k$  tangents, one of them coinciding with the tangent of the circle at  $z_i$ . Moreover, it is simple to check that if  $z_i$  is not on  $G(\mathbf{z})$ , then the tangents of  $C_\theta(P)$  at  $z_i$  are not tangent to  $\mathcal{U}$ .

## REFERENCES

- [1] J. MARTIN, D. SAVITT, T. SINGER, *Harmonic algebraic curves and noncrossing partitions*, Discrete and Computational Geometry **37**, no. 2 (2007), 267–286.
- [2] D. SAVITT, *Polynomials, meanders, and paths in the lattice of noncrossing partitions*, arXiv:math/0606169.
- [3] J. STILLWELL, *Mathematics and its history. Undergraduate Texts in Mathematics*. Springer-Verlag, New York, 1989.

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