IDEALS OF QUASI-SYMMETRIC FUNCTIONS AND SUPER-COVARIANT POLYNOMIALS FOR $\mathcal{S}_n$

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ABSTRACT. The aim of this work is to study the quotient ring $\mathbf{R}_n$ of the ring $\mathbb{Q}[x_1, \ldots, x_n]$ over the ideal $\mathcal{J}_n$ generated by non-constant homogeneous quasi-symmetric functions. This article is a sequel of [2], in which we investigated the case of infinitely many variables. We prove here that the dimension of $\mathbf{R}_n$ is given by C_n, the n^{th} Catalan number. This is also the dimension of the space $\mathbf{SH}_n$ of super-covariant polynomials, defined as the orthogonal complement of $\mathcal{J}_n$ with respect to a given scalar product. We construct a basis for $\mathbf{R}_n$ whose elements are naturally indexed by Dyck paths. This allows us to understand the Hilbert series of $\mathbf{SH}_n$ in terms of number of Dyck paths with a given number of factors.

1. INTRODUCTION

We study, in this paper, a natural analog of the space $\mathbf{H}_n$ of covariant polynomials of $\mathcal{S}_n$. Let X denote the n variables $x_1, \ldots, x_n$ and $\mathbb{Q}[X]$ denote the ring of polynomials in the variables X. Let $\mathcal{I}_n$ denote the ideal of $\mathbb{Q}[X]$ generated by all symmetric polynomials with no constant term. That is

$$\mathcal{I}_n = \langle h_k(X),\ k > 0 \rangle,$$

where $h_k(X)$ is the k^{th} homogeneous symmetric polynomials in the variables X. (*cf.* [18]). We consider the following scalar product on $\mathbb{Q}[X]$:

$$\langle P, Q \rangle = P(\partial X) Q(X)\big|_{X=0} \tag{1.1}$$

where ∂X stands for $\partial x_1, \ldots, \partial x_n$ and in the same spirit $X = 0$ stands for $x_1 = \cdots = x_n = 0$. The space $\mathbf{H}_n$ is defined as the orthogonal complement, denoted by $\mathcal{I}_n^{\perp}$, of the ideal $\mathcal{I}_n$ in $\mathbb{Q}[X]$.

Equivalently (*cf.* [11], Proposition I.2.3), covariant polynomials (also known as $\mathcal{S}_n$-harmonic polynomials) can be defined as polynomials P such that $Q(\partial X)P = 0$, for any symmetric polynomial Q with no constant term. Since elements of $\mathbf{H}_n$ satisfy the Laplace equation

$$(\partial x_1^2 + \cdots + \partial x_n^2)\, P = \Delta\, P = 0,$$

every covariant polynomial is also harmonic.

Classical results [1, 22] state that the space $\mathbf{H}_n$ affords a graded $\mathcal{S}_n$-module structure and is isomorphic (as a representation of $\mathcal{S}_n$) to the left regular representation.

Date: December 12, 2002.

F. Bergeron is supported in part by NSERC and FCAR.

N. Bergeron is supported in part by CRC, NSERC and PREA.

Furthermore, as a graded $\mathcal{S}_n$-module, $\mathbf{H}_n$ is isomorphic to the quotient

$$Q_n = \mathbb{Q}[X]/\mathcal{I}_n.$$

The space Q_n appears naturally in other contexts; for instance, as the cohomology ring of the variety of complete flags [7]. The discussion above implies that

$$\dim \mathbf{H}_n = n! \,. \tag{1.2}$$

Part of the interesting results surrounding the study of $\mathbf{H}_n$ involve the fact that it can also be described as the linear span of all partial derivatives of the Vandermonde determinant. This is a special case of a general result for finite groups generated by reflections [22].

By analogy, we consider here the space $\mathbf{SH}_n = \mathcal{J}_n^\perp$ of *super-covariant* polynomials, where $\mathcal{J}_n$ is the ideal generated by *quasi-symmetric* polynomials with no constant term. Since the ring of symmetric polynomials is a subring of the ring of quasi-symmetric polynomials, we have $\mathcal{I}_n \subseteq \mathcal{J}_n$ hence $\mathcal{J}_n^\perp \subseteq \mathcal{I}_n^\perp$, thus

$$\mathbf{SH}_n \subseteq \mathbf{H}_n \,,$$

which justifies the terminology. Quasi-symmetric polynomials were introduced by Gessel in 1984 [13] and have since appeared as a crucial tool in many interesting algebraico-combinatorial contexts (*cf.* [6, 12, 19, 20, 21]).

As in the corresponding symmetric setup, we have a graded isomorphism

$$\mathbf{SH}_n \simeq \mathbf{R}_n = \mathbb{Q}[X]/\mathcal{J}_n \tag{1.3}$$

and the approach used in the following work concentrates on this alternate description. We construct a basis of $\mathbf{R}_n$ by giving an explicit set of monomial representatives. This set is naturally indexed by *Dyck paths* of length n, hence we obtain the following main theorem.

Theorem 1.1. *The dimension of* $\mathbf{SH}_n$ *is given by the well known Catalan numbers:*

$$\dim \mathbf{SH}_n = \dim \mathbf{R}_n = C_n = \frac{1}{n+1}\binom{2\,n}{n} \,. \tag{1.4}$$

In fact, taking into account the grading (with respect to degree), we have the Hilbert series

$$\sum_{k=0}^{n-1} \dim \mathbf{SH}_n^{(k)} t^k = \sum_{k=0}^{n-1} \frac{n-k}{n+k}\binom{n+k}{k} t^k \,. \tag{1.5}$$

The article contains five sections. In Section 2 we recall useful definitions and basic properties. In Section 3 we construct a family $\mathcal{G}$ of generators for the ideal $\mathcal{J}_n$ and state useful properties of this set. Section 4 is devoted to the proof of the first part of Theorem 1.1. We construct an explicit basis for $\mathbf{R}_n$ which allows us in Section 5 to obtain the Hilbert series of $\mathbf{SH}_n$.

Before we start, let us remark that F. Hivert [15] has developed an action of the Hecke algebra on $\mathbb{Q}[X]$ for which a polynomial is invariant if and only if it is quasi-symmetric. One way to reformulate his result is to consider the generators $e_i = \frac{q-T_i}{(1+q)}$

of the Hecke algebra, where T_i are the standard generators and q is an arbitrary parameter. Then

$$e_i e_{i\pm1} e_i - \frac{q}{(1+q)^2} e_i \tag{1.6}$$

acts, via Hivert's action, as zero on the polynomial ring. Hence, the quotient of the Hecke algebra by the relation (1.6), classicaly known as the Temperley-Lieb algebra $TL_n(q)$ (*cf.* [16]), faithfully acts on polynomials. At $q = 1$ this is a subalgebra of the symmetric group algebra. The ring of quasi-symmetric polynomials is then identified as the invariants $\mathbb{Q}[X]^{TL_n(q)}$ of the algebra $TL_n(q)$ for any value of q. The algebra $TL_n(q)$ is known to have dimension equal to C_n. It is striking to discover in this context that the space $\mathbf{R}_n$ above also has dimension C_n. In analogy to the classical covariant theory, one would like to see an action of $TL_n(q)$ on R_n which would identify it as the left regular representation. Unfortunatly, Hivert's action is not compatible with multiplication and does not preserve the ideal $\mathcal{J}_n$, hence it does not induce the desired action on the quotient. This observation raises many new avenues of investigation.

2. Basic definitions

A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of a positive integer d is an ordered list of positive integers (> 0) whose sum is d. We denote this by $\alpha \models d$ and also say that α is a composition of *size* d and denote this by $|\alpha|$. The integers α_i are the *parts* of α, and the length $\ell(\alpha)$ is set to be the number of parts of α. We denote by 0 the unique empty composition of size $d = 0$.

There is a natural one-to-one correspondence between compositions of d and subsets of $\{1, 2, \ldots, d-1\}$. Let $S = \{a_1, a_2, \ldots, a_k\}$ be such a subset, with $a_1 < \cdots < a_k$, then the composition associated to S is $\alpha_d(S) = (a_1 - a_0, a_2 - a_1, \ldots, a_{k+1} - a_k)$, where we set $a_0 := 0$ and $a_{k+1} := d$. We denote by $D(\alpha)$ the set associated to α through this correspondence. For compositions α and β, we say that β is a *refinement* of α, if $D(\alpha) \subset D(\beta)$, and denote this by $\beta \succeq \alpha$.

We use vector notation for monomials. More precisely, for $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$, we denote X^ν the monomial

$$x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}. \tag{2.1}$$

For a polynomial $P \in \mathbb{Q}[X]$, we further denote $[X^\nu]\, P(X)$ as the coefficient of the monomial X^ν in $P(X)$.

For a *vector* $\nu \in \mathbb{N}^n$, let $c(\nu)$ represent the composition obtained by erasing zeros (if any) in ν. A polynomial $P \in \mathbb{Q}[X]$ is said to be *quasi-symmetric* if and only if, for any ν and μ in $\mathbb{N}^n$, we have

$$[X^\nu]P(X) = [X^\mu]P(X)$$

whenever $c(\nu) = c(\mu)$. The space of quasi-symmetric polynomials in n variables is denoted by $Qsym_n$. The space $Qsym_n^{(d)}$ of homogeneous quasi-symmetric polynomials of degree d admits as linear basis the set of *monomial* quasi-symmetric polynomials

indexed by compositions of d. More precisely, for each composition α of d with at most n parts, we set

$$M_\alpha = \sum_{c(\nu)=\alpha} X^\nu. \tag{2.2}$$

For the 0 composition, we set $M_0 = 1$. Another important linear basis is that of the *fundamental* quasi-symmetric polynomials (*cf.* [13]):

$$F_\alpha = \sum_{\beta \succeq \alpha} M_\beta \tag{2.3}$$

with $\alpha \models n$ and $\ell(\alpha) \leq n$. For example, with $n = 4$,

$$\begin{aligned} F_{21}(x_1, x_2, x_3, x_4) &= M_{21}(x_1, x_2, x_3, x_4) + M_{111}(x_1, x_2, x_3, x_4) \\ &= {x_1}^2 x_2 + {x_1}^2 x_3 + {x_1}^2 x_4 + {x_2}^2 x_3 + {x_2}^2 x_4 + {x_3}^2 \, , x_4 \\ &\qquad + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4. \end{aligned}$$

Part of the interest of fundamental quasi-symmetric functions comes from the following properties. The first is trivial, but very useful and the second comes from the theory of P-partitions [20, 21].

Proposition 2.1. *For* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models d$,

$$F_\alpha(X) = \begin{cases} x_1 F_{(\alpha_1 - 1, \alpha_2, \ldots, \alpha_k)}(X) + F_\alpha(x_2, \ldots, x_n) & \text{if } \alpha_1 > 1, \\ \\ x_1 F_{(\alpha_2, \alpha_3, \ldots, \alpha_k)}(x_2, \ldots, x_n) + F_\alpha(x_2, \ldots, x_n) & \text{if } \alpha_1 = 1. \end{cases} \tag{2.4}$$

Let $u = u_1 \cdots u_l \in \mathcal{S}_\ell$ and $v = v_1 \cdots v_m \in \mathcal{S}_{[\ell+1, \ell+m]}$. Let $u \sqcup v$ denote the set of *shuffles* of the words u and v, *i.e.* $u \sqcup v$ is the set of all permutations w of $\ell + m$ such that u and v are subwords of w. In particular $u \sqcup v$ contains $\binom{\ell+m}{m}$ permutations. Let $\mathcal{D}(u) = \{i, \ u_i > u_{i+1}\}$ denote the *descent set* of u. If β and γ are the two compositions such that $D(\beta) = \mathcal{D}(u)$ and $D(\gamma) = \mathcal{D}(v)$, then

Proposition 2.2 ([21], Exercise 7.93)**.**

$$F_\beta \, F_\gamma = \sum_{w \in u \sqcup v} F_{\alpha_{\ell+m}(D(w))}. \tag{2.5}$$

In (2.1), the monomials are in correspondence with vectors $\nu \in \mathbb{N}^n$. Just as for compositions, the size $\nu_1 + \cdots + \nu_n$ of ν is denoted by $|\nu|$. It is also convenient to denote by $\ell(\nu)$ the position of its last non-zero component. As usual, $\nu + \mu$ is the componentwise addition of vectors.

For ease of reading, we reserve the use of α, β and γ to represent compositions, and the other Greek letters to represent vectors. We use the same symbol α for both the composition $(\alpha_1, \ldots, \alpha_\ell)$ and the word $\alpha_1 \cdots \alpha_\ell$, likewise for vectors.In general, the length of vectors (or number of variables) is fixed and equal to n. If w is a word of integers (that is an element of $\mathbb{N}^k$ for $0 \leq k \leq n$) we denote by $w0^* = w0^{n-k}$ the vector whose first k parts are the *letters* of w, to which are added $n - k$ zeros at the end. If $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_m$ are words of integers, the word

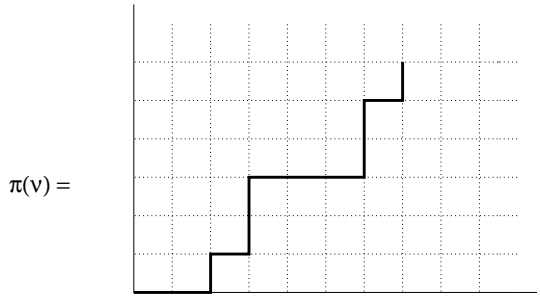
$$u \, v := u_1 \cdots u_k v_1 \cdots v_m$$

is the *concatenation* of u and v.

We next associate to any vector ν a path $\pi(\nu)$ in the $\mathbb{N}\times\mathbb{N}$ plane with steps going north or east as follows. If $\nu=(\nu_1,\dots,\nu_n)$, the path $\pi(\nu)$ is

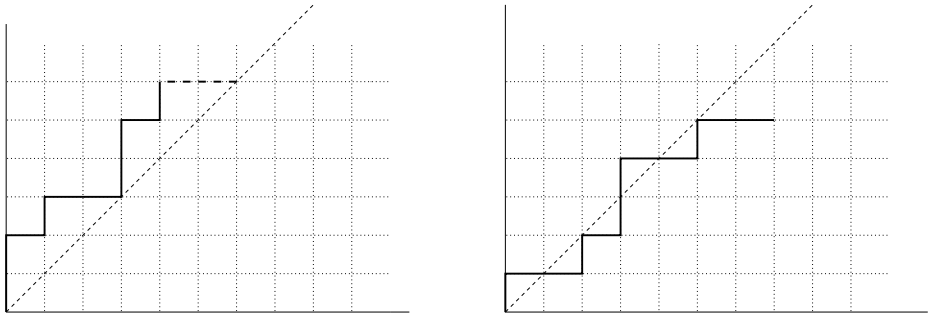
$$(0,0)\to(\nu_1,0)\to(\nu_1,1)\to(\nu_1+\nu_2,1)\to(\nu_1+\nu_2,2)\to\cdots$$
$$\to(\nu_1+\cdots+\nu_n,n-1)\to(\nu_1+\cdots+\nu_n,n).$$

For example the path associated to $\nu=(2,1,0,3,0,1)$ is



Observe that the height of the path is always n, whereas its width is $|\nu|$.

We distinguish two kinds of paths, thus two kinds of vectors, with respect to their "behavior" with respect to the diagonal $y=x$. If the path remains above the diagonal, we call it a *Dyck path*, and say that the corresponding vector is *Dyck*. If not, we say that the path (or equivalently the associated vector) is *transdiagonal*. For example $\eta=(0,0,1,2,0,1)$ is Dyck and $\varepsilon=(0,2,1,0,2,2)$ is transdiagonal.



Observe that $\nu=\nu_1\cdots\nu_n$ is transdiagonal if and only if there exists $1\le m\le n$ such that

$$m<\nu_1+\cdots+\nu_m. \tag{2.6}$$

Recall that the classical lexicographic order, on monomials of same degree, is

$$X^\nu\ge_{\text{lex}}X^\mu \qquad\text{iff}\qquad \nu\ge_{\text{lex}}\mu, \tag{2.7}$$

where we say that ν is lexicographically larger than μ, $\nu>_{\text{lex}}\mu$, if the first non-zero part of the vector $\nu-\mu$ is positive. For example

$$x_1^3>_{\text{lex}}x_1^2x_2>_{\text{lex}}x_1x_2^2>_{\text{lex}}x_2^3 \quad\text{since}\quad (3,0)>_{\text{lex}}(2,1)>_{\text{lex}}(1,2)>_{\text{lex}}(0,3)\,.$$

3. The $\mathcal{G}$ basis

Following [2], we exploit relations (2.4) to construct a family

$$\mathcal{G} = \{G_\varepsilon\} \subset \mathcal{J}_n$$

indexed by vectors that are transdiagonal. For α any composition of $k \leq n$, the polynomial G_ε, with $\varepsilon := \alpha 0^*$, is defined to be

$$G_\varepsilon := F_\alpha. \tag{3.1}$$

When $\alpha \neq 0$, the vector $\varepsilon = \alpha 0^*$ is clearly transdiagonal. For a general vector ε (not of the form $\alpha 0^*$), the polynomial G_ε is defined recursively in the following way. Let $\varepsilon = w0a\beta 0^*$ be the unique factorization of ε such that w is a word of $k-1$ non-negative integers, $a > 0$ is a positive integer, and β is a composition (parts > 0). Then we set

$$G_\varepsilon = G_{wa\beta 0^*} - x_k\, G_{w(a-1)\beta 0^*}. \tag{3.2}$$

By induction on the length of the indexing vectors, both terms on the right of (3.2) are well defined, and we have

- $\ell(wa\beta 0^*) = \ell(w(a-1)\beta 0^*) = \ell(\varepsilon) - 1$;
- $wa\beta 0^*$ and $w(a-1)\beta 0^*$ are transdiagonal as soon as ε is transdiagonal.

In fact, let m be the first ordinate where $\pi(\varepsilon)$ crosses the diagonal, this is to say the smallest integer such that $m < \varepsilon_1 + \cdots + \varepsilon_m$. Then the second assertion follows from

$$\varphi_1 + \cdots + \varphi_m > \psi_1 + \cdots + \psi_m = \varepsilon_1 + \cdots + \varepsilon_m - 1 > m - 1,$$

where $\varphi = wa\beta 0^*$ and $\psi = w(a-1)\beta 0^*$.

For example,

$$\begin{aligned} G_{1020} &= G_{1200} - x_2\, G_{1100} \\ &= F_{12}(x_1,x_2,x_3,x_4) - x_2\, F_{11}(x_1,x_2,x_3,x_4) \\ &= x_1\,{x_2}^2 + x_1\,{x_3}^2 + x_1\,{x_4}^2 + x_2\,{x_3}^2 + x_2\,{x_4}^2 + x_3\,{x_4}^2 + x_1\,x_2\,x_3 + x_1\,x_2\,x_4 \\ &\qquad + x_1\,x_3\,x_4 + x_2\,x_3\,x_4 - x_2\,(x_1\,x_2 + x_1\,x_3 + x_1\,x_4 + x_2\,x_3 + x_2\,x_4 + x_3\,x_4) \\ &= x_1\,{x_3}^2 + x_1\,x_3\,x_4 + x_1\,{x_4}^2 - {x_2}^2\,x_3 - {x_2}^2\,x_4 + x_2\,{x_3}^2 + x_2\,{x_4}^2 + x_3\,{x_4}^2. \end{aligned}$$

We observe on this example that the leading monomial (in lex order) of G_{1020} is $X^{1020} = x_1^1 x_2^0 x_3^2 x_4^0$. This holds in general for the $\mathcal{G}$ family as stated in the following proposition, for which all technical details can be found in [2].

Proposition 3.1 ([2], Corollary 3.4)**.** *The leading monomial $LM(G_\varepsilon)$ of G_ε is X^ε.*

4. Proof of the main theorem

We now give an explicit basis for the space $\mathbf{R}_n$ naturally indexed by Dyck paths. This proves the first part of Theorem 1.1.

Theorem 4.1. *The set of monomials*

$$\mathcal{B}_n = \{X^\eta \mid \pi(\eta) \text{ is a Dyck path}\} \tag{4.1}$$

is a basis of the space $\mathbf{R}_n$.

The proof is achieved in a few steps. We start with the following lemma.

Lemma 4.2. *Any $\mathcal{P}(X) \in \mathbb{Q}[X]$ is in the linear span of $\mathcal{B}_n$ modulo $\mathcal{J}_n$. That is*

$$P(X) \equiv \sum_{X^\eta \in \mathcal{B}_n} c_\eta X^\eta \pmod{\mathcal{J}_n}. \tag{4.2}$$

Proof. It clearly suffices to show that (4.2) holds for any monomial X^ν, with ν transdiagonal. We assume that there exists X^ν not reducible of the form (4.2) and we choose X^ε to be the smallest amongst them with respect to the lexicographic order. Let us write

$$\begin{aligned} X^\varepsilon &= LM(G_\varepsilon) \\ &= (X^\varepsilon - G_\varepsilon) + G_\varepsilon \\ &\equiv X^\varepsilon - G_\varepsilon \pmod{\mathcal{J}_n}. \end{aligned}$$

All monomials in $(X^\varepsilon - G_\varepsilon)$ are lexicographically smaller than X^ε, thus they are reducible. This contradicts our assuption on X^ε and completes our proof. □

Thus $\mathcal{B}_n$ spans the space $\mathbf{R}_n$. We now prove its linear independence. This is equivalent to showing that the set $\mathcal{G}$ is a Gröbner basis of the ideal $\mathcal{J}_n$. A crucial lemma is the following one, which is the quasi-symmetric analogue of a classical result is the case of symmetric polynomials ([11], Theorem II.2.2).

Lemma 4.3. *If we denote by $\mathcal{L}[S]$ the linear span of a set S, then*

$$\mathbb{Q}[X] = \mathcal{L}[X^\eta F_\alpha \mid X^\eta \in \mathcal{B}_n,\ \alpha \models r \geq 0]. \tag{4.3}$$

Proof. We have already obtained the following reduction for any monomial X^ε in $\mathbb{Q}[X]$.

$$X^\varepsilon \equiv \sum_{X^\eta \in \mathcal{B}_n} c_\eta X^\eta \pmod{\mathcal{J}_n},$$

which is equivalent to

$$X^\varepsilon = \sum_{X^\eta \in \mathcal{B}_n} c_\eta X^\eta + \sum_{\alpha \models r \geq 1} Q_\alpha F_\alpha. \tag{4.4}$$

We then apply the reduction (4.4) to each monomial of the Q_α's and use Proposition 2.2 to reduce products of fundamental quasi-symmetric functions. We obtain (4.3) in a finite number of operations since degrees strictly decrease at each operation, because $\alpha \models r \geq 1$ implies $\deg Q_\alpha < |\varepsilon|$. □

The next lemma is the final step in our proof of the Theorem 4.1.

Lemma 4.4. *The set $\mathcal{G}$ is a linear basis of the ideal $\mathcal{J}_n$, i.e.*

$$\mathcal{J}_n = \mathcal{L}[G_\varepsilon \mid \varepsilon \textit{ transdiagonal}]. \tag{4.5}$$

Proof. Let us denote by $\mathcal{A}_n$ the set

$$\mathcal{A}_n = \{X^\xi \mid x_1^{\xi_n} x_2^{\xi_{n-1}} \cdots x_n^{\xi_1} \in \mathcal{B}_n\}. \tag{4.6}$$

Now the algebra endomorphism of $\mathbb{Q}[X]$ that *reverses* the variables,

$$x_i \mapsto x_{n-i+1},$$

clearly fixes the subalgebra $Qsym$. In fact it maps F_α to $F_{\alpha'}$, where α' is the reverse composition.

It follows from Lemma 4.3 and the endomorphism above that:

$$\mathbb{Q}[X] = \mathcal{L}[X^\xi F_\alpha \mid X^\xi \in \mathcal{A}_n,\ \alpha \models r \geq 0]. \tag{4.7}$$

Now to prove Lemma 4.4, we reduce the problem as follows. We first use (4.7) and Proposition 2.2 to write

$$\mathcal{J}_n = \langle F_\alpha,\ \alpha \models s \geq 0\rangle_{\mathbb{Q}[X]} = \mathcal{L}[X^\xi F_\alpha F_\beta \mid X^\xi \in \mathcal{A}_n,\ \alpha \models s \geq 0,\ \beta \models t \geq 1]$$

$$= \mathcal{L}[X^\xi F_\gamma \mid X^\xi \in \mathcal{A}_n,\ \gamma \models r \geq 1].$$

It is now sufficient to prove that for all $X^\xi \in \mathcal{A}_n$ and all $\gamma \models r \geq 1$

$$X^\xi F_\alpha \in \mathcal{L}[G_\varepsilon \mid \varepsilon \text{ transdiagonal}]. \tag{4.8}$$

But Lemma 4.2 implies that any monomial of degree greater than n is in $\mathcal{J}_n$. Hence to prove (4.8), we need only show it for ξ and γ such that $|\xi| + |\gamma| \leq n$. To do that, we reduce the product

$$x_n^{\xi_n}(x_{n-1}^{\xi_{n-1}}(\cdots(x_2^{\xi_2}(x_1^{\xi_1} F_\alpha)))) \tag{4.9}$$

recursively, using

$$x_k\, G_{wb\beta 0^*} = G_{w(b+1)\beta 0^*} - G_{w0(b+1)\beta 0^*} \tag{4.10}$$

or

$$x_k\, G_{w0^*00^*} = G_{w0^*10^*} - G_{w0^*010^*}. \tag{4.11}$$

Relations (4.10) and (4.11) are immediate consequences of the definition of the $\mathcal{G}$ basis (Relation (3.2)).

We have to show that the vectors ε generated in this process are all transdiagonal and that the length $\ell(\varepsilon)$ always remains at most equal to n. Let us first check that the transdiagonal part. This is obvious in the case of relation (4.11). In the other case (relation (4.10)), for $\varphi = wb\beta 0^*$, it is sufficient to observe that if m is such that

$$\varphi_1 + \cdots + \varphi_m > m$$

with $m > \ell(w)$ (if not, it is evident), then

$$\varphi'_1 + \cdots + \varphi'_m > m+1 > m \qquad \text{and} \qquad \varphi''_1 + \cdots + \varphi''_{m+1} > m+1$$

where $\varphi' = w(b+1)\beta 0^*$, and $\varphi'' = w0(b+1)\beta 0^*$. We shall now prove that the length of the ε's always remains at most equal to n. For this we need to keep track of the term $\varepsilon_{\ell(\varepsilon)}$. Two cases have to be considered.

- First case: $\varepsilon_{\ell(\varepsilon)}$ comes from $\alpha_{\ell(\alpha)}$ that has shifted to the right by Relation (4.10). It could move at most $|\xi|$ steps to the right, whence

$$\ell(\varepsilon) \leq \ell(\alpha) + |\xi| \leq |\alpha| + |\xi| \leq n.$$

- Second case: $\varepsilon_{\ell(\varepsilon)}$ is a "1" generated by relation (4.11) that has shifted to the right. If it is generated by a multiplication by x_k, then we consider the vector

$$\eta = \xi_n \xi_{n-1} \cdots \xi_k 0^*.$$

Since $X^\xi \in \mathcal{A}_n$ implies $\pi(\eta)$ is a Dyck path, we have

$$|\eta| < \ell(\eta) = n - k + 1$$

hence the generated "1" can shift at most to position

$$k + |\eta| \leq k + n - k = n.$$

□

The recursive process used to reduce a product of form (4.9) is illustrated in the following example, where $n = 5$.

$$\begin{aligned} x_1\, x_3\, F_{21} &= x_3(x_1\, F_{21}) \\ &= x_3(G_{31000} - G_{03100}) \\ &= x_3\, G_{31000} - x_3\, G_{03100} \\ &= G_{31100} - G_{31010} - G_{03200} + G_{03020}. \end{aligned}$$

End of proof of Theorem 4.1. By Lemma 4.2, the set $\mathcal{B}_n$ spans the quotient $\mathbf{R}_n$. Assume we have a linear dependence relation modulo $\mathcal{J}_n$, *i.e.* there exists P

$$P = \sum_{X^\xi \in \mathcal{B}_n} a_\xi X^\xi \in \mathcal{I}_n.$$

By Lemma 4.4, $\mathcal{J}_n$ is linearly spanned by the G_ε's, thus

$$P = \sum_{\varepsilon \text{ transdiagonal}} b_\varepsilon G_\varepsilon.$$

This implies $LM(P) = X^\varepsilon$, with ε transdiagonal, which is absurd. □

A consequence of Lemma 4.4 and Theorem 4.1 is that the set $\mathcal{G}$ is a Gröbner basis of $\mathcal{J}_n$ with respect to the lex order. From this we see bellow that a minimal Gröbner basis of $\mathcal{J}_n$ is obtained from $\mathcal{G}$ if we select the $G_\varepsilon \in \mathcal{G}$ such that $\pi(\varepsilon)$ has exactly one step under the line $y = x$ and no other horizontal steps after that.

Corollary 4.5. *A minimal Gröbner basis for $\mathcal{J}_n$ is given by*

$$\{G_\varepsilon \in \mathcal{G} \mid \varepsilon = w0^*,\ \ell(w) = |w| + 1,\ w_1 + \cdots + w_s \leq s,\ \textit{for } s < \ell(w)\}. \tag{4.12}$$

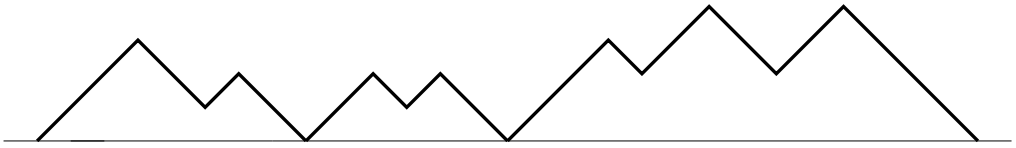
Proof. Theorem 4.1 implies that the monomial ideal $LT(\mathcal{J}_n)$ of leading terms of $\mathcal{J}_n$ is generated by all monomials X^η where $\pi(\eta)$ is transdiagonal. For any such η let m be the smallest integer such that $m < \eta_1 + \cdots + \eta_m$ and let $\varepsilon = \eta_1 \cdots \eta_{m-1} a 0^*$ where $a = m - 1 - \eta_1 - \cdots - \eta_{m-1}$. The monomial X^ε divides X^η which shows that $LT(\mathcal{J}_n)$ is generated by the leading monomials of the G_ε in (4.12). This gives that (4.12) is a Gröbner basis. To show minimality, consider X^ξ a monomial that stricly divides the leading monomial of a G_ε in (4.12). Since $\pi(\varepsilon)$ has exactly one step under the line $y = x$, we have that $\pi(\xi)$ is not transdiagonal and $X^\xi \notin LT(\mathcal{J}_n)$. Hence the leading monomials of the G_ε in (4.12) is a minimal set of generators for $LT(\mathcal{J}_n)$. □

5. Hilbert series

Since Theorem 4.1 gives us an explicit basis for the quotient $\mathbf{R}_n$, which is isomorphic to $\mathbf{SH}_n$ as a graded vector space, we are able to refine relation (1.4) by giving the Hilbert series of the space of super-covariant polynomials. For $k \in \mathbb{N}$, let $\mathbf{SH}_n^{(k)}$ and $\mathbf{R}_n^{(k)}$ denote the projections

$$\mathbf{SH}_n^{(k)} = \mathbf{SH}_n \cap \mathbb{Q}^{(k)}[X] \simeq \mathbf{R}_n \cap \mathbb{Q}^{(k)}[X] = \mathbf{R}_n^{(k)} \tag{5.1}$$

where $\mathbb{Q}^{(k)}[X]$ is the vector space of homogeneous polynomials of degree k together with zero. Here, we represent Dyck paths horizontally, with n rising steps $(1,1)$ and n falling steps $(1,-1)$. Let us denote by $D_n^{(k)}$ the number of Dyck paths of length $2n$ ending by exactly k falling steps and by $C_n^{(k)}$ the number of Dyck paths of length $2n$ which have exactly k factors, *i.e.* $k+1$ points on the axis. The next figure gives an example of a Dyck path of length 28, ending with 4 falling steps and made of 3 factors.



It is well known that

$$D_n^{(k)} = C_n^{(k)} = \frac{k\,(2n-k-1)!}{n!\,(n-k)!}, \tag{5.2}$$

where the first equality is classical (*cf.* [23] for example for a bijective proof), and the second corresponds to [17], formula (7).

Let us denote by $F_n(t)$ the Hilbert series of $\mathbf{SH}_n$, *i.e.*

$$F_n(t) = \sum_{k\geq 0} \dim \mathbf{SH}_n^{(k)}\, t^k. \tag{5.3}$$

Theorem 5.1. *For $0 \leq k \leq n-1$, the dimension of $\mathbf{SH}_n^{(k)}$ is given by*

$$\dim \mathbf{SH}_n^{(k)} = \dim \mathbf{R}_n^{(k)} = D_n^{(n-k)} = C_n^{(n-k)} = \frac{n-k}{n+k}\binom{n+k}{k}. \tag{5.4}$$

For $k \geq n$ the dimension of $\mathbf{SH}_n^{(k)}$ is 0.

Proof. By Theorem 4.1, we know that the set

$$\mathcal{B}_n = \{X^\eta \mid \pi(\eta) \text{ is a Dyck path}\}$$

is a basis for $\mathbf{R}_n$. It is then sufficient to observe that the path $\pi(\eta)$ associated to η ends by exactly $n - |\eta|$ falling steps. □

For example, we have:

n	$F_n(t)$
1	1
2	$1+t$
3	$1+2t+2t^2$
4	$1+3t+5t^2+5t^3$
5	$1+4t+9t^2+14t^3+14t^4$
6	$1+5t+14t^2+28t^3+42t^4+42t^5$
7	$1+6t+20t^2+48t^3+90t^4+132t^5+132t^6$

This gives

$$F_n(t) = \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} t^k \,. \tag{5.5}$$

from which one easily deduces that the generating series for the $F_n(t)$'s is

$$\sum_n F_n(t)\, x^n = \frac{1-\sqrt{1-4\,t\,x}-2\,x}{2\,(t+x-1)}. \tag{5.6}$$

Remark 5.2. The study of various filtrations of the space $\mathbb{Q}[X]$, with respect to family of ideals of quasi-symmetric polynomials, will be the object of a forthcoming paper [3].

Acknowledgments

We thank Adriano Garsia for stimulating discussions and suggestions about this work. We also thank Mike Zabrocki for his careful reading.

References

[1] E. Artin, *Galois Theory*, Notre Dame Mathematical Lecture **2** (1944), Notre Dame, IN.
[2] J.-C. Aval and N. Bergeron, *Catalan Paths and Quasi-Symmetric Functions*, Proc. Amer. Math. Soc., to appear.
[3] J.-C. Aval, F. Bergeron, N. Bergeron and A. Garsia, *Ideals of Quasi-symmetric polynomials and related varieties*, in preparation.
[4] F. Bergeron, N. Bergeron, A. Garsia, M. Haiman and G. Tesler, *Lattice Diagram Polynomials and Extended Pieri Rules*, Adv. Math., **142** (1999), 244-334.
[5] F. Bergeron, A. Garsia and G. Tesler, *Multiple Left Regular Representations Generated by Alternants*, J. of Comb. Th., Series A, **91, 1-2** (2000), 49–83.
[6] N. Bergeron, S. Mykytiuk, F. Sottile, and S. van Willigenburg, *Pieri Operations on Posets*, J. of Comb. Theory, Series A, **91** (2000), 84–110 .
[7] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts*, Ann. of Math., **57** (1953), 115–207.
[8] C. de Concini and C. Procesi, *Symmetric functions, conjugacy classes and the flag variety*, Invent. Math. **64** (1981), 203–230.
[9] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms*, Springer-Verlag, New-York, 1992.
[10] A. M. Garsia and M. Haiman, *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90** (1993), no. 8, 3607–3610.
[11] A. Garsia, M. Haiman, *Orbit Harmonics and Graded Representations*, Éditions du Lacim, to appear.

[12] I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J.-Y. Thibon, *Non-commutative symmetric functions*, Adv. in Math., **112** (1995), 218–348.
[13] I. Gessel, *Multipartite P-partitions and products of skew Schur functions*, in Combinatorics and Algebra (Boulder, Colo., 1983), C. Greene, ed., vol. 34 of Contemp. Math., AMS, 1984, pp. 289–317.
[14] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Amer. Math. Soc., **14** (2001), 941-1006.
[15] F. Hivert, *Hecke algebras, difference operators, and quasi-symmetric functions*, Adv. in Math., **155** (2000), 181–238.
[16] V. Jones, *Index for subfactors*, Inv. Math. **72** (1983) 1–25.
[17] G. Kreweras, *Sur les éventails de segments*, Cahiers du BURO, **15** (1970), 3–41.
[18] I. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, 1995, second edition.
[19] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, Journal of Algebra, **177** (1995), 967–982.
[20] R. Stanley, *Enumerative Combinatorics, Vol. 1*, Wadsworth and Brooks/Cole, 1986.
[21] R. Stanley, *Enumerative Combinatorics Vol. 2*, no. 62 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999. Appendix 1 by Sergey Fomin.
[22] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. Amer. Math. Soc., **112** (1964), 392–400.
[23] J. Vallé, *Une bijection explicative de plusieurs propriétés ramarquables des ponts*, Europ. J. Combinatorics, **18** (1997), 117–124.

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