

A Color Model for Rendering Linear Passive Graphic 2D Objects

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PROOF OF LEMMA 6.3

We show that for each matrix element, the expressions obtained by first combining X and Y and then combining the result with Z are equivalent to the corresponding expressions obtained by first combining Y and Z and then combining X with the result.

Using the definition of the \circ operation, the resulting matrix for $(X \circ Y) \circ Z$ becomes:

$$\begin{pmatrix} A_{FF} & A_{BF} \\ A_{FB} & A_{BB} \end{pmatrix}$$

with:

$$\begin{aligned} A_{FF} &= \left(x_{FF} + \frac{x_{FB}x_{BF}y_{FF}}{1 - x_{BB}y_{FF}} \right) + \frac{\left(\frac{x_{FB}y_{FB}}{1 - x_{BB}y_{FF}} \right) \left(\frac{x_{BF}y_{BF}}{1 - x_{BB}y_{FF}} \right) z_{FF}}{1 - \left(y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1 - x_{BB}y_{FF}} \right) z_{FF}} \\ A_{BF} &= \frac{\left(\frac{x_{BF}y_{BF}}{1 - x_{BB}y_{FF}} \right) z_{BF}}{1 - \left(y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1 - x_{BB}y_{FF}} \right) z_{FF}} \\ A_{FB} &= \frac{\left(\frac{x_{FB}y_{FB}}{1 - x_{BB}y_{FF}} \right) z_{FB}}{1 - \left(y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1 - x_{BB}y_{FF}} \right) z_{FF}} \\ A_{BB} &= z_{BB} + \frac{\left(y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1 - x_{BB}y_{FF}} \right) z_{FB}z_{BF}}{1 - \left(y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1 - x_{BB}y_{FF}} \right) z_{FF}} \end{aligned}$$

Similarly, the resulting matrix for $X \circ (Y \circ Z)$ is:

$$\begin{pmatrix} B_{FF} & B_{BF} \\ B_{FB} & B_{BB} \end{pmatrix}$$

with:

$$\begin{aligned}
B_{FF} &= x_{FF} + \frac{x_{FB}x_{BF}(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}})}{1 - x_{BB}(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}})} \\
B_{BF} &= \frac{x_{BF}(\frac{y_{BF}z_{BF}}{1-y_{BB}z_{FF}})}{1 - x_{BB}(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}})} \\
B_{FB} &= \frac{x_{FB}(\frac{y_{FB}z_{FB}}{1-y_{BB}z_{FF}})}{1 - x_{BB}(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}})} \\
B_{BB} &= (z_{BB} + \frac{y_{BB}z_{FB}z_{BF}}{1 - y_{BB}z_{FF}}) + \frac{x_{BB}(\frac{y_{FB}z_{FB}}{1-y_{BB}z_{FF}})(\frac{y_{BF}z_{BF}}{1-y_{BB}z_{FF}})}{1 - x_{BB}(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}})}
\end{aligned}$$

We need to show equality between these two matrices. Splitting up the equality into an equation for each matrix element gives the following four equations:

$$\begin{cases}
A_{FF} = B_{FF} \\
A_{BF} = B_{BF} \\
A_{FB} = B_{FB} \\
A_{BB} = B_{BB}
\end{cases}$$

Let us consider the first equation. Subtracting x_{FF} from both sides, we get:

$$\left(\frac{x_{FB}x_{BF}y_{FF}}{1 - x_{BB}y_{FF}}\right) + \frac{(\frac{x_{FB}y_{FB}}{1-x_{BB}y_{FF}})(\frac{x_{BF}y_{BF}}{1-x_{BB}y_{FF}})z_{FF}}{1 - (y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1-x_{BB}y_{FF}})z_{FF}} = \frac{x_{FB}x_{BF}(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}})}{1 - x_{BB}(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}})}$$

Clearly, if either x_{FB} or x_{BF} is 0 both sides are 0, so the equality holds. If instead either y_{FB} or y_{BF} is 0, the equation can be simplified to:

$$\frac{x_{FB}x_{BF}y_{FF}}{1 - x_{BB}y_{FF}} = \frac{x_{FB}x_{BF}y_{FF}}{1 - x_{BB}y_{FF}}$$

which obviously holds. If neither x_{FB} nor x_{BF} is zero, we can divide both sides by these quantities, and the equation simplifies to:

$$\left(\frac{y_{FF}}{1 - x_{BB}y_{FF}}\right) + \frac{(\frac{y_{FB}}{1-x_{BB}y_{FF}})(\frac{y_{BF}}{1-x_{BB}y_{FF}})z_{FF}}{1 - (y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1-x_{BB}y_{FF}})z_{FF}} = \frac{(y_{FF} + \frac{z_{FF}y_{BF}y_{FB}}{1-y_{BB}z_{FF}})}{1 - x_{BB}(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}})}$$

After simplification, both sides yield:

$$\frac{y_{FF} - y_{FF}y_{BB}z_{FF} + y_{BF}y_{FB}z_{FF} - x_{BB}y_{FF}(y_{FF} + y_{FF}y_{BB}z_{FF} - y_{BF}y_{FB}z_{FF})}{(1 - x_{BB}y_{FF})(1 - x_{BB}y_{FF} - y_{BB}z_{FF} + x_{BB}y_{FF}y_{BB}z_{FF} + x_{BB}y_{BF}y_{FB}z_{FF})}$$

For the second equation, we need to show that:

$$\frac{\left(\frac{x_{BF}y_{BF}}{1-x_{BB}y_{FF}}\right)z_{BF}}{1 - \left(y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1-x_{BB}y_{FF}}\right)z_{FF}} = \frac{x_{BF}\left(\frac{y_{BF}z_{BF}}{1-y_{BB}z_{FF}}\right)}{1 - x_{BB}\left(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}}\right)}$$

However, we have assumed that at least one of x_{BF}, y_{BF}, y_{BF} is 0. Both sides of this equation are therefore 0. This means that in the combined object $X \circ Y \circ Z$, no light is transmitted from the back to the front, which is to be expected when at least one of the component objects blocks all such light.

Similarly, for the third equation we need to show that:

$$\frac{\left(\frac{x_{FB}y_{FB}}{1-x_{BB}y_{FF}}\right)z_{FB}}{1 - \left(y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1-x_{BB}y_{FF}}\right)z_{FF}} = \frac{x_{FB}\left(\frac{y_{FB}z_{FB}}{1-y_{BB}z_{FF}}\right)}{1 - x_{BB}\left(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}}\right)}$$

But since at least one of x_{FB}, y_{FB}, y_{FB} is 0, again both sides are trivially 0.

Finally, for the fourth equation, we need to show that:

$$\begin{aligned} z_{BB} + \frac{\left(y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1-x_{BB}y_{FF}}\right)z_{FB}z_{BF}}{1 - \left(y_{BB} + \frac{x_{BB}y_{FB}y_{BF}}{1-x_{BB}y_{FF}}\right)z_{FF}} &= \\ \left(z_{BB} + \frac{y_{BB}z_{FB}z_{BF}}{1 - y_{BB}z_{FF}}\right) + \frac{x_{BB}\left(\frac{y_{FB}z_{FB}}{1-y_{BB}z_{FF}}\right)\left(\frac{y_{BF}z_{BF}}{1-y_{BB}z_{FF}}\right)}{1 - x_{BB}\left(y_{FF} + \frac{y_{FB}y_{BF}z_{FF}}{1-y_{BB}z_{FF}}\right)} \end{aligned}$$

But this equation has the same form as the first one, except with symmetrical values for the parameters. Similar simplifications thus apply.