Laboratoire Bordelais de Recherche en Informatique
UMR 5800 - Université Bordeaux I, 351, cours de la Libération,
33405 Talence CEDEX, France

Research Report RR-1291-03

CW complexes: Topological Mainframe for Numerical Representations of Objects

by Pascal DESBARATS and Stefka GUEORGUIEVA

Mars, 2003

CW complexes: Topological Mainframe for Numerical Representations of Objects

Pascal DESBARATS and Stefka GUEORGUIEVA *

Abstract

Dimensionally non-homogeneous pointsets with internal structure are the focus in a consequent number of studies in both computational geometry and discrete (or digital) geometry, and lead to various practical applications in geometric modeling and computer imagery. Our motivation is to revisit the well known notion of algebraic topology, the cell *CW* complex, and to use it as an abstract framework for numerical representation of inhomogeneous objects. Two representational issues, respectively in non-manifold solid modeling and in discrete object boundary reconstruction, are discussed in illustration of this general setting.

*The authors wish to thank Prof. A.Zvonkin for discussions and criticisms that have given many insights of the notions and definitions illustrated in this work.

Contents

1 Introduction

Mathematical fundamentals allow a rigorous characterization of the representational schemes and modeling operations of the continuously growing domain of objects in interest. With respect to geometric modeling and computational geometry for example, starting with the so called "manifold solids" and the corresponding topological data structures, manipulated with the set of the Euler Operators [53], more general classes of solids, presenting degeneracies, came up to view: "pseudo-manifolds" [29], "quasi-manifolds" [42, 12], and in general "non-manifolds" [72, 50, 24, 7, 23, 73, 46, 47, 6, 65]. An emerging fundamental concept of combinatorial topology, the cellular complex, becomes undeniabley a universal tool for construction and manipulation of dimensionally non-homogeneous objects with internal structures and produces the convergence of methods in generic domains such as the geometric modeling and computational geometry, and the computer imagery and discrete geometry [35, 27, 36, 18, 19, 28].

The aim of the present study is to revisit the "classic" cellular structure of the CW complexes [11, 55, 52, 45]. We investigate to which extent the underlying topological structures, the *semisimplicial complexes*, and the corresponding *semisimplicial constructions* (product, quotient and adjunction) form a basis in the modeling space of interest. The motivation is to establish and illustrate the genuineness of the CW complexes as a unified theoritical concept and as practical utility tool for object representation and manipulation. The presentation is organized as follows. First, some basic notions are overviewed. Next, the mathematical framework utilization for object representation with respect to two computer graphics and image processing applications is discussed: the topology-based geometric modeling and the boundary reconstruction of discrete objects. Finally, a synthesis of the related results and future work are reported.

2 Basic Notions

Introductory material on general topology could be found in [3, 30, 70, 22, 26, 33]. A more advanced topic textbooks in algebraic (or combinatorial) topology are [11, 31]. The terminology from the topology of CW complexes mostly follows the textbook of Lundell [45]. The exceptions would be explicitly referenced. A good reading on the subject could also be found in [52].

- Closed Euclidean $n-cell$ E^n.

  E^n is a homeomorphic image of the Euclidean n-cube I^n, the cartesian product of n copies of the closed unit interval $I = \{t \in R | 0 \leq t \leq 1\}$

  Other $n-cells$:

  1. A $n-disc$ $D^n = \{(t_0, t_1, \ldots, t_{n-1}) \in R^n | \Sigma t_i^2 \leq 1\}$
  2. A *standard* $n-simplex$ $\delta^n = \{(t_0, t_1, \ldots, t_n) \in R^{n+1} | t_i \geq 0, 0 \leq i \leq n \text{ and } \Sigma_i t_i = 1\}$.

3. A *boundary* $\dot{E}^n$ of a closed Euclidean $n-cell$ embedded as a convex set in R^n is its topological frontier and is homeomorphic to the standard $(n-1)$-sphere $S^{n-1} = \{(t_0, \ldots, t_{n-1}) \in R^n | \Sigma_i t_i^2 = 1\}$.
4. An *open* $n-cell$ is a homeomorphic image of E^n-$\dot{E}^n$. It is homeomorphic to the Euclidean space R^n.

- *Disjoint union* of a collection of topological spaces $\{X_\alpha | \alpha \in A\}$ is a topological space, the underlying set of which is the disjoint union of the sets X_α and the topology of which can be described as follows. A set $U \subset \cup_\alpha X_\alpha$ is open if and only if each intersection $U \cap X_\alpha$ is open in the topology of X_α.

  1. If we are given a map $f_\alpha : X_\alpha \to Y$ for each $\alpha \in A$, the *disjoint union* of this family of maps is the map $f = \cup_\alpha f_\alpha : \cup_\alpha X_\alpha \to Y$ such that $f|X_\alpha = f_\alpha$. Clearly the disjoint union of a family of continuous maps is a continuous map. Note that for each $\beta \in A$, the inclusion $i_\beta : X_\beta \to U_\alpha X_\alpha$ is a continuous injection.
  2. If a covering $\beta = \{B_\alpha | \alpha \in A\}$ of a space X is such that a subset $U \subset X$ is open if and only if it meets each B_α in an open subset of B_α, the covering β is said to *dominate* the space X or that X has the **weak topology** with respect to β. Clearly a disjoint union of spaces has the weak topology with respect to the family of summands.

- A topological space X is a **topological n-manifold** provided each point $x \in X$ has a neighborhood homeomorphic to Euclidean n-space R^n.
  Standard examples of topological n-manifolds are R^n and the n-sphere S^n. As a neighborhood of $x \in S^n$ which is homeomorphic to R^n we may take $S^n - \{-x\}$, with the homeomorphism given by the stereographic projection from the point $-x$.

  1. Theorem **Invariance of domain**
     Let A be a subset of the topological n-manifold X, let B be a subset of the topological n-manifold Y, and let $f : A \to B$ be a homeomorphism. If A is open in X, then B is open in Y.

- If $f, g : X \to Y$ are continuous maps, **a homotopy** between f et g is a map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. If there is a homotopy between the maps $f, g : X \to Y$, we say that f and g are homotopic and write $f \simeq g$

  1. Homotopy defines an equivalence relation on the set of all maps from X to Y. The set of equivalence classes is denoted by $\pi(X, Y)$.
  2. Two spaces X and Y are said to have the same *homotopy type* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that the composition fg and gf are homotopic to the identity maps of X and Y, respectively. The maps are called *homotopy equivalence* and we say that one is *homotopy inverse* of the other.

3. Let X be a space and let $x \in X$. We write $X = (X, x)$ and $S^n = (E^n, \dot{E}^n)$. Then the homotopy class of maps $\pi(S^n; X) = \pi(E^n, \dot{E}^n; X, x)$ of $(E^n, \dot{E}^n)$ into (X, x) is denoted $\pi_n(X, x)$, or by $\pi_n(X)$, if X is *pathwise connected.*
   If X is pathwise connected and $\pi_1(X)$ is a single element, X is said to be *simply connected.*
   If $x \in A \subset X$, and $e \in \dot{E}^n$, then $\pi(E^n, \dot{E}^n, e; X, A, x)$ is denoted by $\pi_n(X, A, x)$, or by $\pi_n(X, A)$ if A is pathwise connected.
   $\pi_n(X)$ and $\pi_n(X, A)$ have group structures when $n \geq 1$.
   With this structure, the set $\pi_n(X)$ is called *nth homotopy group* of the pathwise connected space X,
   and $\pi_n(X, A)$ the *nth relative homotopy group* of the pair (X, A) at x.

3 Combinatorial Cell Complexes

- Cell structure.

Let X be a set. A *cell structure* on X is a pair (X, ϕ), where ϕ is a collection of maps of closed Euclidean cells into X satisfying the following conditions.

- If $\varphi \in \phi$ has domain E^n, then ϕ is injective on $E^n - \dot{E}^n$.
- The images $\{\varphi(E^n - \dot{E}^n) | \varphi \in \phi\}$ partition X, i.e., they are disjoint and have union X.
- If $\varphi \in \phi$ has domain E^n, then $\varphi(\dot{E}^n) \subset \bigcup\{\psi(E^k - \dot{E}^k) | \psi \in \phi$ has domain E^k and $k \leq n-1\}$.

1. If $\varphi \in \phi$ and φ has domain E^n, the image set $\varphi(E^n) = \sigma^n$ is called $n-$*cell* or *closed* $n-$*cell* of (X, ϕ), and we say φ is a *characteristic map* for the cell σ^n. Thus ϕ is a set of characteristic maps for the cells of (X, ϕ). Also $\varphi(\dot{E}^n)$ is called *boundary* of the cell σ^n, and $\varphi(E^n - \dot{E}^n)$ is called *interior*.
   If $n > 0$, $\varphi(E^n - \dot{E}^n)$ is called an *open* $n-$*cell*.
2. The union $\bigcup\{\psi(E^k - \dot{E}^k) | \psi \in \phi$ has domain E^k and $k \leq n-1\} = X^{n-1}$ is called $(n-1)-$*skeleton* of the cell structure. Thus for each n, $\dot{}\sigma^n = \varphi(\dot{E}^n) \subset X^{n-1}$.
3. We say that two structures (X, ϕ) and (X, ϕ') are strictly equivalent if there is a one-to-one correspondance between ϕ and ϕ' such that a characteristic function with domain E^n corresponds to a characteristic function with domain E^n, and corresponding functions differ only by a reparametrization of their domain. That is if φ and φ' are corresponding functions of ϕ and ϕ', respectively, then $\varphi' = \varphi.h$, where $h : (E^n, \dot{E}^n) \to (E^n, \dot{E}^n)$ is a homeomorphism of pairs.

If (X, ϕ) is a cell structure, let S_ϕ consist of all pairs $(\sigma^n, [\varphi])$, where $\sigma^n = \varphi(E^n)$ and $[\varphi]$ is the strict equivalence class of $\varphi \in \phi$.

4. A *cell complex* on a set X or a *cellular decomposition* of a set X is an equivalence class of cell structures (X, ϕ) under the equivalence relation of strict equivalence.
   A cell complex on X will be denoted by a pair (X, S), where $S = S_\phi$ for some representative cell structure (X, ϕ). The set S is called the set of (closed) cells of (X, S).
   To simplify the notations , when dimension of a member of S is not important, we will write $\sigma = \sigma^n$.Thus if $\sigma \in S$, it has a characteristic map φ_σ and $\varphi_\sigma : E_\sigma \to X$, where E_σ is an Euclidean cell.

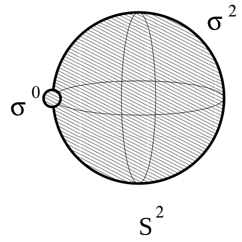
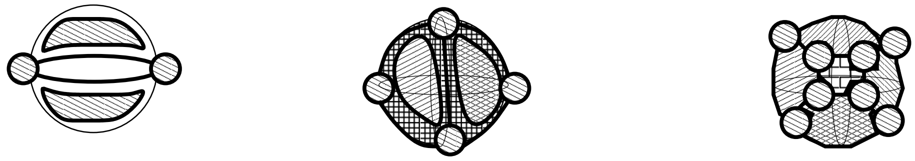


Figure 1: Example of a cell complex (S^2, S) on the sphere $S^2 = \{x = (x_0, x_1, x_2) \in R^3| < x, x >= x_0^2 + x_1^2 + x_2^2 = 1\}$. The set of cells $S = \{\sigma^0, \sigma^2\}$ is defined as follows. Cell $\sigma^0 = (1, 0, 0)$ and $\sigma^2 = S^2$. The set ϕ consists of $\varphi^0 : D^0 \to S^2$ and $\varphi^2 : D^2 \to S^2$ where $\varphi^0(x) = (1, 0, 0)$ and $\varphi^2(x) = (2 < x, x > -1, 2x_0 \sqrt[2]{1- < x, x >}, 2x_1 \sqrt[2]{1- < x, x >}), < x, x >= x_0^2 + x_1^2$. Thus $\sigma^0 = (1, 0, 0)$ and $\sigma^2 = \varphi^2(D^2) = S^2$. The cell complex (D^3, Υ) on D^3 is constructed with the characteristic maps φ^0 and φ^2 as defined above together with the identity map $\varphi^3 : D^3 \to D^3$, $\Upsilon = \{\sigma^0, \sigma^2, \sigma^3\}, \sigma^3 = D^3$. The example could be generalized for a n-sphere S^n and $(n+1)$-disc D^{n+1}. The complex (S^n, S) is defined by analogy: $S = \{\varphi^0, \varphi^n\}$ where $\varphi^0 : D^0 \to S^n$, $\varphi^0(x) = (1, 0, \ldots, 0)$, and $\varphi^n : D^n \to S^n$, $\varphi^n(x) = (2 < x, x > -1, 2x_0 \sqrt[2]{1- < x, x >}, \ldots, 2x_{n-1} \sqrt[2]{1- < x, x >}), < x, x >= x_0^2 + \ldots + x_{n-1}^2$. To get the complex (D^{n+1}, Υ), the characteristic maps φ^0 and φ^n are taken together with the identity map $\varphi^{n+1} : D^{n+1} \to D^{n+1}$.

Some examples are given on Fig.1 and Fig.2.

5. A *subcomplex* (A, ι) of a cell complex (X, S), denoted by $(A, \iota) \subset (X, S)$, is a cell complex such that $A \subset X$ and $\iota \subset S$.
   If (X, S) is any complex and $S^n = \{\sigma^p \in S | p \leq n\}$. Then (X^n, S^n) is a subcomplex of (X, S) called $n - skeleton$.
   The empty subcomplex $(\oslash, \oslash) = (X^{-1}, S^{-1}) \subset (X^0, S^0) \subset \ldots \subset (X, S)$.

6. A cell complex (X, S) is $finite$ or countable if S is a finite or countable set. It is *locally* $finite$ or locally countable if each closed cell



(a) (b) (c)

Figure 2: Examples of cell complexes on the sphere S^2.

meets only a finite or countable number of cells, and *closure finite* if each n-cell meets only a finite number of open cells $\sigma^p - \dot{\sigma}^p$ with $p < n$.

7. A cell is called *regular cell* if it has one characteristic map bijective on all E^n.
   A cell complex is *regular* if all its cells are regular.
   An example of a regular cell complex is given on Fig.3 and Fig.4.
   The cell complex shown on Fig.1 is not regular. The cell σ^3 is a regular cell but σ^2 is not.

   A cell complex is *normal* if, for each cell σ, the subset σ carries the structure of a subcomplex.
   A normal cell complex on the *projective spaces* is illustrated on Fig.5.

   The cell structure (X, S) illustrated on Fig.6 is an example of a non-normal cell stucture.

8. The cell complex (X, S) has dimension n if it has no cells σ^p of dimension p greater than n, and at least one cell of dimension n.

- Simplicial Complex

  1. A subset A of a real vector space V is independent if $t_0\alpha_0 + t_1\alpha_1 + \ldots + t_n\alpha_n \neq 0$, where $\alpha_i \in A, t_i \in R$ and $\Sigma_i t_i = 1$.
     If A is an independent set, the *simplex* of A is the set $\triangle(A) = \{\Sigma_i t_i \alpha_i | \alpha_i \in A, t_i \geq 0 \text{ and } \Sigma_i t_i = 1\}$.
     The elements of A are called *vertices* of the simplex $\triangle(A)$.
     For any set A, the collection of functions $\beta_\alpha : A \to R$ defined by $\beta_\alpha(a') = 0$ for $a \neq a'$ and $\beta_\alpha(a) = 1$, forms an independent subset of the real vector space R^A. Clearly the collection of β_α is in one-to-one correspondence with the elements of A. Thus for any set there is a simplex whose vertices are in one-to-one correspondence with the set.

  2. A simplex has a canonical cell structure.
     Suppose A is an independent finite set, say $A = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$. Let $\triangle^n \subset R^{n+1}$ be the standard n-simplex and define $\varphi_A : \triangle^n \to \triangle(A)$ by $\varphi_A(t_0, t_1, \ldots, t_n) = \Sigma_i t_i \alpha_i$. Then φ_A is a bijective characteristic

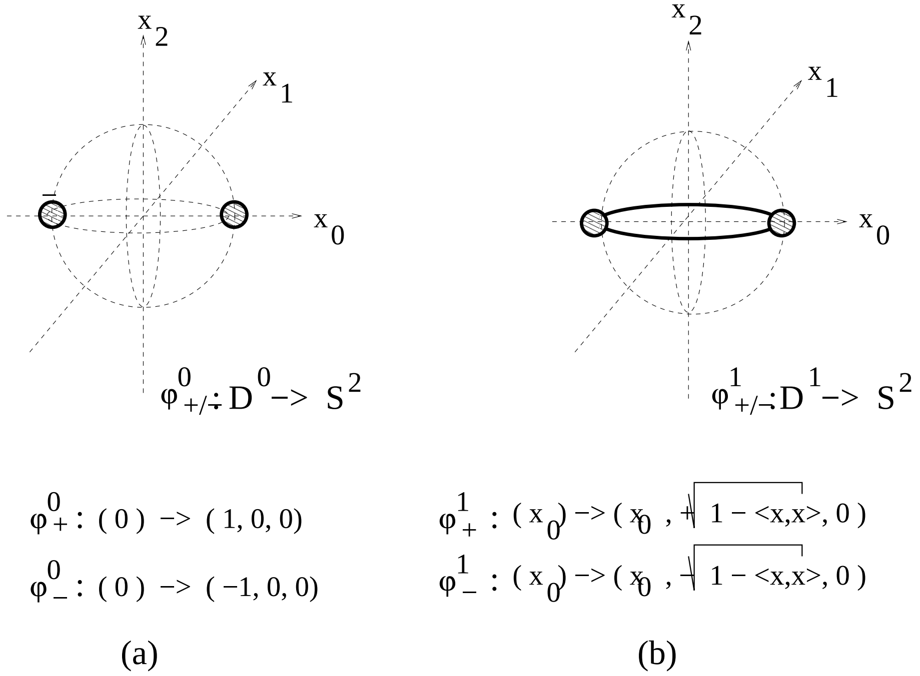
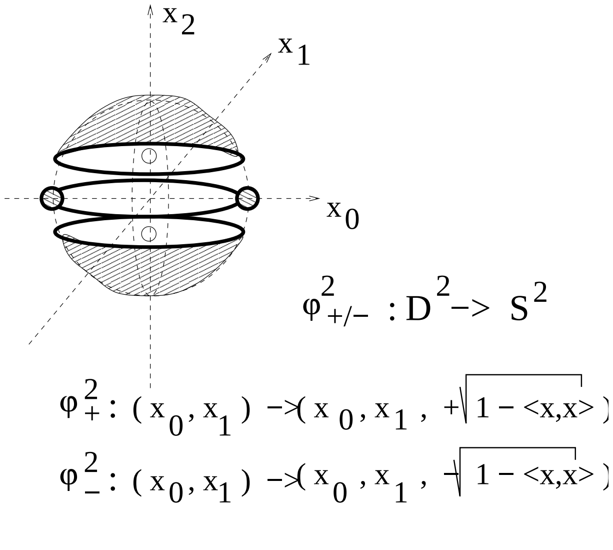


Figure 3: Example of a regular cell complex (S^n, Φ) on the sphere $S^n = \{x = (x_0, x_1, \ldots, x_n) \in R^{n+1} | < x, x >= x_0^2 + \ldots + x_n^2 = 1\}$ where $\Phi = \{\varphi_+^0, \varphi_+^0, \varphi_+^1, \varphi_-^1, \ldots, \varphi_+^n, \varphi_+^n\}$ and $\varphi_{+/-}^k(x) : D^k = \{x = (x_0, x_1, \ldots, x_{k-1}) \in R^k | < x, x >= x_0^2 + \ldots + x_{k-1}^2 \leq 1\} \rightarrow S^n$, $\varphi_+^k(x) = (x_0, \ldots, x_{k-1}, \sqrt[2]{(1- < x, x >}, 0, \ldots, 0)$, $\varphi_-^k(x) = (x_0, \ldots, x_{k-1}, -\sqrt[2]{(1- < x, x >}, 0, \ldots, 0)$, $< x, x >= \sum_{i=0}^{k-1} x_i^2$, $k = 1, \ldots, n$. The illustration for $n = 2$ is detailed. (a) Cells σ_+^0 and σ_-^0. (b) Cells σ_+^1 and σ_-^1.



(c)

Figure 4: Example of a regular cell complex on the sphere S^2. (c) Cells σ^2_+ and σ^2_-.

map for the n-cell of $\triangle(A)$. In the general case, a characteristic map $\varphi_{A'}$ is chosen for each finite subset $A' \subset A$, and let $\phi_A = \{\varphi_{A'} | A' \subset A$ is finite$\}$.
$(\triangle(A), \phi_a)$ is a regular cell structure on $\triangle(A)$.

3. Let $\gamma \in \triangle(A)$. Then $\gamma = \Sigma_{\alpha \in A} t_\alpha \alpha$, where $t_\alpha \geq 0, t_\alpha = 0$ for all but finitely many $\alpha \in A$, and $\Sigma_{\alpha \in A} t_\alpha = 1$. The number t_α is uniquely determined by γ and is called the *barycentric coordinate* of γ corresponding to the vertex α. The *barycentric coordinate functions* are the functions $b_\alpha : \triangle(A) \rightarrow R$ defined by $b_\alpha(\gamma) = t_\alpha$.

4. An *abstract simplicial complex* K consists of a set A, called the vertices of K, together with a collection S of finite subsets of A with the following properties:

   – For each $a \in A$, the set $\{a\}$ belongs to S.
   – If A' belongs to S, then so does each subset. The elements of S are called the *simplexes* of K, and if a simplex contains $n \geq 1$ distinct elements, it is called a $(n-1) - simplex$.

   Clearly, there is an one-to-one correspondence between the simplicial complexes embedded in $\triangle(A)$ and the abstract simplicial complexes which have A for a set of vertices.

5. A simplicial map of one simplex $\triangle(A)$ into a second $\triangle(B)$ is a map

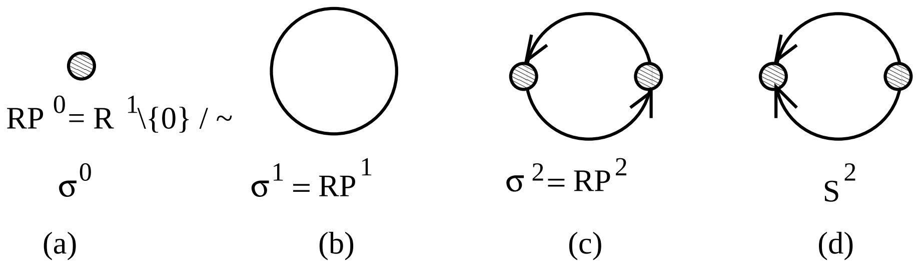


Figure 5: Example of a normal cell complex (X, S) where $X = RP^n$ known as *real projective space* defined as follows. Let us consider $R^{n+1} = \{x = (x_0, \ldots, x_n)\}$ and $W = R^{n+1} \backslash \{\circ\}$, $\circ = (0, \ldots, 0)$ is the origin of R^{n+1}. Introduce the equivalence relation $\sim$ in W. If $x = (x_0, \ldots, x_n)$ and $y = (y_0, \ldots, y_n)$ are two points $\in R^{n+1}$ then $x \sim y$ iff there exists a nonzero real number λ such that $y_i = \lambda x_i, i = 0, \ldots, n$. The quotient space RP^n, $RP^n = W/\sim$ under this relation is called *real projective space*. Let $\pi : W \to RP^n$ denote the quotient map and $\pi(x_0, \ldots, x_n) = [x_0, \ldots, x_n]$. Define the characteristic maps $\varphi^k : D^k \to RP^n$, $\varphi^k(x) = [x_0, \ldots, x_{k-1}, \sqrt[2]{1- < x, x >}, 0, \ldots, 0]$, $< x, x >= \sum_{i=0}^{k-1} x_i^2$, $k = 1, \ldots, n$. The set of cells $S = \{\sigma_0, \ldots, \sigma^n\}$ where $\sigma^k = \varphi^k(D^k) = RP^k$. If we take the cell complex on Fig.3 and Fig.4, the projection π identifies the cells σ^k_+ and σ^k_-. In other words the projective space RP^n is homeomorphic to the space obtained from D^n by identification of the points x and $-x$, $x \in S^{n-1} = \dot{\mathrm{D}}^n$. In terms of adjunction spaces, RP^n is homeomorphic to the space obtained by adjunction of RP^{n-1} to D^n under the projection $S^{n-1} = \dot{\mathrm{D}}^n \to RP^{n-1}$. The cases for $n \leq 2$ are detailed bellow. (a) RP^0 is a a single point of RP^2. (b) RP^1 , the real projective line, homeomorphic to S^1, obtained from D^1 by identifying -1 and $1 \in S^0 = \dot{\mathrm{D}}^1$. (c) RP^2 , the real projective plane also called "twisted sphere", represented with its plane model, obtained from D^2 by identifying $-x$ and $x \in S^1 = \dot{\mathrm{D}}^2$. (d) The plane model of the sphere S^2.

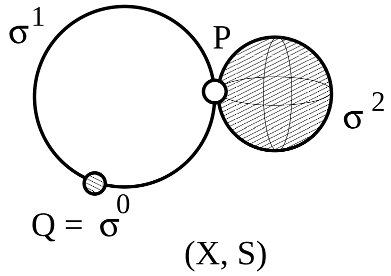


Figure 6: Example of a nonnormal cell structure (X, S) where $X = Q \cup S^1 \cup S^2$, $Q \in S^1$, $S^1 \cap S^2 = P$ and $S = \{\sigma^0, \sigma^1, \sigma^2\}$, $\varphi^k : E^k \to X$, $\varphi^k(E^k) = \sigma^k$. The maps $\varphi^k, k = 0, 1, 2$ are injective on $E^k - \dot{\mathrm{E}}^k$, the images $\varphi^k(E^k - \dot{\mathrm{E}}^k)$ partitition X and $\varphi^1(\dot{\mathrm{E}}^1) = \sigma^0$, $\varphi^2(\dot{\mathrm{E}}^2) = P \in \varphi^1(E^1 - \dot{\mathrm{E}}^1)$ but (X, S) is not normal. For example, σ^2 is not a subcomplex, $P \neg \in S$. The smallest subcomplex of (X, S) containing σ^2 is the whole complex.

$f : A \to B$ which is linearly extended over the subset $\triangle(A)$ of the vector space V to the subset $\triangle(B)$ of the vector space V' .
A map $f : K \to L$ of abstract simplicial complexes which maps vertices into vertices and sends the simplex $\{a_0, a_1, \ldots, a_n\}$ into $\{f(a_0), f(a_1), \ldots, f(a_n)\}$ is also called a *simplicial map*.
Simplicial maps between two simplicial complexes correspond one to one with the simplicial maps between corresponding abstract simplicial complexes.

- *Carrier topology*

  The *carrier topology* of a cell complex (X, S) is the topology on X in which the closed sets are the subcomplexes of (X, S).
  The *closure* of a set A is the smallest closed set containing it in the carrier topology, hence it is the smallest subcomplex of (X, S) whose underlying set contains A.

  1. If (X, S) is a cell complex, the *carrier* of $A \subset X$ is the intersection of all subcomplexes of (X, S) whose underlying set contains A. The carrier of A will be denoted by $(C(A), S|A)$, or by just $C(A)$ if the cellular decomposition is fixed.
  2. A cell complex (X, S) is *connected* if it is connected in the carrier topology.
  3. Let (X, S) be a cell complex. A *combinatorial path* or $c-path$ of (X, S) is a sequence $(x_0, x_1, \ldots, x_n)$ of points of X such that there exist cells $\sigma_i \in S$ for which $\{x_{i-1}, x_i\} \subset \sigma_i, i = 1, 2, \ldots, n$. We say that the c-path $(x_0, x_1, \ldots, x_n)$ is a c-path from the *initial* point x_0 to the *terminal* point x_n.
     If $p = (x_0, x_1, \ldots, x_n)$ and $q = (x_{n+1}, x_{n+2}, \ldots, x_{n+m})$ are c-paths of

(X,S) we define the product of the c-paths p and q as $p \vee q = (x_0, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$.

4. Let (X,S) be a cell complex and let A be a subset of X. The *star* or the *open star* of A, written $st(A)$, is the complement of the union of all cell carriers $C(\sigma)$ which do not meet A. The *closed star* $\bar{\mathrm{st}}(A)$ is the union of all cell carriers which do meet A.

- Function

  1. Let (X,S) and (Y,ι) be cell complexes. A function $f: X \to Y$ is said to be *cellular* provided that, for each n, $f(X^n) \subset Y^n$.
  2. A function $f: X \to Y$ is regular if f is cellular, and for each $\sigma^n \in S, f(\sigma^n) = \tau^m \in \iota$ and $f(\sigma^n - \dot{}\sigma^n) = \tau^m - \dot{}\tau^m$.
  3. If (X,S) and (Y,ι) are cell complexes and if there is a regular cellular bijection $f: X \to Y$, (X,S) and (Y,ι) are said to be *isomorphic*.
  4. If (X,S) and (Y,ι) are isomorphic then they are *homeomorphic* in the carrier topologies.

- Product cell complex

  1. Let (X,S) and (Y,ι) be cell complexes and ϕ,ϕ' representative collections of characteristic maps. Let α^n be a n-cell of S, and ι^m a m-cell of ι, and φ, φ' their characteristic maps in ϕ,ϕ'. Finally, let $\varphi \times \varphi' : E^n \times E^m \to X \times Y$ be the product map onto $\sigma^n \times \iota^m$.
     Note that $\varphi \times \varphi'(\dot{\mathrm{E}}^{n+m}) = \varphi \times \varphi'((E^n \times \dot{\mathrm{E}}^m) \bigcup (\dot{\mathrm{E}}^n \times E^m)) = (\sigma^n \times \dot{}\iota^m) \bigcup (\dot{}\sigma^n \times \iota^m)$.
     Let $\varphi \times \varphi'$ be the collection of all maps $\varphi \times \varphi'$ for $\varphi \in \phi$ and $\varphi' \in \phi'$. The pair $(X \times Y, \phi \times \phi')$ is a cell structure. The equivalence class of ϕ' depends only on S and ι, and we therefore may note it by $S \times \iota$.
  2. If (X,S) and (Y,ι) are cell complexes, the cell complex $(X \times Y, \phi \times \phi')$ is called *product cell complex*.
     The *$n-$skeleton* of the product is the set $(X \times Y)^n = \bigcup_p (X^p \times Y^{n-p})$.

- Equivalence relation and quotient cell complex.

  1. If X is a set and $\Re$ is an equivalence relation on X, let $\rho : X \to X/\Re$ be the *quotient map* onto the set of equivalence classes $X/\Re$.
  2. Let (X,S) be a cell complex and $\Re$ an equivalence relation on X. Then $\Re$ is a *cellular equivalence relation* provided the following are satisfied.
     - If $\sigma \in S$ then $\rho^{-1}\rho(\sigma - \dot{}\sigma)$ is a union of open cells $\sigma' - \dot{}\sigma'$ of the cellular partition of X.

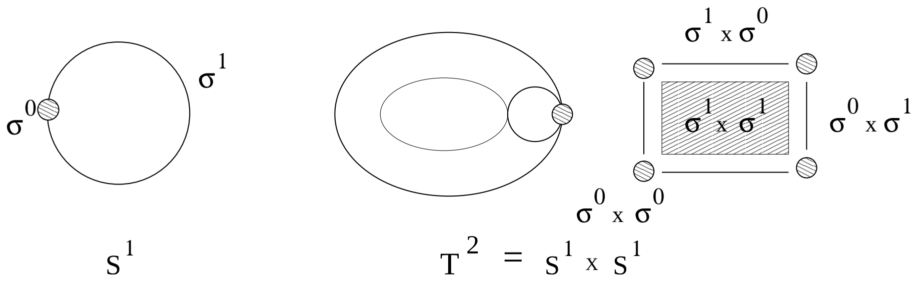


Figure 7: Example of a cartesian product of sphere cell complexes $X = S^m \times S^n$. Let $p_0 = (1, 0, \ldots, 0) \in S^m$ and $q_0 = (1, 0, \ldots, 0) \in S^n$. The product complex (X, Ψ) is defined as follows: $\Psi = \{\psi^0, \psi^m, \psi^n, \psi^{n+m}\}$ and $\psi^k : D^k \to X$. Let $h_{m,n} : (D^{m+n}, \dot{D}^{m+n}) \to (D^m \times D^n, D^m \times \dot{D}^n \cup \dot{D}^m \times D^n)$ be a homeomorphism of pairs, $\psi^0(x) = (p_0, q_0), \psi^m(x) = (\varphi^m(x), q_0), \psi^n(x) = (p_0, \varphi^n(x))$ and $\psi^{n+m}(x) = (\varphi^m \times \varphi^m)h_{m,n}(x)$ where $\varphi^k(x)$ are defined as on Fig.1. The construction is illustrated for the torus $T^2 = S^1 \times S^1$.

- If $\sigma' - \dot{\sigma}' \in \rho^{-1}\rho(\sigma - \dot{\sigma})$ is of minimal dimension among all such open cells in the union, then $\rho|(\sigma' - \dot{\sigma}')$ is a bijection onto $\rho(\sigma - \dot{\sigma})$ and $\rho(\sigma') = \rho(\sigma)$. Such a cell σ' will be called $\Re$-*minimal* for the cell σ.
- If σ' and σ'' are both $\Re$-minimal for the cell σ and if φ' and φ'' are the respective characteristic functions, then there is a homemorphism $h : E_{\sigma'} \to E_{\sigma''}$ such that $\rho\varphi' = \rho\varphi'' h$.

3. If (X, S) is a cell complex and $\Re$ is a cellular equivalence relation on X, define $S/\Re = \{\rho(\sigma) | \sigma \in S \text{ and } \sigma \text{ is } \Re - minimal\}$. The complex $(X/\Re, S/\Re)$ is the *quotient* or *identification complex* with respect to $\Re$. Illustrations are given on Fig.8.
4. Let (X, S) be a cell complex and (A_γ, ι_γ) a family of disjoint subcomplexes.
   We say that the quotient complex $(X/\Re, S/\Re)$, established from this relation, is obtained from (X, S) by *shrinking* or *collapsing* the subcomplexes (A_γ, ι_γ) to vertices of (X, S).
5. Let $I = (I, \jmath)$ denotes the standard regular cell complex on the unit interval I consisting of two 0-cells, denoted by 0 and 1 and called endpoints of I, and one 1-cell which has the identity map as a characteristic map.
   If (X, S) is a cell complex, the following are all cell complexes:
   - The *cone over* X, $c(X)$, is obtained from $X \times I$ by collapsing the subcomplex $X \times \{1\}$ to a vertex. An example is given on Fig.9.
   - The *reduced cone over* X, $\overline{c}(X)$, is obtained from $X \times I$ by collapsing the subcomplex $(X \times \{1\}) \cup (\{\overline{x}\} \times I)$ to a vertex

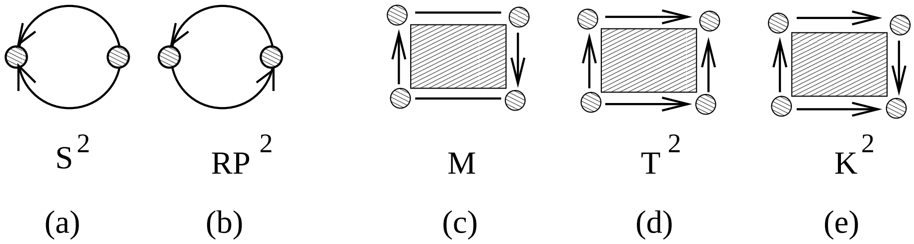


Figure 8: Examples of quotient spaces, illustrated in terms of plane models. Underlying complex structure constructions on S^n and I follow respectively Fig.1 and Fig.9. (a)The quotient space, obtained from D^2 by the identity on the boundary S^1, is homeomorphic to S^2. (b)The quotient space, obtained from D^2 by identifying antipodal points on the boundary S^1, is homeomorphic to the 2-dimensional projective space RP^2 (see also Fig.5). (c) Consider the unit square $I^2 = I \times I$. The quotient space, obtained from I^2 by identifying $(0,t)$ with $(1, 1-t)$ for every $t \in I$, is called the Möbius *strip* M. (d) The quotient space, obtained from I^2 by identifying $(0,t)$ with $(1,t)$, and $(t,0)$ with $(t,1)$ for every $t \in I$, is the *torus* T^2. (e) The quotient space obtained from I^2 by identifying $(0,t)$ with $(1,1-t)$, and $(t,0)$ with $(t,1)$, for every $t \in I$, is called the *Klein bottle* K^2.

where $\overline{x}$ is a vertex of X. One may also obtain it from the cone by collapsing a generator, the image of the subcomplex $\{\overline{x}\} \times I$ in $c(X)$.

– The *suspension* of x, $s(X)$, is obtained from $X \times I$ by collapsing the subcomplexes $X \times \{0\}$ and $X \times \{1\}$ to distinct vertices. See Fig.10.
– The *reduced suspension* of X, $\overline{s}(X)$, is obtained from $X \times I$ by collapsing the subcomplex $(X \times \{0\}) \cup (\{\overline{x}\} \times I) \cup (X \times \{1\})$ to a point, where $\overline{x}$ is a vertex of X.
– If $(X, S) = X$ and $(Y, \iota) = Y$ are cell complexes, the *reduced product* or *smash product* $X \wedge Y$ is obtained from $X \times Y$ by collapsing the subcomplex $(X \times \{\overline{y}\}) \cup (\{\overline{x}\} \times Y)$ to a point. Here $\{\overline{x}\}$ and $\{\overline{y}\}$ are vertices of X and Y, respectively.

- Adjunction complex

1. Let X and Y be sets, let $A \subset X$, and let $f : A \to Y$ be a function. Let us define the set $Y \cup_f X$ as the quotient of the disjoint union of X and Y under the equivalence relation which identifies $a \in A$ with $f(a) \in Y$. The set $Y \cup_f X$ is called the *adjunction* of X to Y by the map f.
   Let $p : Y \cup X \to Y \cup_f X$ be quotient map. The subsets Y and $X - A$ of the disjoint union $Y \cup X$ are mapped bijectively into $Y \cup_f X$.

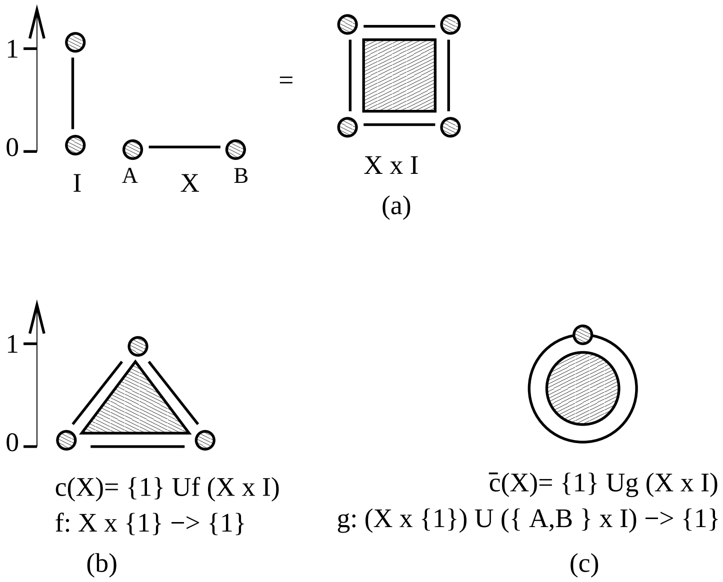


Figure 9: Let us consider the standard regular complex (I, ι) on the unit interval $I = [0, 1]$. The complex has two 0-cells denoted by $\iota_0^0 = 0$ and $\iota_1^0 = 1$, with caracteristic maps $\varphi_i^0 : i \to I, i = 0, 1$, (the end points 0 and 1), and one 1-cell, denoted by $\iota^1 = 01$ with the identity characteristic map. By analogy we construct the complex (X, S) with cells $\sigma_A^0 = A, \sigma_B^0 = B$ and $\sigma_{AB}^1 = AB$. The following are examples of topological constructions on the product of (X, S) and (I, ι). (a) The product of (X, S) and (I, ι). (b) The cone $c(X)$ is obtained from $X \times I$ by collapsing the subcomplex $(X \times \{1\})$ to a vertex. In terms of adjunction complexes, $c(X)$ is constructed adjoining $X \times I$ to $\{1\}$ by $f : X \times I \to \{1\}$. (c) The reduced cone $\overline{c}(X)$ is obtained from $X \times I$ by collapsing the subcomplex $(X \times \{1\}) \cup (\{\overline{x}\} \times I)$ to a vertex. In other words, one adjoins $X \times I$ to $\{1\}$ by $g : (X \times \{1\}) \cup (\{\overline{x}\} \times I) \to \{1\}$.

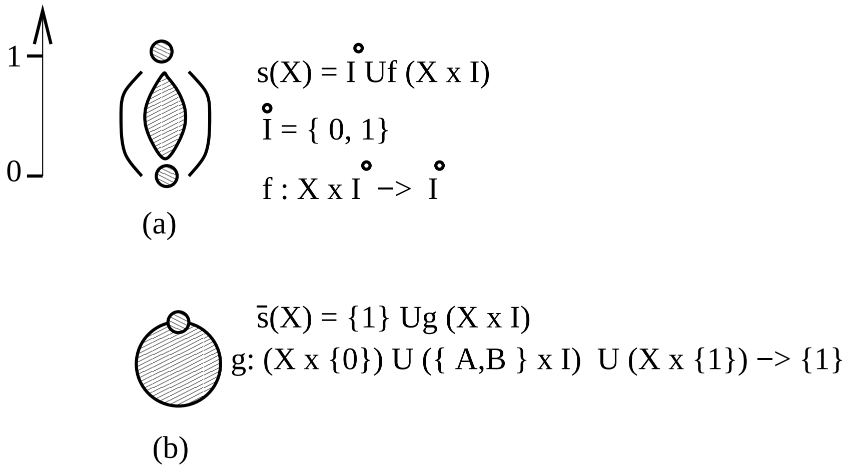


Figure 10: Let (I, ι) and (X, S) be the complexes illustrated on Fig.9. (a) The suspension $s(X)$ is obtained from $X \times I$ by collapsing the subcomplexes $X \times \{0\}$ and $X \times \{1\}$ to distinct vertices. Let $\dot{\mathrm{I}} = \{0, 1\}$ be a two point complex and define $f : X \times \dot{\mathrm{I}} \to \dot{\mathrm{I}}$ by $f(x, 0) = 0$ and $f(x, 1) = 1$. In terms of adjunction complexes, $s(X)$ is constructed adjoining $X \times I$ to $\dot{\mathrm{I}}$ by f. (b) The reduced suspension $\overline{s}(X)$ is obtained from $X \times I$ by collapsing the subcomplex $(X \times \{0\}) \cup (\{\overline{x}\} \times I) \cup (X \times \{1\})$ to a vertex. In other words, one adjoins $X \times I$ to a point complex $\{1\}$ by the map g, $g((X \times \{0\}) \cup (\{\overline{x}\} \times I) \cup (X \times \{1\})) = 1$.

2. The *adjunction* cell complex is obtained from the cell complexes (X, S) and (Y, ι) by adjunction of X to Y as above, and denoted as $(Y \cup_f X, S \cup_f \iota)$.
For example, the n-sphere S^n is the adjunction space obtained by adjoining the closed unit n-cell D^n to a singleton space Y by the means of the constant map $f : S^{n-1} \to Y$.

3. In the following $(I, \jmath)$ is the standard cell complex on the unit interval I, and (X, S) and (Y, ι) are cell complexes.

   – The *mapping cylinder*.
   Let $f : X \to Y$ be a cellular map. The complex $X \times I$ is adjoined to Y by the map $\overline{f} : X \times \{1\} \to Y$ defined by $\overline{f}(x, 1) = f(x)$. The resulting complex $(M_f, \Im_f)$ is called the *mapping cylinder*. An illustration is given on Fig.11.

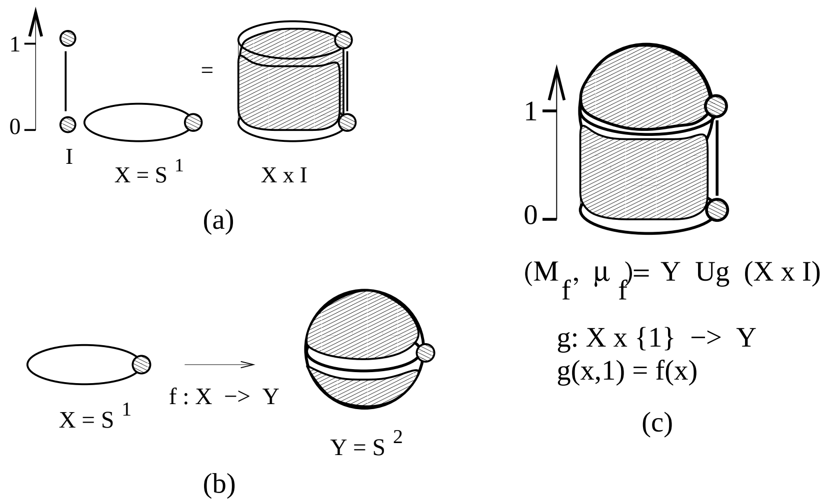


Figure 11: Let us consider the standard cell complex (I, ι) on the unit interval and the complexes (X, S) and (Y, ι) respectively on S^1 and S^2. (a) The complex product $X \times I$. (b) Let $f : X \to Y$ be a cellular map. (c) The *mapping cylinder* (M_f, μ_f) is obtained adjoining $X \times I$ to Y by the map $g : X \times \{1\} \to Y$ defined by $g(x, 1) = f(x)$.

   – The *join*.
   Let us define $f : (X \times \{0\} \times Y) \cup (X \times \{1\} \times Y) \to X \cup Y$ (disjoint union) by $f(x, 0, y) = x$ and $f(x, 1, y) = y$. Then the complex obtained by adjoining $X \times I \times Y$ to $X \cup Y$ by f is a complex called the *join* of X and Y and usually denoted by $(X \star Y, S \star \iota)$. The construction is illustrated on Fig.12.

   – The *cone* and *reduced cone*.
   Let $\{1\}$ be a point complex and $f : X \times \{1\} \to \{1\}$. The complex

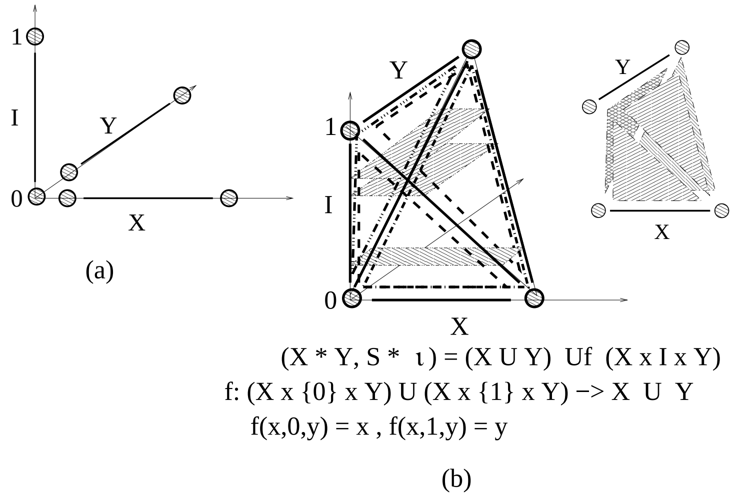


Figure 12: (a) Let us consider the standard cell complex (I, ι) on the unit interval and the complexes (X, S) and (Y, ι) on D^1. (b) The join complex $(X \star Y, S \star \iota)$ results from adjoining $X \times I \times Y$ to $X \cup Y$ by f, defined by $f : (X \times \{0\} \times Y) \cup (X \times \{1\} \times Y) \to X \cup Y$, $f(x, 0, y) = x$ and $f(x, 1, y) = y$.

obtained by adjoining $X \times I$ to $\{1\}$ by f is the *cone* $c(X)$. To get the reduced cone $\overline{c}(X)$, one adjoins $X \times I$ to $\{1\}$ by $f : (X \times \{1\}) \cup (\{\overline{x}\} \times I) \to \{1\}$, where $\overline{x}$ is a vertex of X. See the illustration on Fig.9.

– The *suspension* and *reduced suspension*.
Let $\dot{\text{I}} = \{0, 1\}$ be two point complex and define $f : X \times \dot{\text{I}} \to \dot{\text{I}}$ by $f(x, 0) = 0$ and $f(x, 1) = 1$. The suspension $s(X)$ is obtained by adjoining $X \times I$ to $\dot{\text{I}}$ by f. To get the reduced suspension $\overline{s}(X)$, adjoin $X \times I$ to one point complex $\{1\}$, by the map $f((X \times \{0\}) \cup (\{\overline{x}\} \times I) \cup (X \times \{1\}) = 1$ where $\overline{x}$ is a vertex of X. An example is given on Fig.10.

– The *smash product*.
Adjoin $X \times Y$ to the point complex $\{1\}$ by the map $f((X \times \{\overline{y}\}) \cup \{\overline{x}\} \times Y)) = 1$ where $\overline{x}$ and $\overline{y}$ are vertices of X and Y, respectively.

4. Let X and Y be spaces, let A be a closed subset of X, and let $f : A \to Y$ be a continuous map. The adjunction space $Y \cup_f X$ is the quotient of the disjoint union of Y and X by the smallest equivalence relation which identifies each $a \in A$ with its image $f(a) \in Y$. If $\phi : Y \cup X \to Y \cup_f X$ be the quotient map, then $\phi|Y$ is a homeomorphism of Y onto the closed subset $\phi(Y)$, such that $\phi|(X - A)$ is a homeomorphism of $X - A$ onto the open subset $\phi(X - A)$, and $Y \cup_f X = \phi(Y) \cup \phi(X - A)$ is a partition of $Y \cup_f X$. If A is the empty set, the adjunction space

is just the disjoint union.

5. Suppose that X is a space, and for each $\lambda \in \Lambda$, there is a map $f_\lambda : \dot{\mathrm{E}}_\lambda \to X$ of the boundary of the Euclidean cell E_λ into X. Let $\varepsilon = \cup\{E_\lambda | \lambda \in \Lambda\}$ and $\dot{\varepsilon} = \cup\{\dot{\mathrm{E}}_\lambda | \lambda \in \Lambda\}$ be disjoint unions, and let $F = \cup_\lambda f_\lambda : \dot{\varepsilon} \to X$ be the union map. The adjunction space $X \cup_F \varepsilon$ is said to be obtained by *attaching* the cells E_λ to X. The maps $f_\lambda : \dot{\mathrm{E}}_\lambda \to X$ are called the *attaching maps* for the cells E_λ of $X \cup_F \varepsilon$. The adjunction of a family of 0-cells is just the disjoint union.

4 CW Complexes

The cellular structure of the cell complex (X, S) is related to a topology on the set X with the concept of *CW complex* due to J.H.C Whitehead, 1953.

- If (X, S) is a cell complex. Choose a set of characteristic functions $\{\varphi_\sigma | \sigma \in S\}$ for the cells of S. Then the *weak topology* on X with respect to S is obtained by:
  - Giving each cell $\sigma \in S$ the quotient topology with respect to its characteristic function;
  - Giving X the weak topology with respect to the subsets $\sigma \in S$, i.e., a set $F \in X$ is closed if and only if $F \cap \sigma$ is closed in σ for each $\sigma \in S$.
- A cell complex (X, S) is *closure finite* if each cell $\sigma \in S$ meets finitely many cells of lower dimension.
- A Hausdorff space X is a *CW complex* with respect to a family of cells S provided:
  - The pair (X, S) is a cell complex such that each cell $\sigma \in S$ has a continuous characteristic function;
  - The space X has the weak topology with respect to S;
  - The cell complex (X, S) is closure finite.
- If (X, S) is a cell complex such that X satisfies the above conditions with respect to S, we usually say that X is a *CW complex with cells S*. Illustrations of cellular structures with respect to CW complex are given on Fig. 13.
- The following characterization of CW complex is useful in checking properties, skeleton by skeleton, rather than cell by cell:
  A space X is a CW complex if and only if there is a sequence of closed subspaces $X_0 \in X_1 \in \ldots \in X$ such that $X = \cup_n X_n$ and:
  - The set X_0 is discrete;
  - For each n, X_n is obtained from X_{n-1} by attaching $n - cells$;

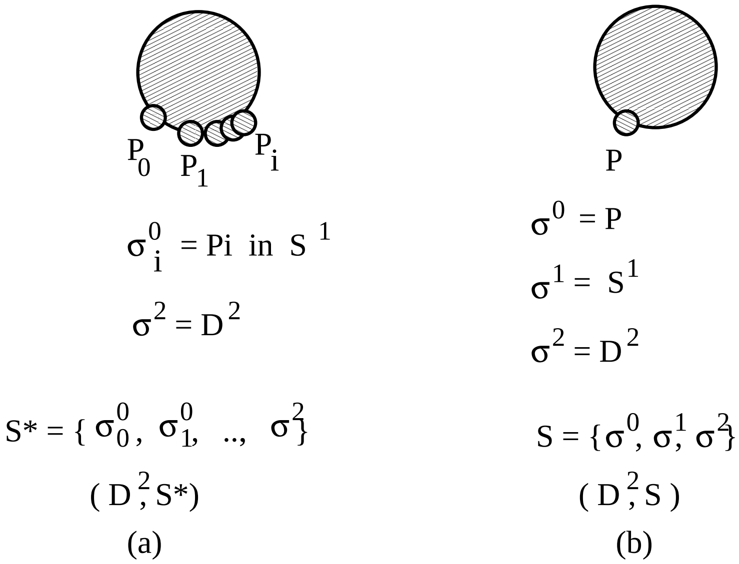


Figure 13: (a) Cell structure (D^2, S^*) on $D^2 = \{(x_0, x_1), x_0^2 + x_1^2 \leq 1\}$, $S^* = \{\sigma_0^0, \ldots, \sigma_i^0, \ldots, \sigma^2\}$, with one 0-cell σ_i^0 for each point P_i in $S^1 = \dot{D}^2$ and one 2-cell $\sigma^2 = D^2$. This structure is not a CW complex, the condition of closure finiteness being not verified. (b) Example of a CW complex (D^2, S).

- The space X has the weak topology with respect to the closed sets X_n.

- Regular CW complex

  1. A CW complex X is *regular* if each closed cell is homeomorphic to a closed Euclidean n-cell.
  2. A CW complex is *normal* if each cell is a subcomplex.

     A normal CW complex need not to be regular as schown on Fig. 5 with the example of the decomposition of the real projective space RP^n where each cell is homeomorphic to a projective space.

     It could be proved that

     - If X is a regular CW complex then X is a normal CW complex.
     - If X is a normal CW complex which is a topological n-manifold then
       * for $q < n$, each q-cell is a face of some n-cell, and
       * if X is regular, each $(n-1)$-cell is a face of exactly two n-cells.
  3. If X is a CW complex and σ, τ are cells of X, then τ is called a $face$ of σ provided $\tau \in \sigma$. If τ is a face of σ and $\tau \neq \sigma$, τ is called a *proper* $face$ of σ. A cell σ is a *principal cell* of X if it is not a proper face of any cell of X.

- If X_1 and X_2 are CW complexes. Let $X_1 \bigotimes X_2$ denote the space for which the underlying set is the product $X_1 \times X_2$ and the topology is the weak topology with respect to the product cells. Then $X_1 \bigotimes X_2$ is a CW complex.
  The spaces $X_1 \bigotimes X_2$ and $X_1 \times X_2$ differ only in that the CW topology may be finer than the product topology.

- Let X and Y be two CW complexes. If either:

  - X (or Y) is finite; or
  - both X and Y are countable;

  Then the map $\eta : X \bigotimes Y \to X \times Y$ is a homeomorphism. Thus the CW topology and the product topology on the product set $X \times Y$ coincide.

- The following are CW complexes:

  - The mapping cylinder of a continuous cellular map betweeen CW complexes.
  - The join of two CW complexes whenever their product is a CW complex.
  - The cone and the reduced cone over a CW complex.
  - The suspension and reduced suspension of a CW complex.

- The smash product of two CW complexes whenever their product is a CW complex.

- Let X be a CW complex with respect to two families of cells S and S'. Then (X, S') is a *subdivision* of (X, S) provided for each $\sigma' \in S'$ there is a cell $\sigma \in S$ such that $\sigma' - \dot{\sigma}' \in \sigma - \dot{\sigma}$.

  1. Let X be CW complex with cells S, let $A \in X$ be a subcomplex with cells ι, and let (X, S') be a subdivision of (X, S). It could be proved that if $\iota' = \{\sigma' \in S' | \sigma' \in A\}$, (A, ι') is a subcomplex of (X, S'), and is a subdivision of (A, ι).
  2. If X be a CW complex and $x \in X$, there is a subdivision of X in which x is a vertex.
  3. If X is a regular normal CW complex, there is a triangulation of X such that the resulting simplicial CW complex is a subdivision of X.

Regular CW complexes are of great interest because like the simplicial complexes they support simple algorithms for algebraic boundary computation, see [17] for examples. In parallel, they have the advantage to represent topological spaces with fewer cells than in the simplicial decompositions.
Our interest is directed towards a certain type of CW complexes associated with *semisimplicial complexes*.

5 Semisimplicial complexes

- Semisimplicial complex

  1. A *semisimplicial complex* X consists of a sequence $\{X_n | n = 0, 1, \ldots\}$ of disjoint sets together with a collection of maps in each dimension n:
     - $d_i : X_{n+1} \to X_n, i = 0, 1 \ldots n+1$, the ith *face* operator ;
     - $s_j : X_n \to X_{n+1}, j = 0, 1 \ldots n$, the jth *degeneracy* operator ;

     such that the following *semisimplicial identities* are satisfied:

     (a) $d_i d_j = d_{j-1} d_i$, $i < j$;
     (b) $d_i s_j = s_{j-1} d_i$, $i < j$;
     (c) $d_i s_j = 1$, $i = j, j+1$;
     (d) $d_i s_j = s_j d_{i-1}$, $i > j+1$;
     (e) $s_i s_j = s_{j+1} s_i$, $i \leq j$.

     The elements of X_n are called the $n - simplexes$ of X.
  2. If X and Y are semisimplicial complexes, a *semisimplicial map* is a map $f : X \to Y$ mapping X_n into Y_n for each n and such that $d_i f(x) = f(d_i x), s_j f(x) = f(s_j x)$ for each simplex x of X and each map d_i, s_j defined on x.

A simplex of form $s_i x$ is *degenerate*; one not of this form is *nondegenerate*. Examples of nondegenerate and degenerate simplexes are given on Fig.14. One can see that the face and degeneracy identities are verified.

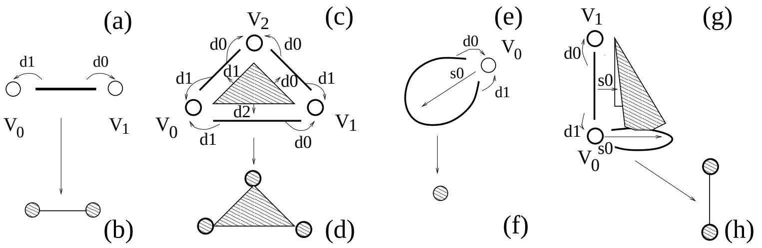


Figure 14: Examples of nondegenerate and degenerate 1D and 2D simplexes, and the corresponding face and degeneracy operators. (a) Nondegenerate 1D simplex $< V_0, V_1 >$ with face operators $d_0 < V_0, V_1 >= V_1, d_1 < V_0, V_1 >= V_0$. (b) Geometric realization of a nondegenerate 1D simplex $< V_0, V_1 >$. (c) Nondegenerate 2D simplex $< V_0, V_1, V_2 >$ with face operators $d_0 < V_0, V_1, V_2 >=< V_1, V_2 >$, $d_1 < V_0, V_1, V_2 >=< V_0, V_2 >$ and $d_2 < V_0, V_1, V_2 >=< V_0, V_1 >$, and semisimplicial identities $d_0 d_1 = d_0 d_0$, $d_0 d_2 = d_1 d_0$ and $d_1 d_2 = d_1 d_1$. One can see for example that $d_0 d_1 < V_0, V_1, V_2 >= d_0 < V_0, V_2 >=< V_2 >$ and $d_0 d_0 < V_0, V_1, V_2 >= d_0 < V_1, V_2 >=< V_2 >$ i.e. $d_0 d_1 = d_0 d_0$. (d) Geometric realization of a nondegenerate 2D simplex $< V_0, V_1, V_2 >$. (e) Degenerate 1D simplex $< V_0, V_0 >$. It could be represented in the form $< V_0, V_0 >= s_0 < V_0 >$. (f) Geometric realization of a degenerate 1D simplex $< V_0, V_0 >$. (g) Degenerate 2D simplex $< V_0, V_0, V_1 >$. It could be represented as $< V_0, V_0, V_1 >= s_0 < V_0, V_1 >$. (h) Geometric realization of a degenerate 2D simplex $< V_0, V_0, V_1 >$.

3. If X is a topological space, let $S_n(X)$ be the collection of all continuous maps of the standard n-simplex $\triangle^n$ into X, and let $S(X)$ be the sequence of $S_n(X)$. This collection of maps along with X define a complex known as *singular complex* on X.
   Define $d_i^* : \triangle^{n-1} \to \triangle^n$ to be the map sending $(t_0, \ldots, t_{n-1})$ to $(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$ and $s_j^* : \triangle^{n+1} \to \triangle^n$ the one sending $(t_0, \ldots, t_{n+1})$ to $(t_0, \ldots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \ldots, t_{n+1})$.
   Let $d_i : S_n(X) \to S_{n-1}(X)$ be the map sending the simplex $x_n : \triangle^n \to X$ into $x_n d_i^*$, and $s_j : S_n(X) \to S_{n+1}(X)$ be the map sending the simplex $x_n : \triangle^n \to X$ into $x_n s_j^*$.
   Then $S(X)$ is a semisimplicial complex with face and degeneracy operators respectively $d_i, i = 0, \ldots, n+1$ and $s_j, j = 0, \ldots, n$.
4. Let K be a simplicial complex with vertex set $V = \{v_\alpha\}$. Let $\sharp$ denotes some partial order on V which has the property that any set of vertices which lies in a simplex of K is linearly ordered by

$\sharp$. Let $K_n^\sharp$ denotes the collection of all nondecreasing linearly ordered sequences $< v_0, \dots, v_n >$ of elements of V which lie in a simplex of K, and let $d_i < v_0, \dots, v_n > = < v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n >$, $s_j < v_0, \dots, v_n > = < v_0, \dots, v_j, v_j, \dots, v_n >$. Then the sequence $K^\sharp$ of the $K_n^\sharp$ together with these maps d_i and s_j is a semisimplicial comlex. Such a semisimplicial comlex is referred as *ordered simplicial complex*. See an example on Fig. 15.

5. The *standard semisimplicial complex* $\sum(n)$ is the ordered simplicial complex whose vertices are the elements $\{0, 1, \dots, n\}$ with the usual ordering, and whose k-simplexes are all nondecreasing sequences with $k+1$ elements. The i-th face operator is given by deleting the $(i+1)$-st vertex, and the j-th degeneracy by repeating the $(j+1)$-st vertex. The semisimplicial maps $\varphi : \sum(n) \to \sum(m)$ correspond to monotone fonctions from $\{0, 1, \dots, n\}$ to $\{0, 1, \dots, m\}$. The i-th face map d_i^* and the j-th degeneracy map s_j^* are defined as follows:
$d_i^*(k) = k, k < i$ and $d_i^*(k) = k + 1, k \geq i$,
$s_j^*(k) = k, k \leq j$ and $s_j^*(k) = k - 1, k > j$.

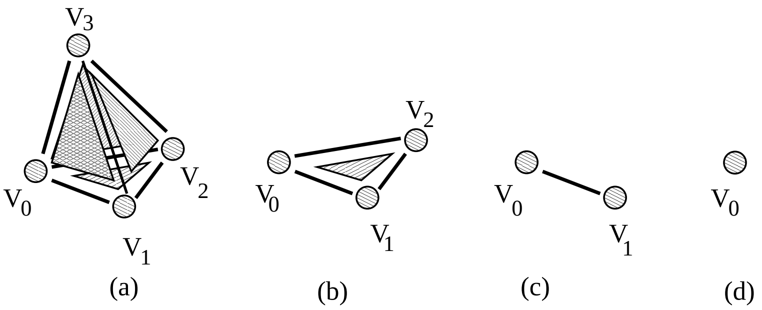


Figure 15: Examples of nondecreasing face operator sequences. (a) 3D nondegenerate simplex $< V_0, V_1, V_2, V_3 >$. (b) 2D nondegenerate simplex $< V_0, V_1, V_2 > = d_3 < V_0, V_1, V_2, V_3 >$. (c) 1D nondegenerate simplex $< V_0, V_1 > = d_2 d_3 < V_0, V_1, V_2, V_3 > = d_2 d_2 < V_0, V_1, V_2, V_3 >$. (d) 0D simplex $< V_0 > = d_1 d_2 d_3 < V_0, V_1, V_2, V_3 > = d_1 d_2 d_2 < V_0, V_1, V_2, V_3 > = d_1 d_1 d_3 < V_0, V_1, V_2, V_3 > = d_1 d_1 d_2 < V_0, V_1, V_2, V_3 >$.

- Realization of semisimplicial complexes

1. Given a semisimplicial complex X, to each n-simplex $x_n \in X$ let us associate a copy $(\triangle^n, x_n)$ of the standard n-simplex corresponding to x_n.
Let $t = (t_0, \dots, t_n)$ be a point of $\triangle^n$ and (t, x_n) be the corresponding point of $(\triangle^n, x_n)$.
Let $M(X)$ be the disjoint union of (t, x_n).
An equivalence $\Re$ on $M(X)$ is defined as follows:

$(d_i^* t, x_n) \sim (t, d_i x_n)$ for all $t \in \triangle^{n-1}$,
$(s_j^* t, x_n) \sim (t, s_j x_n)$ for all $t \in \triangle^{n+1}$.
Then $(t,x) \sim (u,y)$ if there exists a finite chain of such elementary equivalences beginning at (t,x) and ending at (u,y).
Let $|X|$ be the quotient space of $M(X)$ by $\Re$, let $\eta : M(X) \to |X|$ be the quotient map and let us denote $[t,x] = \eta(t,x)$.

2. The *geometric realization* $|X|$ of a semisimplicial complex X is a normal CW complex with one n-cell $|x_n|$ corresponding to each nondegenerate n-simplex x_n of X, and the faces of $|x_n|$ corresponding to the nondegenerate faces of x_n. The characteristic map of the cell $|x_n|$ is the composition of the inclusion of $(\triangle^n, x_n)$ into $M(X)$ followed by the projection onto the quotient space $|X|$.

   An illustration of the geometric realization is shown on Fig.14.

3. The subcomplexes of the semisimplicial complex X are in one-to-one correspondance with the subcomplexes of the CW complex $|X|$.

- Semisimplicial constructions

  1. Product .
     If X, Y are semisimplicial complexes, their *product* is a semisimplicial complex with set of n-simplexes $(X \times Y)_n = X_n \times Y_n$, and face and degeneracy maps given by operation on the coordinates: $d_i(x \times y) = d_i x \times d_i y$ and $s_j(x \times y) = s_j x \times s_j y$.
     An example is given on Fig.16.

  2. Quotient .
     If (X, A) is a semisimplicial pair then the *quotient complex* X/A is the one with n-simplex $(X/A)_n$ the quotient of the sets X_n by the sets A_n (identifying all the elements of the subsets A_n to a single element $[A_n]$).
     The face and the degeneracy operators are the quotient operations: $d_i[x] = [d_i x], s_j[x] = [s_j x]$.

  3. Adjunction .
     Let $f : A \to Y$ be a semisimplicial map where A is a subcomplex of X. The *adjunction complex* $Y \cup_f X$ is the quotient of the disjoint union $Y \cup X$ by the equivalence relation that identifies simplexes of A with their images under f, and whose semisimplicial operations are the quotient operations.
     The quotient map $q : X \cup Y \to Y \cup_f X$ is a semisimplicial map.
     The realization of the disjoint union $|X \cup Y|$ is the disjoint union of the realizations $|X| \cup |Y|$.
     The adjunction complex of a simplicial map is a semisimplicial complex.
     Some illustrations are given on Fig.17.

The following observations [45] should be emphasized:

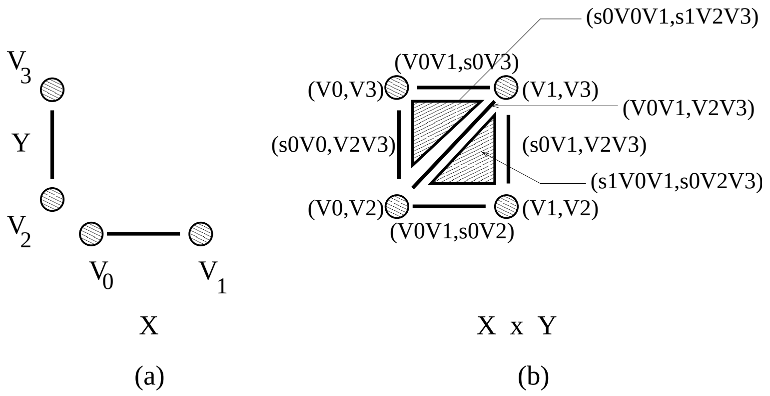


Figure 16: Example of product semisimplicial construction. (a) Let $X = \{< V_0 >, < V_1 >, < V_0, V_1 >\}$ and $Y = \{< V_2 >, < V_3 >, < V_2, V_3 > \}$. (b) The product complex $X \times Y$ is composed by the $0D$ simplexes $(V0, V2), (V1, V2), (V0, V3)$ and $(V1, V3)$, where $(Vi, Vj) = (< V_i >, < V_j >)$, the $1D$ simplexes $(s0V0, V2V3) = (s_0 < V_0 >, < V_2, V_3 >)$, $(s0V1, V2V3) = (s_0 < V_1 >, < V_2, V_3 >)$, $(V0V1, s0V2) = (< V_0, V_1 >, s_0 < V_2 >)$, $(V0V1, s0V3) = (< V_0, V_1 >, s_0 < V_3 >$ and $(V0V1, V2V3) = (< V_0, V_1 >, < V_2, V_3 >)$, and two $2D$ simplexes $(s0V0V1, s1V2V3) = (s_0 < V_0, V_1 >, s_1 < V_2, V_3 >)$ and $(s1V0V1, s0V2V3) = (s_1 < V_0, V_1 >, s_0 < V_2, V_3 >)$. We can verify that semisimplicial identities are fulfilled. For example let us test if $d_0 d_1(s0V0V1, s1V2V3) = d_0 d_0(s0V0V1, s1V2V3)$. We have $d_0 d_1(s0V0V1, s1V2V3) = d_0 d_1(s_0 < V_0, V_1 >, s_1 < V_2, V_3 >) = d_0 d_1(< V_0, V_0, V_1 >, < V_2, V_3, V_3 >) = (d_0 d_1 < V_0, V_0, V_1 >, d_0 d_1 < V_2, V_3, V_3 >= (d_0 < V_0, V_1 >, d_0 < V_2, V_3 >= (< V_1 >, < V_3 >) = (V1, V3)$. By analogy $d_0 d_0(s0V0V1, s1V2V3) = d_0 d_0(s_0 < V_0, V_1 >, s_1 < V_2, V_3 >) = d_0 d_0(< V_0, V_0, V_1 >, < V_2, V_3, V_3 >) = (d_0 d_0 < V_0, V_0, V_1 >, d_0 d_0 < V_2, V_3, V_3 >) = (d_0 < V_0, V_1 >, d_0 < V_3, V_3 >) = (< V_1 >, < V_3 >) = (V1, V3)$.

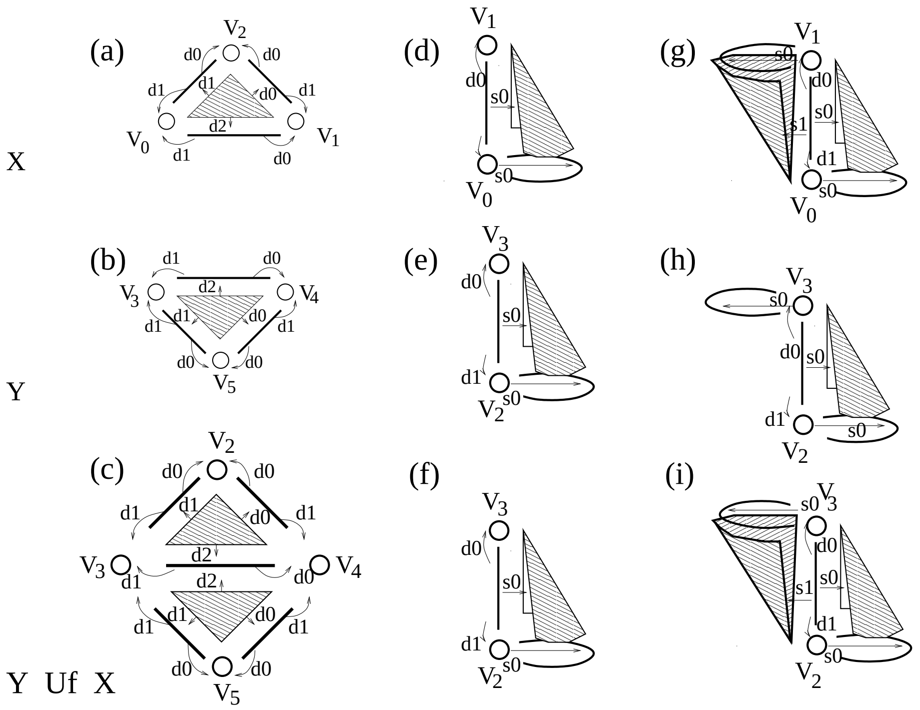


Figure 17: Examples of adjunction semisimplicial construction. (a) Semisimplicial complex X. (b) Semisimplicial complex Y. (c) Semisimplicial adjunction complex $Y \cup_f X$ where $f : A \to Y$, $A = (< V_0 >, < V_1 >, < V_0, V_1 >) \subset X$, $f(< V_0 >) =< V_3 >$, $f(< V_1 >) =< V_4 >$, $f(< V_0, V_1 >) =< V_3, V_4 >$, is a semisimplicial map. (d) Semisimplicial complex X with degenerate simplexes: $s_0 < V_0 >=< V_0, V_0 >$ and $s_0 < V_0, V_1 >=< V_0, V_0, V_1 >$. (e) Semisimplicial complex Y with degenerate simplexes: $s_0 < V_2 >=< V_2, V_2 >$ and $s_0 < V_2, V_3 >=< V_2, V_2, V_3 >$. (f) Semisimplicial adjunction complex $Y \cup_f X$ where $f : A \to Y$, $A = (< V_0 >, < V_1 >, < V_0, V_1 >) \subset X$, $f(< V_0 >) =< V_2 >$, $f(< V_1 >) =< V_3 >$, $f(< V_0, V_1 >) =< V_2, V_3 >$. Corresponding degenerate simplexes are also identified: $s_0 < V_0 >$ with $s_0 < V_2 >$, and $s_0 < V_0, V_1 >$ with $s_0 < V_2, V_3 >$. (g) Semisimplicial complex X with degenerate simplexes $< V_0, V_0 >$, $< V_1, V_1 >$, $< V_0, V_0, V_1 >$ and $< V_0, V_1, V_1 >$. (h) Semisimplicial complex Y with degenerate simplexes: $< V_2, V_2 >$, $< V_3, V_3 >$ and $< V_2, V_2, V_3 >$. (i) Semisimplicial adjunction complex $Y \cup_f X$ where $f : A \to Y$, $A = (< V_0 >, < V_1 >, < V_0, V_1 >) \subset X$, $f(< V_0 >) =< V_2 >$, $f(< V_1 >) =< V_3 >$, $f(< V_0, V_1 >) =< V_2, V_3 >$, is a semisimplicial map. By analogy to (f), $s_0 < V_0 >$ is identified with $s_0 < V_2 >$ and $s_0 < V_0, V_1 >$ with $s_0 < V_2, V_3 >$. While simplexes $s_0 < V_1 >$ and $s_1 < V_0, V_1 >$ from X are adjoined to the result complex.

1. Each semisimplicial complex has a subdivision into a simplicial complex i.e. polyhedron (or triangulable space).
2. Semisimplicial constructions followed by realization lead to triangulable space.
3. Unlike the case of simplicial complex where each simplex injects into the realization, semisimplicial constructions may identify points of $\triangle^n$ for some cells. That is, some cells of the geometric realization of the semisimplicial complex need not to be homeomorphic to E^n.
4. Thus the realization gives a universal object through which simplicial or semisimplicial complexes can be factored.

In this way, CW complexes could support uniform and factorized numerical representations of solids underlying inhomogeneous structured pointsets.

6 CW complexes and object representations in solid modeling

Topology has ever been a leading part in the foundations of solid modeling for more than thirty years ([61], [56],[53], [29].) See for example the applications of the pointset topology and the leading concepts of regularized r-sets and regularized boolean operations for the Constructive Solid Geometry. Boundary representations and surface based models have been developed on the basis of the algebraic topology of surfaces ([41, 71]). Combinatorial topology and cellular complexes emerged as a powerful background for numerous solid representations in a wide range of applications. See the examples in 2D and 3D mesh generation and refinement [13, 6] , boolean operations [57, 66, 67, 51, 68, 54], feature based modeling [64, 48, 60, 25, 4], multiresolution and object modeling [16, 37, 44, 49].

6.1 Related work

Pioneer researches in application of CW complexes in geometric modeling of area guide map are due to [69, 38]. An area guide map is considered as assembly of surface patches, each one represented in terms of its singular points (peaks, passes and pits). Using Morse function and Reeb graph, each patch is represented as a CW complex. Further, surface patches are blended along common boundaries using homotopy between corresponding cellular characteristic maps. Finally, the composite surface, encoding the area guide map, is represented as a differentiable manifold. Results of area guide map automatic generation are reported. Despite of the Morse theory restrictions in surface knots and links representation, as it is pointed out by [69], the elaborated approach shows that CW complexes provide a basic format to code the shape of a solid or a surface in the space.

A significant contribution to cellular complex, and in general, to topological subdivision study and applications in geometric modeling is made by [40, 41, 42, 43]. A wide set of combinatorial cellular structures has been proposed [43] to handle "irregular assemblies of regular cells" (semisimplicial sets [39], simploidal sets [20, 21]) and "irregular assemblies of irregular cells" (chains [12], generalized maps [42]). A classical but still tedious problem of topology-based representations is the embedding definition i.e. how to ensure the consistency between the combinatorial structure ant its embedment. With this respect, the reported results (linear embedment and Bezier curves and surfaces) still present challenges. Besides, a purely combinatorial approach is experimented to report [43] performance evaluation i.e. independent of actual implementation.

Simplicial decompositions of dimension independent polyhedra are studied by [58, 59]. As stated by the author, simplicial decompositions can be considered as a preliminary step toward the introduction of more general complexes where cells are convex or affine sets. Operations for both linear polyhedra and simplicial approximations of curved polyhedra generation and combination are proposed, such as the boolean operations [57], the boundary evaluation and the extrusion [15]. In fact, the elaborated Winged Representation covers a subclass of CW complexes, the regular CW complexes, where a $(n-1)$-face is adjacent to at most two n-simplexes. Authors acknowledge that a critical point of the proposed approach is the high representation fragmentation deriving from extrusion and set operations, and thus necessitating a reduction step.

A method of solid representation as simplicial chains is proposed by [14]. Simplexes are obtained directly from the solid boundary and the simplicial covering may contain overlapping simplexes. Solids are given in function of simplicial chains as canonical expressions on the basis of a "graphical object algebra". This formalism is applied for the computation of boolean operations as operations on simplexes. Unfortunately, there is no results on model fragmentation evaluation. Domain enlargement to free-form solids is acknowledged as a future development.

A framework, formalism and practical software kernel, based on chain models, is put forward for consideration by [10]. The concept of "generalized cell", called n-grouping is introduced. A n-grouping is a finite collection of $(n-1)$-groupings. There is not necessarily any topological relationship amongst the $(n-1)$-groupings forming a n-grouping and in general it may be of arbitrary topological form. The 0-groupings are taken as primitives. Further, solids are represented as formal sum of i-groupings, $0 \leq i \leq n$. No general assumption is made that cells have geometric realization. The motivation is that chains and complexes can be used in applications where no geometry is involved as in the given example of mass-spring network and tissue modeling. However, under certain constraints, system geometry could also be defined in terms of chains. This research is a good example of a unification treatment of geometric and non-geometric aspects in object modeling but the integrity structure verification in the case of model geometric embedment remains open.

A great number of researches brings into special prominence geometric complexes: "Selective Geometric Complexes" [66], "Complete Geometric Complexes"

[6], "Djinn Complexes" [54] and "Multi-Complexes" [16]. Relevant contribution in formalism definition of spatial decomposition is achieved. The fundamental topic of the geometric realization validation of cellular complexes remains crucial. Moreover, while comparative analysis in storage requirements or objects characteristic computations are reported [58, 43] , little is done for the evaluation of the cellular structure complexity with respect to this fundamental question.

Along this work we use the CW complexes as a formal model for geometric modeling that allows us to elaborate a unified representational frame, to encode object degeneracies with no complementary costs, and to implement a software kernel in order to evaluate cell structure performances.

6.2 Our approach

Our approach, in opposition with the above cited works, is in some way minimalist: to revisit the CW complexes and the corresponding semisimplicial constructions as defined by [11, 45] and to investigate to which extent this "minimalist" set of representation and manipulation tools is a complete and minimal set i.e. forms a basis in the solid modeling space of interest (manifolds, pseudomanifolds, quasimanifolds, non-manifolds).

6.3 Realization and Examples

Solid model in terms of CW complexes is a cellular structure of iD-cells, $0 \leq i \leq 3$. Model construction is a two step procedure.
First, a list of principal cells belonging to the object is built, each iD principal cell being constructed in a recursive way in increasing order of dimension of the proper faces. A graphical illustration for a $2D$ simplex stepwise construction is given on Fig.18. The geometric embedment is defined for each proper face, and face and degeneracy operators are initialized while semisimplicial identities are ensured.

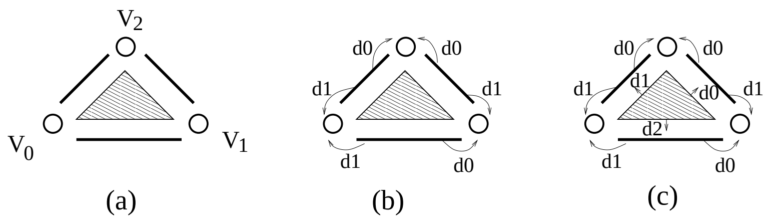


Figure 18: 2D nondegenerate simplex construction.

Second, a merging operation is fulfilled in order to regroup cells with identical geometric embedment through adjunction and quotient semisimplicial constructions.
In fact, the first phase corresponds to a semisimplicial complex construction and

Solids	0D cells	1D cells	2D cells	Number of tests	CPU time
cube	$36 \to 8$	$36 \to 24$	$12 \to 12$	1250	0.0000001
sphere	$393 \to 63$	$393 \to 247$	$131 \to 131$	60709	2 sec
torus	$702 \to 96$	$702 \to 403$	$234 \to 234$	24677	4 sec
star	$1296 \to 224$	$1296 \to 1156$	$432 \to 432$	23301	21 sec
spiral	$1890 \to 288$	$1890 \to 1156$	$630 \to 630$	63649	35 sec

Table 1: CW complex model evaluation for solids. The CPU time is calculated for a Pentium III, 866 MHz, RAM 256 MB

the second one, to its geometric realization. It should be emphasized that a iD degenerate cell is realized to a lower dimensional nondegenerate cell but owed to the corresponding sequence of degeneracy operators, the initial cell structure is maintained. For example, the geometric realization on Fig.14(b) according to degeneracy operators could represent cell on either Fig.14(a) or Fig.14(g).

Some examples are given on Fig.20 and Fig.21. Next we will detail the model construction for the cube on Fig.20(a) and its evaluation is given in the first line in Table 1. The cube is constructed as a $2D$ quasi-manifold regular complex using the extrusion operation. Initially, the list of principal cells includes 12 $2D$-cells (each square is represented as 2 triangles). Each principal $2D$-cell has 6 proper faces, 3 $0D$-cells and 3 $1D$-cells. Working out the total for the whole complex, we have 36 $0D$-cells and 36 $1D$-cells. Further, the geometric realization is performed and the number of cells is reduced to 8 $0D$-cells and 24 $1D$-cells. The number of $2D$-cells remains unchanged as long as the final result is a $2D$ complex. The cost of the construction is given in number of geometric tests (point/point, point/line, etc.) with no accelerating technics. The CPU time for this construction for a Pentium III, 866MHz, RAM 256 MB, is quasi-instantaneous.

An illustration of quasi-manifold , pseudo-manifold and non-manifold $2D$ complexes is given on Fig.19

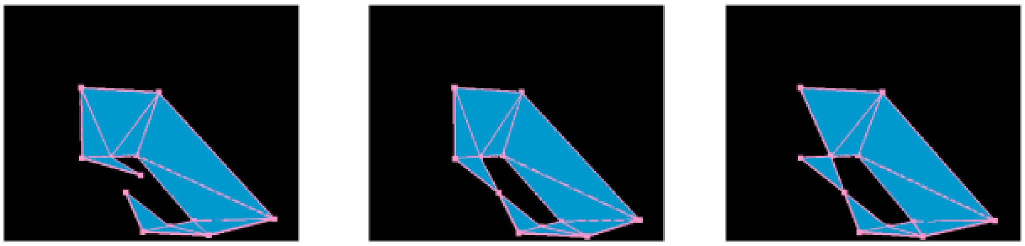


Figure 19: (Left) $2D$ quasi-manifold. (Center) $2D$ pseudo-manifold. (Right) $2D$ non-manifold.

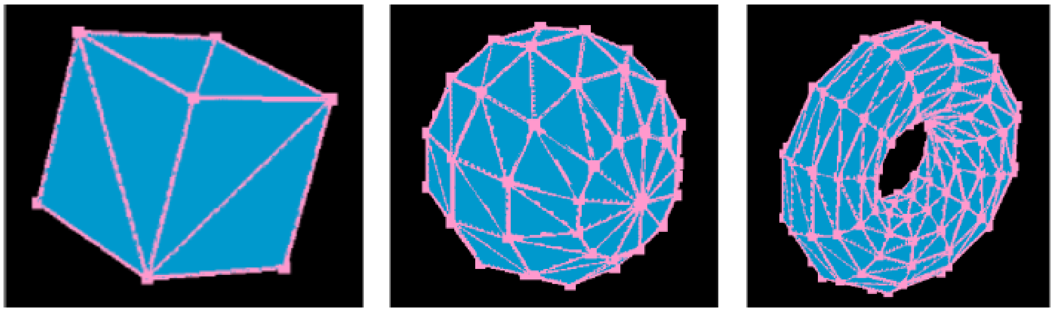


Figure 20: Extrusion solid primitives: cube, sphere, torus

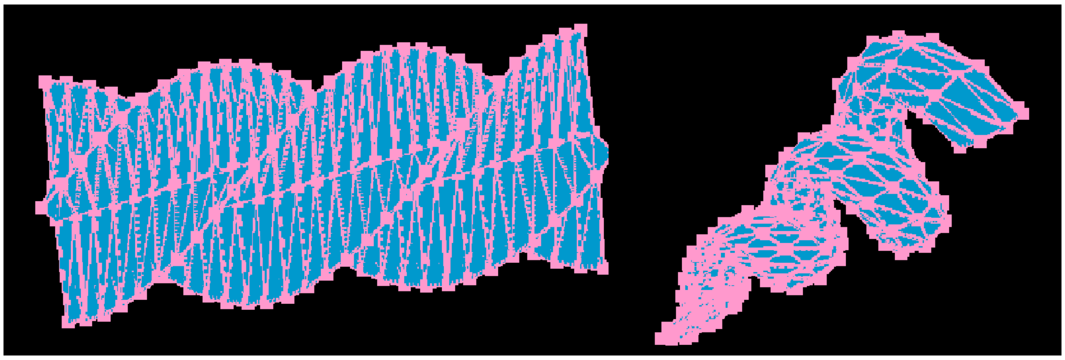


Figure 21: Extrusion solids: star and spiral

6.4 Synthesis results and future work

We carry out the construction of solids as CW complexes belonging to the mostly used classes of solids, and in general to the non-manifolds. The performance evaluation of the model in relation to different modeling operations is in progression. Illustrations are given for the extrusion with respect to model fragmentation reduction and the established results are promising. Besides, the well known tiresome problem of handling the model structure embedment in case of "authorized" degenerate cells, resulting in numerous applications as long as computational geometry calculations are performed, gives us the strong belive that this representational scheme and the underlying basic operations could become a "standard" encoding format. Actual studies go further on cellular subdivision algorithms development and implementation, and will be reported in the future.

7 CW complexes and object representations in computer imagery

In this section we will discuss the application of the semisimplicial complexes and their geometric realization, the CW complexes, as representational mainframe in computer imagery for object boundary reconstruction procedure.

7.1 Related work

In opposition to the classic algebraic topology mainly considering sets with infinitely many elements, computer imagery manipulates sets with finite numbers of points. A first attempt to formulate consistent topology notions equivalent to the continuum notions of neighborhoods is due to [62]. An image is regarded as a graph, called *neighborhood graph*, whose nodes are the pixels and whose branches are linking the adjacent pixels to each other. This notion has the advantage to be intuitive. However, the definition of connectivity and boundary by means of the neighborhood graph fails owed to the so-called *connectivity paradox* and some other contradictions (the Jordan curve paradox, the infinitesimal thin object, the ambiguous boundary definition, see [1, 2]).

Substential efforts to overcome these difficulties and further discrete topology development have been made by [35, 32, 63, 27, 34, 36, 2, 18, 19, 28].

Appropriate definition and application of cellular complexes in computer imagery is presented by [35]. An abstract cellular complex is defined as $C = (E, B, dim)$, where E is a set of abstract elements provided with an assymmetric, irreflexive, and transitive binary relation $B \subset E \times E$, called *bounding relation* (or face relation), and with a dimension function $dim : E \to I$, from E into the set I of non-negative integers such that $dim(e') < dim(e")$ for all pairs $(e', e") \in B$. In $2D$ image processing, $2D$ elements are the surface elements associated with the notion of pixel. $0D$ and $1D$ elements are not represented in the hardware realizations of the image. It is proved that finite topological space in which at least one smallest neighborhood consists of more than one element is an abstract cellular complex. The simplest open subset of an element e in a complex, called *open star* of e, consists of e itself and all elements bounded by e. An example is given on Fig.22(b).

Using the definition of a discrete cellular complex , "spel-manifold" objects, and in general "kλ-manifolds" could be described [28]. Nevertheless, the underlying object definition is limited to the discrete representational domain.

Cellular complex applications in image segmentation are reported in [5, 8, 9]. Discrete objects are described in terms of nD-maps [42] and thus restrained to oriented nD quasi-manifolds. In relation with the cited researches, our approach supports a more general class of discrete objects, the non-manifolds. Thus in image segmentation for example, the region inclusion tree could be calculated allowing a $3D$ region reconstruction.

7.2 Our approach

Our aim is to establish a correspondance between discrete cellular complexes used for $3D$ discrete image representation and CW complexes, and to illustrate a general boundary reconstruction procedure for discrete objects represented in terms of CW complexes.

7.3 Realization and examples

A $3D$ discrete object is regarded as a grid of $3D$ discrete points (see Fig.22). This grid can be used to define the corresponding $3D$ CW complex as follows. A $3D$ discrete point corresponds to a $3D$-cell bounded by six $2D$-cells. Each $2D$-cell being limited by four $1D$-cells and finally, each $1D$-cell being bounded by two $0D$-cells. It should be pointed out that discrete points correspond to non-degenerate cells and the geometric embedment of each iD-cell, $0 \leq i \leq 3$, is explicitly encoded in the grid. For example, let us consider a $1D$-cell, denoted as e. The topological star of e is illustrated on Fig.22(b). The face operators of e encode the $0D$-cells belonging to the star of e and shown on Fig.22(c). The degeneracy operators of e point to the $2D$-cells highlighted on Fig.22(d).

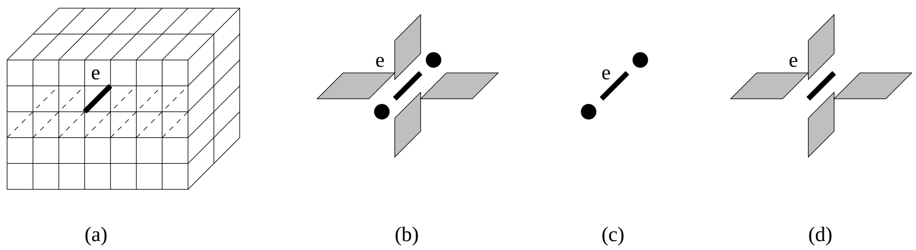


Figure 22: $3D$ discrete image. (a) Grid. (b) Topological star of the $1D$ cell e. (c) Face operators. (d) Degeneracy operators.

Objects	3D discrete points	0D cells	1D cells	2D cells	CPU time
sphere	485	138	414	207	0.0000001
torus	132	96	288	144	0.0000001
brain	25691	26470	79410	39705	1 sec

Table 2: CW complex model evaluation for discrete objects. The CPU time is calculated for a Pentium III, 500 MHz, RAM 128 MB

In fact, the grid could be regarded as the list of principal $3D$-cells belonging to the object. Further, iD-cells, $0 \leq i \leq 2$, with identical embedment are joined up on the basis of quotient and adjunction operations. The result is a $3D$ CW complex corresponding to the initial object.
This approach has been implemented as a software kernel to reconstruct object boundary in discrete images. We are interested in boundary reconstruction of objects spanning pointsets structured as maximum isovalued connected sets. For this purpose, discrete image is segmented into regions using a homogeneity criterion [9]. The regions of interest are produced as grid of discrete points. Then, the CW complex corresponding to this grid is constructed. In practice, the explicit set of pixels and voxels belonging to the regions allows to replace adjacency determination by detection of a cell presence in a given discrete point. Next, the inclusion region tree is computed. Finally a traversal procedure reconstructs region boundary.
For the discrete object given on Fig. 23(c), the brain is segmented into regions corresponding to the white matter, the gray matter, the corpus callosum and the hypothalamus. The model supports boundary reconstruction either of each region boundary separately or of the aggregate. The brain outer boundary is illustrated. Model evaluation in terms of total number of iD-cells is given on Table 2. The reconstruction procedure takes $1sec$ CPU time for a Pentium III, 500 MHz, RAM 128 KB.

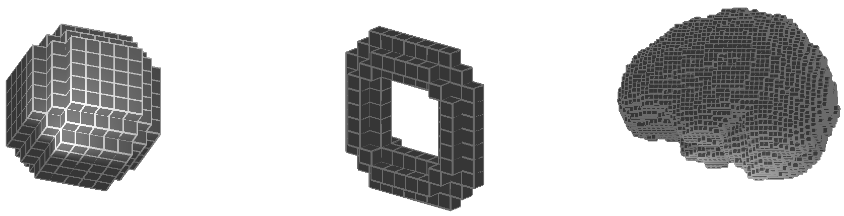


Figure 23: Examples of object boundary reconstruction for the sphere, the torus and the MRI brain image

The second column in the Table.2, contains the raw data i.e. the total number of points included in the discrete object. The next three columns describe the number of $0D$ cells (vertices), $1D$ cells (edges) and $2D$ cells (faces) as parts of the elaborated model. It should be pointed out that the growth of the total amount of data is compensated by the pointset structuration and in particular by the boundary object reconstruction.

7.4 Synthesis results and future work

The produced software kernel was originally developed to provide both geometrical and topological information on discrete images as an extension of a 2D kernel [5]. A drawback of the original method is the restriction to orientable manifold and quasi-manifold surfaces reconstruction. Using CW complexes as mathematical framework, the field of possible object reconstructions is extended including the non-manifolds.

8 Conclusion

Along this paper we have described a framework, based on CW complexes, for the representation and manipulation of solid and discrete objects. The framework is more general than previous approaches with respect to two main points: it provides a unified representation of solid and discrete objects and it enables explicit object degeneracy encoding. We have shown examples illustrating its utility in both geometric modeling and computer imagery.

Realizations of solid and discrete object definition in terms of CW complexes are implemented.

First, the realization for the topology-based solid modeling is illustrated on Table.1. Extruded solids are generated using the product semisimplicial operation and represent $2D$ regular CW complexes. The numbers of nD cells, $0 \leq n \leq 2$, are evaluated before and after the geometric realization. Model fragmentation is reduced on the basis of cell identification using the quotient and the adjunction operations. The number of the underlying geometric tests is a "rough" value i.e. no accelerating geometric intersection computing technics are used.

Second, the realization for the discrete object boundary reconstruction is provided on Table.2. Discrete objects are initially specified as unstructured sets of $3D$ discrete points. Through a boundary reconstruction procedure, these sets of points are structured as a $3D$ regular CW complexes. As one can see, the total amount of nD cells, $0 \leq n \leq 3$, increases but multiple adjacencies, inclusion tree of the regions, and other useful information are enabled.
According to our experience, CW complexes provide a unified representational issue while being structurally simple. Object degeneracies are naturally encoded.

Actual studies go further on cellular subdivision algorithm development and "split and merge" operations where the geometric insight of degeneracy cases, supported by the CW complexes, will facilitate the study of singular [17] cases and will improve their robustness and efficiency.

References

[1] E. Ahronovitz, J.P. Aubert, and C. Fiorio. Représentation topologique associée à l'analyse d'images. Technical Report 95040, Laboratoire d'Informatique, de Robotique et de Microlectronique de Montpellier, 1995.

[2] E. Ahronovitz, J.P. Aubert, and C. Fiorio. The star-topology: a topology for image analysis. In *5th Discrete Geometry for Computer Imagery, Proceedings*, pages 107–116, September 1995. Groupe GDR PRC/AMI du CNRS.

[3] Paul Alexandroff. *Elementary Concepts of Topology.* Dover,Inc., 1961.

[4] R. Bidarra, K.J. Kraker, and W.F Bronsvoort. Representation and management of feature information in a cellular model. *Computer-Aided Design*, 30(4):301–313, 1998.

[5] J.P. Braquelaire and L. Brun. Image segmentation with topological maps and inter-pixel representation. *Journal on Visual Communication and Image Representation*, 9(1):62–79, 1998.

[6] P.R. Cavalcanti, P.C.P Carvalho, and L.F. Martha. Non-manifold modelling: an approach based on spatial subdivision. *Computer-Aided Design*, 29(3):209–220, 1997.

[7] G.A. Crocker and W.F.Reinke. An editable non-manifold boundary representation. *IEEE ComputerGraphics&Applications*, pages 39–51, March 1991.

[8] G. Damiand. *Définition et étude d'un modèle topologique minimal de reprsentation d'images 2D et 3D.* PhD thesis, Université Montpellier II, December 2001.

[9] Pascal Desbarats. *Structuration d'images segmentées 3D discrètes.* PhD thesis, Université Bordeaux I, 2001.

[10] R. Egli and N.F. Stewart. A framework for system specification using chains on cell complexes. *Computer-Aided Design*, 32:447–459, 2000.

[11] Samuel Eilenberg and Norman Steenrod. *Foundations of Algebraic Topology.* Princeton University Press, 1952.

[12] E. Elter and P. Lienhardt. Cellular complexes as structured semi-simplicial sets. *International Journal of Computational Geometry &Applications*, 1(2):191–217, 1994.

[13] T. Fang and L.A. L.A.Piegl. Delaunay triangulation in three dimensions. *IEEE Computer Graphics and Applications*, pages 62–69, Septembre 1995.

[14] F.R. Feito and M. Rivero. Geometric modelling based on simplicial chains. *Computer & Graphics*, 22:611–619, 1998.

[15] V. Ferrucci and A. Paoluzzi. Extrusion and boundary evaluation for multidimensional polyhedra. *Computer- Aided- Design*, 23(1):40–50, January - February 1991.

[16] L. Floriani, P. Magillo, and E. Puppo. Multiresolution representation of shapes based on cell complexes. In *Discrete Geometry for Computer Imagery*, 1999.

[17] A.T. Fomenko and S.V.Mateev. *Algorithms and Computer Methods for Three-Manifolds*. Kluwer Academic Publishers, 1997.

[18] J. Françon. Discrete combinatorial surfaces. *CVGIP : Graphical Models and Image Processing*, 57(1):20–26, 1995.

[19] J. Françon. On recent trends in discrete geometry in computer science. In *6th Discrete Geometry for Computer Imagery, Proceedings*, pages 3–16. Springer-Verlag Berlin Heidelberg, 1996.

[20] L. Fuchs. Une spécification formelle des modèles de courbes et de surfaces de forme libre. Phd thesis, Université Louis Pasteur, 1997. 2791.

[21] L. Fuchs and P.Lienhardt. Topological structures for d-dimensional free-form objects. In *Computer Aided Geometric Design*, 1997.

[22] Claude Godbillon. *Éléments de topologie algébrique*. Dover,Inc., 1979.

[23] S. Gueorguieva and D. Marcheix. Non-manifold boundary representation for solid modeling. In *Proc. of the International Computer Symposium*, 1994.

[24] E.L. Gursoz, Y. Choi, and B. Prinz. Vertex-based representation of non-manifold boundaries. In *Geometric modeling for product engineering.*, 1990.

[25] J.H. Hang and A.G.G. Requicha. Feature recognition from cad models. *IEEE Computer Graphics and Applications*, pages 80–94, March/April 1998.

[26] Michael Henle. *A Combinatorial Introduction to Topoly*. Herman, Paris, 1979.

[27] G.T. Herman. Discrete multidimensional jordan surfaces. *CVGIP : Graphical Models and Image Processing*, 54(6):507–515, November 1992.

[28] T.H. Herman. An abstract theoretical foundation of the geometry of digital spaces. In *Discrete Geometry for Computer Imagery*, pages 279–288. Springer-Verlag Berlin Heidelberg, 2002. LNCS 2301.

[29] Ch.M Hoffmann. *Geometric and Solid Modeling: An Introduction*. Morgan Kaufmann, 1989.

[30] Sze-Tsen Hu. *Elements of General Topology*. Holden-Day, Inc., 1964.

[31] Sze-Tsen Hu. *Homology Theory: A First Course in Algebraic Topology*. Holden-Day, Inc., 1966.

[32] E. Khalimski, R. Koppermann, and P.R. Meyer. Computer graphics and connected topology on finite ordered sets. In *Topology and its applications*, pages 27–55, 1990.

[33] L.Ch. Kinsey. *Topology of surfaces*. Springer Verlag, 1993.

[34] T.Y. Kong, A.W. Roscoe, and A. Rosenfeld. Concept of digital geometry. In *Special Issue on digital topology, Topology Applications*, pages 219–262, 1992.

[35] V.A. Kovalevski. Finite topology as applied in image analysis. volume 46, pages 141–161, 1989.

[36] V.A. Kovalevsky. Digital geometry based on the topology of abstract cell complexes. In *Discrete Geometry for Computer Imagery*. Springer-Verlag Berlin Heidelberg, 1993. LNCS 2301.

[37] V. Kumar, D. Burns, D. Dutta, and C. Hoffmann. A framework for object modeling. *Computer-Aided Design*, 31:541–556, 1999.

[38] T.L. Kunii and S. Takahashi. Area guide map modeling by manifolds and cw-complexes. In *IFIP TC5/WG2 Working Conference on Geometric Modeling*, 1993.

[39] Veronique Lang. *Une étude de l'utilisation des ensembles semi-simpliciaux en modélisation géométrique interactive*. PhD thesis, Université Louis Pasteur de Strasbourg, 1995.

[40] P. Lienhardt. Subdivisions of surfaces and generalized maps. In *EUROGRAPHICS'89*, 1989.

[41] P. Lienhardt. Topological models for boundary representation: A comparison with n- dimensional generalized maps. *Computer-Aided Design*, pages 59–82, January/February 1991.

[42] P. Lienhardt. N-dimensional generalized combinatorial maps and cellular quasi-manifolds. *International Journal of Computational Geometry &Applications*, 4(3):275–324, 1994.

[43] P. Lienhardt. Aspect in topology-based geometric modeling: Possible tools for discrete geometry? In *Proc. 7th DGCI, Montepellier, France*, volume 1347 of *LNCS*, pages 33–48, 1997.

[44] A. Losa and B Cervelle. 3d topological modeling and visualisation for 3d gis. *Computer&Graphics*, 23:469–478, 1999.

[45] A.T. Lundell and S. Weingram. *The Topology of CW Complexes.* Van Nostrand Reinhold Company, 1969.

[46] D. Marcheix and S. Gueorguieva. Topological operators for non-manifold modeling. In *Proc. of the Third International Conference in Central Europe on Comput er Graphics and Visualisation 95*, 1995.

[47] David Marcheix. *Modélisation géométrique d'objets non-variétés: construction, représentation et manipulation.* PhD thesis, Université Bordeuax 1, 1996.

[48] H. Masuda. Form-feature representation based on non-manifold geometric modelling. In *MICAD'92*, 1992.

[49] H. Masuda and R. Ohbuchi. Coding topological structure of 3d cad models. *Computer-Aided Design*, 32:367–375, 2000.

[50] H. Masuda and K. Shimada. A mathematical theory and applications of non-manifold geometric modeling. In *IFIP/GI Advanced Geometric Modeling for Engineering Applications*, 1990.

[51] A. Maulik. An efficient intersection algorithm for polyhedral cellular decompositions. In *ACM Symposium on Solid Modelling and Foundations*, 1991.

[52] J. Peter May. *Simplicial Objects in Algebraic Topology.* Chicago University Press, 1967.

[53] M. Măntylă. *Geometric and Solid Modeling: An introduction.* Computer Science Press, 1988.

[54] A.E. Middleditch, C.M.P. Reade, and A.J. Gomes. Set-combinations of mixed-dimension cellular objects. *Computer-Aided Design*, pages 683–694, 1999.

[55] John Milnor. The geometric realization of semi-simplicial complex. *Annals of Mathematics*, 65(2):357–362, March 1957.

[56] M.E. Mortenson. *Geometric Modeling.* John Wiley & Sons, 1985.

[57] A. Paoluzzi, M. Ramella, and A. Santarelli. Boolean algebra over linear polyhedra. *Computer-Aided Design*, 21(8):474–484, October 1989.

[58] F. Paoluzzi, F. Bernardini, C. Cattani, and V. Ferrucci. Dimension-independent modeling with simplicial complexes. *ACM Transactions on Graphics*, 12(1):56–102, January 1993.

[59] V. Pascucci, V. Ferrucci, and A. Paoluzzi. Dimension- independant convex-cell based hpc: Skeletons and product. Technical report, University La Sapienza , Roma, Italy, 1995.

[60] D.B. Perng and C.F. Chang. Resolving feature interactions in 3d part editing. *Computer-Aided Geometric Design*, 29(10):687–699, 1997.

[61] A.G. Requicha. Representation for rigid solids: Theory, methods and systems. *ACM Computing Surveys*, pages 437–464, December 1980.

[62] A. Rosenfeld and A.C. Kak. *Digital Picture Processing*. Harbour and Brace, 1982.

[63] A. Rosenfeld, T.Y. Kong, and A.Y. Wu. Digital surfaces. *CVGIP : Graphical Models and Image Processing*, 53(6):305–312, July 1991.

[64] J.R. Rossignac. Issues on feature-based editing and interrogation of solid models. *Computer&Graphics*, 14(2):149–172, 1990.

[65] J.R. Rossignac and D. Cardoze. Matchmaker: Manifold breps for non-manifold r-sets. In *ACM Solid Modelling'99*, 1999.

[66] J.R. Rossignac and M.A. O'Connor. Sgc: A dimension- independent- model for pointsets with internal structure and incomplete boundaries. Research report rc14340, IBM T.J. Watson Research Center, 1989. NY 10598.

[67] J.R. Rossignac and A.A.G Requicha. Constructive non-regularized geometry. *Computer-Aided Design*, 23(1):21–32, January/February 1991.

[68] N. Sapidis and R. Perucchio. Solid/solid classification operations for recursive spatial decomposition and domain triangulation of solid models. *Computer-Aided Design*, (10):517–529, October 1992.

[69] Y. Shinagawa, T.L. Kunii, and Y.L. Kergosien. Surface coding based on morse theory. *IEEE Computer Graphics and Applications*, 11(5):66–78, 1991.

[70] I.M. Singer and J.A. Thorpe. *Lecture Notes on Elementary Topology and Geometry*. Springer-Verlag, 1967.

[71] T. Takala. A taxonomy on geometric and topological models. In *Computer Graphics and Mathematics*, 1992.

[72] K. Weiler. The radial edge structure: A topological representation for non-manifold geometric boundary modeling. In *Geometric Modeling for CAD Applications, IFIP'88*, 1988.

[73] Y. Yamaguchi and F Kimura. Non-manifold topology based on coupling entities. *IEEE Computer Graphics&Applications*, pages 42–50, January 1995.