
The [equivalence problem](#)
for Deterministic PushDown Automata

Géraud Sénizergues
FMI-Stuttgart

[URL:http://www.informatik.uni-stuttgart.de/fmi/ti/mitarbeiter/Senizergues](http://www.informatik.uni-stuttgart.de/fmi/ti/mitarbeiter/Senizergues)

0- MAIN RESULT

Decision problem:

INSTANCE: A, B , two deterministic pushdown automata;

QUESTION: $L(A) = L(B)$?

Raised in:

S. Ginsburg and S. Greibach, Deterministic Context-free languages, Information and Control, pages 620-648, 1966

0- MAIN RESULT

G. Sénizergues, $L(A) = L(B)?$, *Technical report*, LaBRI, Université Bordeaux I, report nr1161-97, 1997, Pages 1-71.

G. Sénizergues. The Equivalence Problem for Deterministic Pushdown Automata is Decidable. In *Proceedings ICALP 97*, pages 671–681. Springer, LNCS 1256, 1997.

G. Sénizergues, $L(A) = L(B)?$ In *Proceedings INFINITY 97*, pages 1–26. Electronic Notes in Theoretical Computer Science 9.

0- MAIN RESULT

REFERENCE:

G. Sénizergues. $L(A) = L(B)$? decidability results from complete formal systems. *Theoretical Computer Science*, 251:1–166, 2001.

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1.1- PREFIX MATRICES- OPERATIONS

Let us remark:

$$\langle \mathcal{P}(X^*), \cup, \cdot, \emptyset, \{\varepsilon\} \rangle \approx \langle \mathbb{B}\langle\langle X \rangle\rangle, +, \cdot, 0, 1 \rangle.$$

We rather use the second notation.

We call a series $S = \sum_{w \in X^*} S_w \cdot w$ **prefix-free** iff: for every $u, v \in X^*$,

$$(S_{uv} = 1 \text{ and } S_u = 1) \Rightarrow v = \varepsilon.$$

1.1- PREFIX MATRICES- OPERATIONS

Definition 1 A row-vector $(A_1, \dots, A_i, \dots, A_m) \in (\mathbb{B}\langle\langle X \rangle\rangle)^n$ is said **prefix-free** iff

- 1-the supports of the A_i 's are disjoint
- 2- $\sum_{i=1}^n A_i$ is prefix-free.

Definition 2 A matrix $A \in (\mathbb{B}\langle\langle X \rangle\rangle)^{n \times m}$ is said **prefix-free** iff every row-vector $(A_{i,1}, \dots, A_{i,j}, \dots, A_{i,m})$ is prefix-free.

Example 3 Some prefix matrices:

$$\begin{pmatrix} ab & 0 & a^*c \end{pmatrix}, \begin{pmatrix} ab & 0 & a^*c \\ 1 & 0 & 0 \\ abc & abd & a^*cd \end{pmatrix}, \begin{pmatrix} a \\ ab^*c \\ abc + ac \end{pmatrix}$$

1.1- PREFIX MATRICES- OPERATIONS

Product: defined as usual.

Let $A, B \in (\mathbb{B}\langle\langle X \rangle\rangle)^{1 \times m}$ and $1 \leq j_0 \leq m$.

If $A = (a_1, \dots, a_j, \dots, a_m)$, $B = (b_1, \dots, b_j, \dots, b_m)$ then

$$A \nabla_{j_0} B = (a_1 + a_{j_0} \cdot b_1, \dots, a_{j_0-1} + a_{j_0} \cdot b_{j_0-1}, 0, a_{j_0+1} + a_{j_0} \cdot b_{j_0+1}, \dots, a_m + a_{j_0} \cdot b_m)$$

$$\nabla_{j_0}^* (A) = (a_{j_0}^* \cdot a_1, \dots, a_{j_0}^* \cdot a_{j_0-1}, 0, a_{j_0}^* \cdot a_{j_0+1}, \dots, a_{j_0}^* \cdot a_m)$$

1.1- PREFIX MATRICES- OPERATIONS

We denote by $\mathbb{DB}_{n,m} \langle\langle X \rangle\rangle$ the set of prefix matrices (which will turn out to be a particular case of Deterministic matrices).

Lemma 4 *Let $A \in \mathbb{DB}_{n,m} \langle\langle X \rangle\rangle$, $B \in \mathbb{DB}_{m,s} \langle\langle X \rangle\rangle$. Then $A \cdot B \in \mathbb{DB}_{n,s} \langle\langle X \rangle\rangle$*

Lemma 5 *Let $A, B \in \mathbb{DB}_{1,m} \langle\langle X \rangle\rangle$ and $1 \leq j_0 \leq m$. Then*

- 1- $A \nabla_{j_0} B \in \mathbb{DB}_{1,m} \langle\langle X \rangle\rangle$
- 2- $\nabla_{j_0}^* (A) \in \mathbb{DB}_{1,m} \langle\langle X \rangle\rangle$

1.2- PREFIX MATRICES- RIGHT-ACTION

Definition 6 Let $S = \sum_{w \in X^*} S_w \cdot w$ and $u \in X^*$. Then

$$S \bullet u = \sum_{w \in X^*} S_{u \cdot w} \cdot w,$$

(i.e. $S \bullet u$ is the *left-quotient* of S by u , or the *residual* of S by u).

This right-action is extended *componentwise* to matrices.

Lemma 7 Let $A \in \mathbb{D}\mathbb{B}_{n,m} \langle\langle X \rangle\rangle$ and $u \in X^*$. Then

$$A \bullet u \in \mathbb{D}\mathbb{B}_{n,m} \langle\langle X \rangle\rangle$$

1.2- PREFIX MATRICES- RIGHT-ACTION

Lemma 8 *Let $S \in \text{DB}_{1,m} \langle\langle W \rangle\rangle, T \in \mathbb{B}_{m,s} \langle\langle W \rangle\rangle$ and $u \in W^*$. Exactly one of the following cases is true:*

(1) $\exists j, S_j \bullet u \notin \{\emptyset, \epsilon\}$

in this case $(S \cdot T) \bullet u = (S \bullet u) \cdot T.$

(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', S_{j_0} \bullet u' = \epsilon;$

in this case $(S \cdot T) \bullet u = T_{j_0,*} \bullet u''.$

(3) $\forall j, \forall u' \preceq u, S_j \bullet u = \emptyset, S_j \bullet u' \neq \epsilon;$

in this case $(S \cdot T) \bullet u = \emptyset = (S \bullet u) \cdot T.$

1.3- PREFIX MATRICES- LINEAR IDENTITIES

Fundamental remark((Meitus 1989)):

Let $S_1, \dots, S_j, \dots, S_m \in \text{DB}\langle\langle X \rangle\rangle$ and let $\vec{\alpha}, \vec{\beta} \in \text{DB}_{1,m}\langle\langle X \rangle\rangle$ such that

$$(0) \sum_{j=1}^m \alpha_j \cdot S_j = \sum_{j=1}^m \beta_j \cdot S_j$$

Then one of cases (1),(2) must occur:

$$(1) \vec{\alpha} = \vec{\beta}$$

(2) $\exists j_0 \in [1, m], \exists \vec{\gamma} \in \text{DB}_{1,m}\langle\langle X \rangle\rangle, \gamma_{j_0} = 0$, such that

$$S_{j_0} = \sum_{1 \leq j \leq m} \gamma_j \cdot S_j.$$

1.3- PREFIX MATRICES- LINEAR IDENTITIES

Sketch of proof: Let $S = (S_j)_{1 \leq j \leq m} \in \mathbf{DB}_{m,1} \langle \langle X \rangle \rangle$.

Suppose that $\vec{\alpha} \neq \vec{\beta}$. Let us consider

$$u = \min\{\vec{\alpha}\Delta\vec{\beta}\}.$$

Then there exists, $j_0 \in [1, m]$, such that

$$\vec{\alpha} \bullet u = \epsilon_{j_0}^m \Leftrightarrow \beta_{j_0} \bullet u \neq \epsilon_{j_0}^m.$$

Let us suppose, for example, that $\vec{\alpha} \bullet u = \epsilon_{j_0}^m$ while $\vec{\beta} \bullet u \neq \epsilon_{j_0}^m$ and let $\vec{\gamma} = \vec{\beta} \bullet u$. Let u act (by \bullet) on both sides of hypothesis (0):

$$(\vec{\alpha} \cdot S) \bullet u = (\vec{\beta} \cdot S) \bullet u. \quad (1)$$

Using lemma 8 we obtain:

$$(\vec{\alpha} \cdot S) \bullet u = S_{j_0}. \quad (2)$$

Let us examine now the righthand-side of equality (1). Let $u' \prec u$. By minimality of u , $\vec{\beta} \bullet u'$ is a unit iff $\vec{\alpha} \bullet u'$ is a unit. But if $\vec{\alpha} \bullet u'$ is a unit, then $\vec{\alpha} \bullet u = \emptyset^m$, which is false. Hence $\vec{\beta} \bullet u'$ is not a unit. By lemma 8

$$(\vec{\beta} \cdot S) \bullet u = (\vec{\beta} \bullet u) \cdot S. \quad (3)$$

Let us plug equalities (2) and (3) in equivalence (1) and let us define $\vec{\gamma} = \vec{\beta} \bullet u$. We obtain:

$$S_{j_0} = \vec{\gamma} \bullet S, \text{ where } \vec{\gamma} \neq \epsilon_{j_0}^m.$$

Using Arden's lemma we get:

$$S_{j_0} = \sum_{j \neq j_0} \gamma_{j_0}^* \gamma_j \cdot S_j \text{ i.e. } S_{j_0} = \vec{\gamma}' \bullet S$$

where $\vec{\gamma}' = \nabla_{j_0}^* (\vec{\gamma})$, hence $\gamma'_{j_0} = 0$. \diamond

1.4.1- PREFIX MATRICES- BASES

A subset $V \subseteq \text{DB}\langle\langle X \rangle\rangle$ is said to be a **prefix space** iff it is closed under linear combinations (with **prefix** vector of coefficients).

A set $\mathcal{F} = \{S_1, \dots, S_m\}$ (where $m = \text{Card}\mathcal{F}$) is called :

- a **generating** subset of V iff

$$V = \left\{ \sum_{j=1}^m \alpha_j \cdot S_j \mid \vec{\alpha} \in \text{DB}_{1,m}\langle\langle X \rangle\rangle \right\}.$$

- a **linearly independent** set if, for every $\vec{\alpha}, \vec{\beta} \in \text{DB}_{1,m}\langle\langle X \rangle\rangle$,

$$\vec{\alpha} \cdot \vec{S} = \vec{\beta} \cdot \vec{S} \Leftrightarrow \vec{\alpha} = \vec{\beta}$$

- a **base** of V iff, it is both a generating set of V and it is linearly independent.

1.4.1 - PREFIX MATRICES- BASES

Proposition 9 *Every finitely generated prefix space has a base. Moreover this base is unique.*

Question: For what kind of series (i.e. languages) is the base of $V(\mathcal{F})$ **computable** from the set \mathcal{F} ?

1.4.2- PREFIX MATRICES- MOST GENERAL UNIFIER

Let $\vec{\alpha}, \vec{\beta} \in \text{DB}_{1,m} \langle\langle X \rangle\rangle$. We call **unifier** for the pair $(\vec{\alpha}, \vec{\beta})$ any matrix $U \in \text{DB}_{m,n} \langle\langle X \rangle\rangle$ ($n \geq 1$) such that:

$$\vec{\alpha} \cdot U = \vec{\beta} \cdot U.$$

It is said to be a **Most General Unifier** iff, every unifier has the form $U \cdot T$ for some matrix $T \in \text{DB}_{n,s}$ (where $s \geq 1$).

1.4.2- PREFIX MATRICES- MOST GENERAL UNIFIER

Proposition 10 *Every pair of prefix row-vectors $(\vec{\alpha}, \vec{\beta})$ has a **MGU**. This MGU is unique (except in the case where $0_{m,1}$ is a MGU).*

Question: For what kind of series (i.e. languages) is the MGU **computable** from the pair of prefix row-vectors ?

1.5- PREFIX MATRICES- EXAMPLE

A linear equation:

$$ab \cdot U_1 + 0 \cdot U_2 + aab \cdot U_3, 0 \cdot U_1 + abc \cdot U_2 + (aab + abd) \cdot U_3.$$

Computation of a MGU:

applying $\bullet ab$ on both sides:

$$U_1, c \cdot U_2 + d \cdot U_3.$$

substituting U_1 by the r.h.s:

$$abc \cdot U_2 + (aab + abd) \cdot U_3, abc \cdot U_2 + (abd + aab) \cdot U_3,$$

hence:

$$MGU((ab, 0, aab), (0, abc, (aab + abd))) = \begin{pmatrix} c & d \\ \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$$

1.5- PREFIX MATRICES- EXAMPLE

A prefix column vector:

$$S = \begin{pmatrix} (ca^*b + d)a^*c \\ a^*ba^*c \\ a^*c \end{pmatrix}$$

This vector is a solution of:

$$ab \cdot S_1 + 0 \cdot S_2 + aab \cdot S_3, 0 \cdot S_1 + abc \cdot S_2 + (aab + abd) \cdot S_3.$$

Hence it is obtained by substitution from the MGU:

$$S_1 = cS_2 + dS_3$$

$$V(\{S_1, S_2, S_3\}) = V(\{S_2, S_3\})$$

A **base** of $V(\{S_1, S_2, S_3\})$ is $\{S_3\}$.

2.0- DETERMINISTIC MATRICES- DETERMINISTIC GRAMMARS

(Harrison-Havel, JCSS 1973):

Definition 11 Let $G = \langle X, V, P \rangle$ be some context-free grammar.

G is said *strict deterministic* iff there exists some equivalence relation \sim over $X \cup V$ such that

1- X is some class mod \sim

2- $\forall v_1, v'_1 \in V, \forall \alpha \in (X \cup V)^*, \forall w, w' \in V^*,$

if

$$v_1 \sim v'_1, v_1 \rightarrow \alpha w_1, v'_1 \rightarrow \alpha w'_1$$

then $(w_1 = w'_1 = \epsilon \text{ and } v_1 = v'_1) \text{ or}$

$\exists v_2, v'_2 \in (X \cup V), w_2, w'_2 \in (X \cup V)^*,$

$$v_2 \sim v'_2, w_1 = v_2 w_2, w'_1 = v'_2 w'_2.$$

2.0- DETERMINISTIC MATRICES- DETERMINISTIC GRAMMARS

Theorem 12 (Harrison-Havel,73) *A language $L \subseteq X^*$ is recognized (by empty stack) by some dpda iff it is generated by some strict-deterministic context-free grammar.*

2.1- DETERMINISTIC MATRICES- OPERATIONS

Definition 13 A series $S \in \mathbb{B}\langle\langle V^* \rangle\rangle$ is said *deterministic*

iff, for every words $\alpha, w_1, w'_1 \in V^*$ such that

$\alpha \cdot w_1, \alpha \cdot w'_1 \in \text{supp}(S)$

either $w_1 = w'_1 = \epsilon$

or $(\exists v_2, v'_2 \in V, w_2, w'_2 \in V^*, \text{ such that}$

$$v_2 \smile v'_2, w_1 = v_2 w_2, w'_1 = v'_2 w'_2.$$

Intuition: a *deterministic* series is a generalized configuration of the initial pushdown automaton.

Definition 14 A row-vector $(S_1, \dots, S_i, \dots, S_m)$ is said *deterministic* iff

1-The supports of the S_i 's are disjoint

2- $\sum_{j=1}^m S_j$ is deterministic.

Definition 15 *A matrix of dimension (n, m) , $(S_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$ is said **deterministic** iff every one of its row-vectors is deterministic.*

We denote by $\mathbb{DB}_{n,m} \langle \langle V \rangle \rangle$ the set of **deterministic** matrices.

2.1- DETERMINISTIC MATRICES- OPERATIONS

Lemma 16 Let $A \in \mathbb{DB}_{n,m} \langle \langle V \rangle \rangle$, $B \in \mathbb{DB}_{m,s} \langle \langle V \rangle \rangle$. Then $A \cdot B \in \mathbb{DB}_{n,s} \langle \langle V \rangle \rangle$

Lemma 17 Let $A, B \in \mathbb{DB}_{1,m} \langle \langle V \rangle \rangle$ and $1 \leq j_0 \leq m$. Then

- 1- $A \nabla_{j_0} B \in \mathbb{DB}_{1,m} \langle \langle V \rangle \rangle$
- 2- $\nabla_{j_0}^* (A) \in \mathbb{DB}_{1,m} \langle \langle V \rangle \rangle$

2.2- DETERMINISTIC MATRICES- RIGHT-ACTION

We define a σ -**right-action** \odot of the monoid X^* over the semi-ring $\mathbb{B}\langle\langle V^* \rangle\rangle$ by:

for every $w \in V^*, u \in X^*$,

$$w \odot u = \sum_{\substack{w' \in V^* \\ w \xrightarrow{u}^* w'}} w'$$

for every $S \in \mathbb{B}\langle\langle V^* \rangle\rangle, u \in X^*$,

$$\left(\sum_{w \in V^*} S_w \cdot w \right) \odot u = \sum_{w \in V^*} S_w \cdot (w \odot u).$$

This right-action is extended **componentwise** to matrices.

2.2- DETERMINISTIC MATRICES- RIGHT-ACTION

Lemma 18 *Let $A \in \mathbf{DB}_{n,m} \langle\langle V \rangle\rangle$ and $u \in X^*$. Then*

$$A \odot u \in \mathbf{DB}_{n,m} \langle\langle V \rangle\rangle$$

Lemma 19 *Let $S \in \mathbf{DB}_{1,m} \langle\langle W \rangle\rangle, T \in \mathbf{B}_{m,s} \langle\langle W \rangle\rangle$ and $u \in W^*$.*

Exactly one of the following cases is true:

(1) $\exists j, S_j \odot u \notin \{\emptyset, \epsilon\}$

in this case $(S \cdot T) \odot u = (S \odot u) \cdot T.$

(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', S_{j_0} \odot u' = \epsilon;$

in this case $(S \cdot T) \odot u = T_{j_0,*} \odot u''.$

(3) $\forall j, \forall u' \preceq u, S_j \odot u = \emptyset, S_j \odot u' \neq \epsilon;$

in this case $(S \cdot T) \odot u = \emptyset = (S \odot u) \cdot T.$

2.2- DETERMINISTIC MATRICES- RIGHT-ACTION

A map $\varphi : V \rightarrow \mathbb{B}\langle\langle X^* \rangle\rangle$ is defined by:

$$\varphi(v) = \sum_{v \odot u = \epsilon} u$$

This map is extended as a σ -additive **semi-ring homomorphism** $\varphi : \mathbb{B}\langle\langle V \rangle\rangle \rightarrow \mathbb{B}\langle\langle X \rangle\rangle$ by

$$\varphi\left(\sum_{w \in V^*} S_w \cdot w\right) = \sum_{w \in V^*} S_w \cdot \varphi(w).$$

We set

$$S \equiv T \text{ iff } \varphi(S) = \varphi(T).$$

2.2- DETERMINISTIC MATRICES- RIGHT-ACTION

Lemma 20 *For every*

$$S \in \mathbb{DB}_{n,m} \langle \langle V \rangle \rangle, T \in \mathbb{DB}_{m,s} \langle \langle V \rangle \rangle, u \in X^*,$$

$$1- \varphi(S \cdot T) = \varphi(S) \cdot \varphi(T)$$

$$2- \varphi(S \odot u) = \varphi(S) \bullet u$$

2.3- DETERMINISTIC MATRICES- LINEAR IDENTITIES

Fundamental remark:

Let $S_1, \dots, S_j, \dots, S_m \in \text{DB}\langle\langle X \rangle\rangle$ and let $\vec{\alpha}, \vec{\beta} \in \text{DB}_{1,m}\langle\langle X \rangle\rangle$ such that

$$(0) \sum_{j=1}^m \alpha_j \cdot S_j \equiv \sum_{j=1}^m \beta_j \cdot S_j S$$

Then one of cases (1),(2) must occur:

$$(1) \vec{\alpha} \equiv \vec{\beta}$$

(2) $\exists j_0 \in [1, m], \exists \vec{\gamma} \in \text{DB}_{1,m}\langle\langle X \rangle\rangle, \gamma_{j_0} = 0$, such that

$$S_{j_0} \equiv \sum_{1 \leq j \leq m} \gamma_j \cdot S_j.$$

2.4.1 - DETERMINISTIC MATRICES- BASES

A subset $V \subseteq \mathbb{DB}\langle\langle V \rangle\rangle$ is said to be a **deterministic space** iff it is closed under linear combinations (with **deterministic** vector of coefficients).

A set $\mathcal{F} = \{S_1, \dots, S_m\}$ (where $m = \text{Card}\mathcal{F}$) is called :

- a **generating** subset of V iff

$$V = \left\{ \sum_{j=1}^m \alpha_j \cdot S_j \mid \vec{\alpha} \in \mathbb{DB}_{1,m}\langle\langle V \rangle\rangle \right\}.$$

- a **linearly independent** set if, for every $\vec{\alpha}, \vec{\beta} \in \mathbb{DB}_{1,m}\langle\langle V \rangle\rangle$,

$$\vec{\alpha} \cdot \vec{S} = \vec{\beta} \cdot \vec{S} \Leftrightarrow \vec{\alpha} \equiv \vec{\beta}$$

- a **base** of V iff, it is both a generating set of V and it is linearly independent.

2.4.1- DETERMINISTIC MATRICES- BASES

Proposition 21 *Every finitely generated deterministic space has a base. Moreover this base is unique (up to the equivalence \equiv).*

It is unknown whether linear independence is a **decidable** property. It is unknown whether the base of $V(\mathcal{F})$ is **computable** from the set \mathcal{F} .

2.4.2- DETERMINISTIC MATRICES- MOST GENERAL UNIFIER

Let $\vec{\alpha}, \vec{\beta} \in \mathbf{DB}_{1,m} \langle \langle V \rangle \rangle$. We call **unifier** for the pair $(\vec{\alpha}, \vec{\beta})$ any matrix $U \in \mathbf{DB}_{m,n} \langle \langle V \rangle \rangle$ ($n \geq 1$) such that:

$$\vec{\alpha} \cdot U \equiv \vec{\beta} \cdot U.$$

It is said to be a **Most General Unifier** iff, every unifier is equivalent (mod \equiv) to $U \cdot T$ for some matrix $T \in \mathbf{DB}_{n,s}$ (where $s \geq 1$).

2.4.2- DETERMINISTIC MATRICES- MOST GENERAL UNIFIER

Proposition 22 *Every pair of prefix row-vectors $(\vec{\alpha}, \vec{\beta})$ has a **MGU**. This MGU is unique*

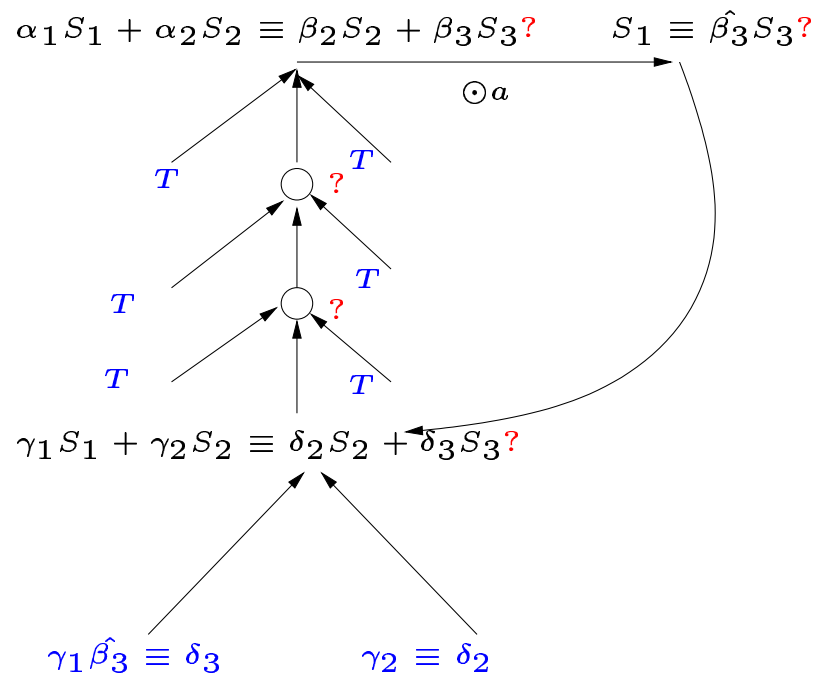
(up to equivalence (mod \equiv), and with the exception of the case where $0_{m,1}$ is a MGU).

- The MGU is **computable** from the pair of prefix row-vectors .
- This will be a *corollary* of the main result ... hence **not directly usable** in its proof ! We shall merely *adapt the underlying idea*.

3.1- DEDUCTION SYSTEMS- MOTIVATION

Let P be a set of equations such that, every equation is **implied** by the others. When can we conclude that they are all **true** ?

3.1- DEDUCTION SYSTEMS- MOTIVATION



3.2- DEDUCTION SYSTEMS- GENERAL DEFINITIONS

General philosophy of (Harrison-Havel-Yehudai79), (Courcelle 83).

Let us call *deduction system* any triple $\mathcal{D} = \langle \mathcal{A}, H, \vdash \rangle$ where

- \mathcal{A} is a denumerable set called the *set of assertions*
- $H : \mathcal{A} \rightarrow \mathbb{N} \cup \{\infty\}$ is the *cost-function*
- \vdash , the *deduction relation* is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$ such that the following axioms are satisfied:

(A 1) $\forall (P, A) \in \vdash, (\inf \{H(p), p \in P\} < H(A))$ or $(H(A) = \infty)$.

(A 2) \vdash is recursively enumerable.

We call *proof* in the system \mathcal{D} , any subset $P \subseteq \mathcal{A}$ fulfilling :

$$\forall p \in P, \exists Q \subseteq P, Q \vdash p.$$

3.2- DEDUCTION SYSTEMS- GENERAL DEFINITIONS

If P is a **proof**, then $\forall A \in P, H(A) = \infty$.

3.3- DEDUCTION SYSTEMS- STRATEGIES

Informal definition:

A strategy is a “method” allowing to enlarge every “partial proof” P into a partial proof Q with $P \subset Q$.

Notice: this method needs **not** be **computable**.

4.1 - THE SYSTEM \mathcal{D}_0 - ASSERTIONS AND COST FUNCTION

Set of assertions:

$$\mathcal{A} = \mathbb{N} \times \text{DRB}\langle\langle V \rangle\rangle \times \text{DRB}\langle\langle V \rangle\rangle$$

The “cost-function”:

$$H(n, S, S') = n + 2 \cdot \text{Div}(S, S').$$

Where the *divergence* between S and S' , is defined by:

$$\text{Div}(S, S') = \inf\{\|u\| \mid u \in \varphi(S) \Delta \varphi(S')\}.$$

4.2-THE SYSTEM \mathcal{D}_0 - THE RULES

(R0)

$$\{(p, S, T)\} \Vdash (p + 1, S, T)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle$

(R1)

$$\{(p, S, T)\} \Vdash (p, T, S)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle$.

(R2)

$$\{(p, S, S'), (p, S', S'')\} \Vdash (p, S, S'')$$

for $p \in \mathbb{N}, S, S', S'' \in \text{DRB}\langle\langle V \rangle\rangle$.

(R3)

$$\emptyset \Vdash (0, S, S)$$

for $S \in \text{DRB}\langle\langle V \rangle\rangle$.

4.2-THE SYSTEM \mathcal{D}_0 - THE RULES

(R0)

$$\{(p, S, T)\} \Vdash (p + 1, S, T)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle$.

(R1)

$$\{(p, S, T)\} \Vdash (p, T, S)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle$.

(R2)

$$\{(p, S, S'), (p, S', S'')\} \Vdash (p, S, S'')$$

for $p \in \mathbb{N}, S, S', S'' \in \text{DRB}\langle\langle V \rangle\rangle$.

(R3)

$$\emptyset \Vdash (0, S, S)$$

for $S \in \text{DRB}\langle\langle V \rangle\rangle$.

4.2-THE SYSTEM \mathcal{D}_0 - THE RULES

(R'3)

$$\emptyset \Vdash (0, S, T)$$

for $S \in \text{DRB}\langle\langle V \rangle\rangle, T \in \{\emptyset, \epsilon\}, S \equiv T,$

(R4)

$$\{(p + 1, S \odot x, T \odot x) \mid x \in X\} \Vdash (p, S, T)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle, (S \neq \epsilon \wedge T \neq \epsilon) .$

(R5)

$$\{(p, S, S')\} \Vdash (p + 2, S \odot x, S' \odot x)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle, x \in X,$

(R6)

$$\{(p, S \cdot T' + S', T')\} \Vdash (p, S^* \cdot S', T')$$

for $p \in \mathbb{N}, (S, S') \in \text{DRB}_{1,2}\langle\langle V \rangle\rangle, T' \in \text{DRB}\langle\langle V \rangle\rangle, S \neq \epsilon,$

4.2-THE SYSTEM \mathcal{D}_0 - THE RULES

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for $S \in \text{DRB}\langle\langle V \rangle\rangle, T \in \{\emptyset, \epsilon\}, S \equiv T,$

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4.2-THE SYSTEM \mathcal{D}_0 - THE RULES

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for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle, (S \neq \epsilon \wedge T \neq \epsilon),$

(R5)

$$\{(p, S, S')\} \Vdash (p + 2, S \odot x, S' \odot x)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle, x \in X,$

(R6)

$$\{(p, S \cdot T' + S', T')\} \Vdash (p, S^* \cdot S', T')$$

for $p \in \mathbb{N}, (S, S') \in \text{DRB}_{1,2}\langle\langle V \rangle\rangle, T' \in \text{DRB}\langle\langle V \rangle\rangle, S \neq \epsilon,$

4.2-THE SYSTEM \mathcal{D}_0 - THE RULES

(R'3)

$$\emptyset \Vdash (0, S, T)$$

for $S \in \text{DRB}\langle\langle V \rangle\rangle, T \in \{\emptyset, \epsilon\}, S \equiv T,$

(R4)

$$\{(p + 1, S \odot x, T \odot x) \mid x \in X\} \Vdash (p, S, T)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle, (S \neq \epsilon \wedge T \neq \epsilon) .$

(R5)

$$\{(p, S, S')\} \Vdash (p + 2, S \odot x, S' \odot x)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}\langle\langle V \rangle\rangle, x \in X,$

(R6)

$$\{(p, S \cdot T' + S', T')\} \Vdash (p, S^* \cdot S', T')$$

for $p \in \mathbb{N}, (S, S') \in \text{DRB}_{1,2}\langle\langle V \rangle\rangle, T' \in \text{DRB}\langle\langle V \rangle\rangle, S \neq \epsilon,$

4.2-THE SYSTEM \mathcal{D}_0 - THE RULES

(R7)

$$\{(p, S, S'), (p, T, T')\} \Vdash (p, S + T, S' + T')$$

for $p \in \mathbb{N}$, $(S, T), (S', T') \in \text{DRB}_{1,2}\langle\langle V \rangle\rangle$,

(R8)

$$\{(p, S, S')\} \Vdash (p, S \cdot T, S' \cdot T)$$

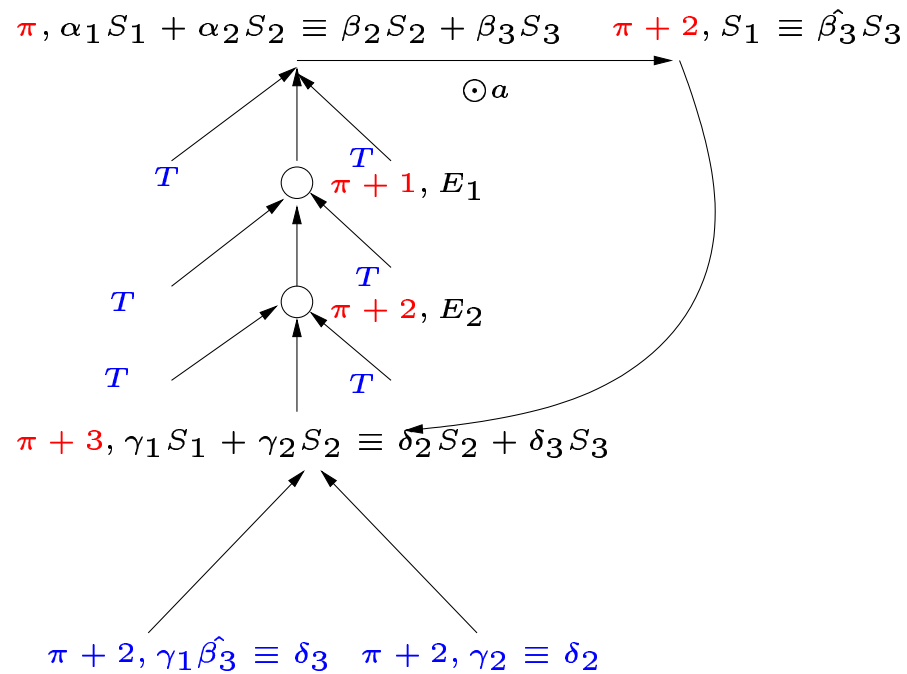
for $p \in \mathbb{N}$, $S, S', T \in \text{DRB}\langle\langle V \rangle\rangle$,

(R9)

$$\{(p, T, T')\} \Vdash (p, S \cdot T, S \cdot T')$$

for $p \in \mathbb{N}$, $S, T, T' \in \text{DRB}\langle\langle V \rangle\rangle$,

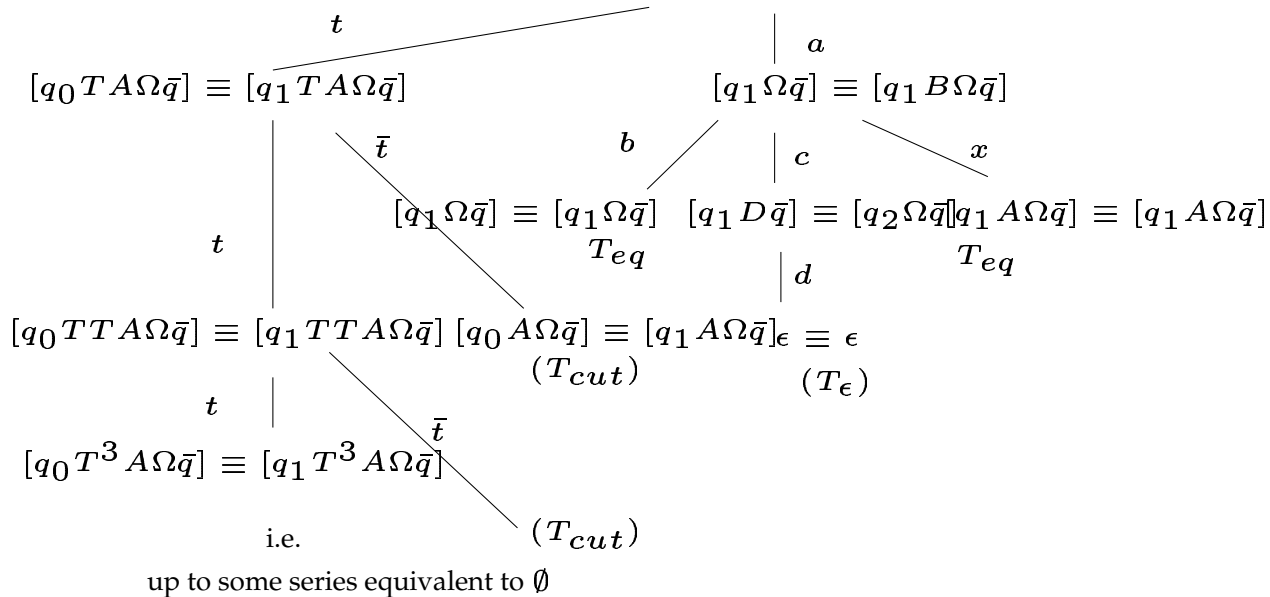
4.3- THE SYSTEM \mathcal{D}_0 - EXAMPLE



$$[q_0 A \Omega \bar{q}] \equiv [q_1 A \Omega \bar{q}] \quad (1)$$

i.e., up to some series equivalent to \emptyset

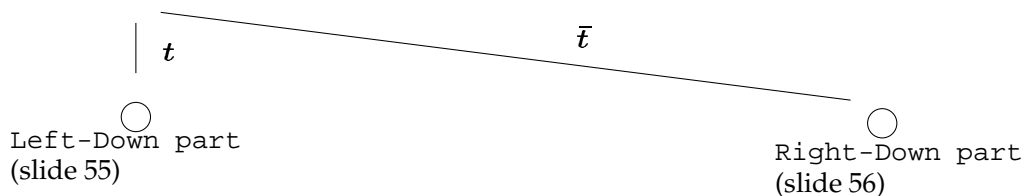
$$[q_0 A q_1][q_1 \Omega \bar{q}] + [q_0 A q_2][q_2 \Omega \bar{q}] \equiv [q_1 A q_1][q_1 \Omega \bar{q}] + [q_1 A q_2][q_2 \Omega \bar{q}]$$



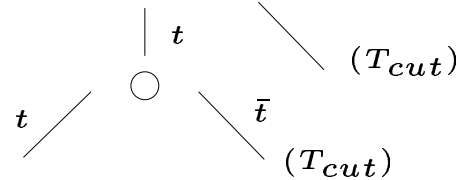
$$[q_0 T^3 A q_1][q_1 \Omega \bar{q}] + [q_0 T^3 A q_2][q_2 \Omega \bar{q}] \equiv [q_1 T^3 A q_1][q_1 \Omega \bar{q}] + [q_1 T^3 A q_2][q_2 \Omega \bar{q}]$$

||| (T_C(1))

$$[q_0 T^3 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^3 A q_2] \equiv [q_1 T^3 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_1 T^3 A q_2] \quad (2)$$



$$[q_0 T^4 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^4 A q_2] \equiv [q_1 T^4 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_1 T^4 A q_2]$$



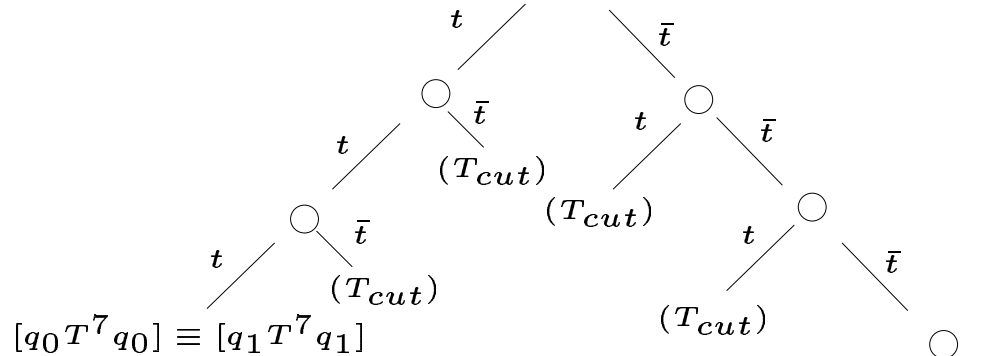
$$[q_0 T^6 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^6 A q_2] \equiv [q_1 T^6 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_1 T^6 A q_2]$$

$\parallel T_B^+(2)$

$$[q_0 T^6 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^6 A q_2] \equiv [q_1 T^4 q_1][q_1 e q_1]([q_0 T^2 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^2 A q_2]) \quad (3)$$

$\parallel T_C(3)$

$$[q_0 T^4 q_0] \equiv [q_1 T^4 q_1] \quad (4)$$



$$[q_0 T^7 q_0] \equiv [q_1 T^7 q_1]$$

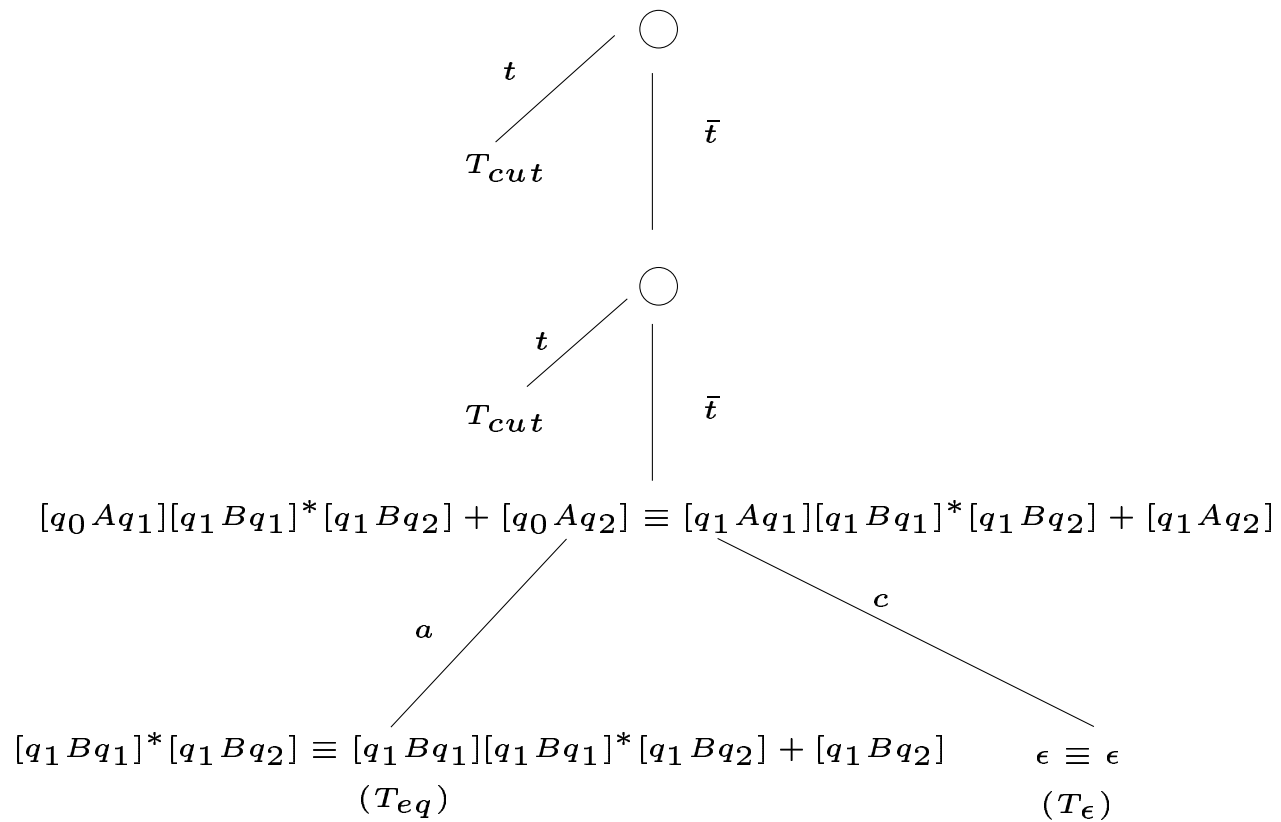
$\parallel T_B^+(4)$

$$[q_0 T^7 q_0] \equiv [q_1 T^4 q_1][q_1 e q_1][q_0 T^3 q_0] \quad (5) \quad (T_{cut}) \quad \epsilon \equiv \epsilon$$

$\parallel T_C(5)$

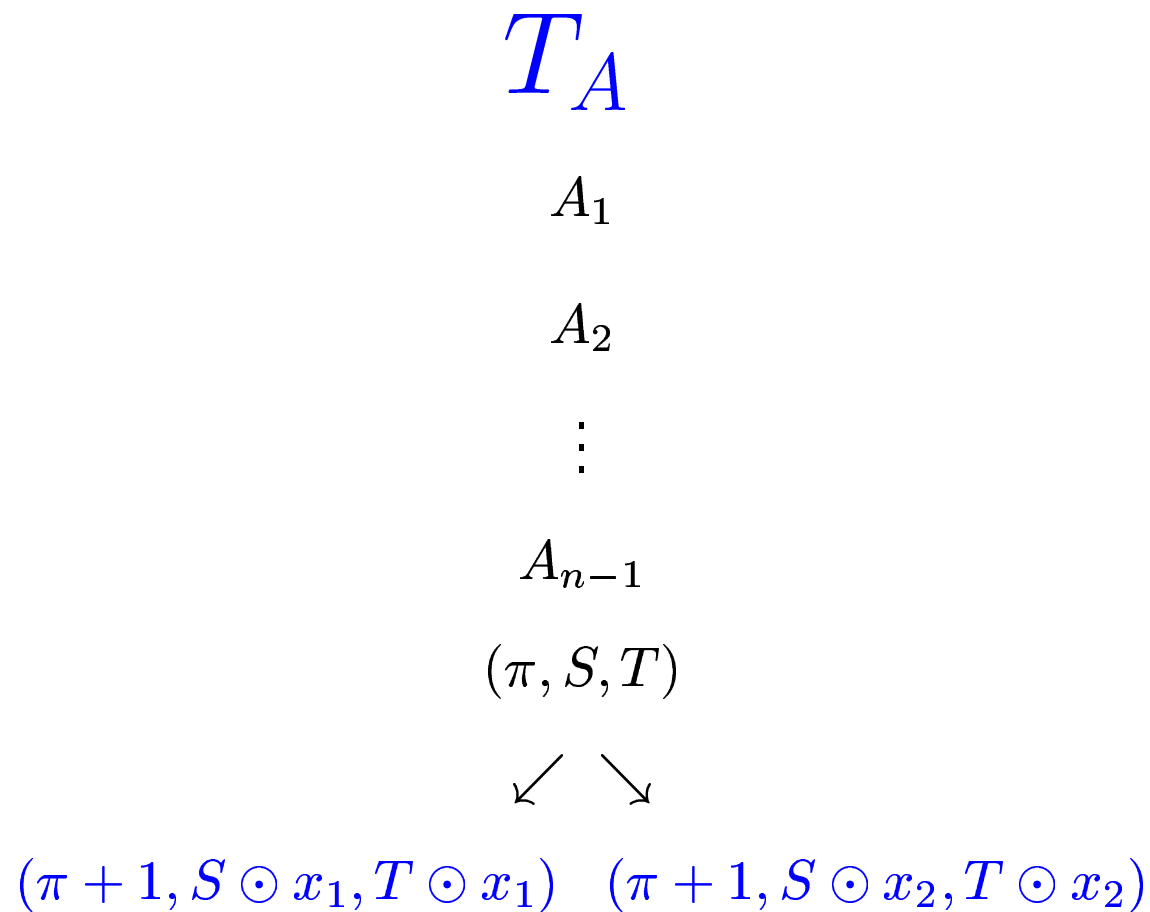
$$[q_0 T^4 q_0] \equiv [q_1 T^4 q_1]$$

$T_{cut}(4)$



4.4- THE SYSTEM \mathcal{D}_0 - STRATEGIES

We suppose $X = \{x_1, x_2\}$.



$$T_B^+$$

$$\vdots$$

$$(\pi, S, \sum_{q \in Q} [\bar{p}zq] \cdot V_q$$

$$\vdots$$

$$\vdots$$

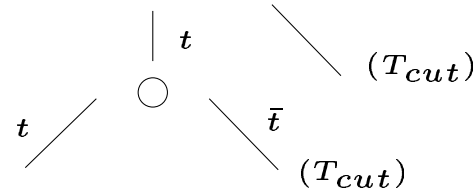
$$(\pi + k_1, S \odot u, \sum_{q \in Q} ([\bar{p}zq] \odot u) \cdot V_q$$

$$\downarrow$$

$$(\pi + k_1 - 1, S \odot u, \sum_{q \in Q'} ([\bar{p}zq] \odot u) \cdot (S \odot u_q))$$

where $Q' = \{q \in Q \mid [\bar{p}zq] \neq \emptyset\}$ and $u_q = \min(\varphi([\bar{p}zq]))$

$$[q_0 T^4 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^4 A q_2] \equiv [q_1 T^4 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_1 T^4 A q_2]$$



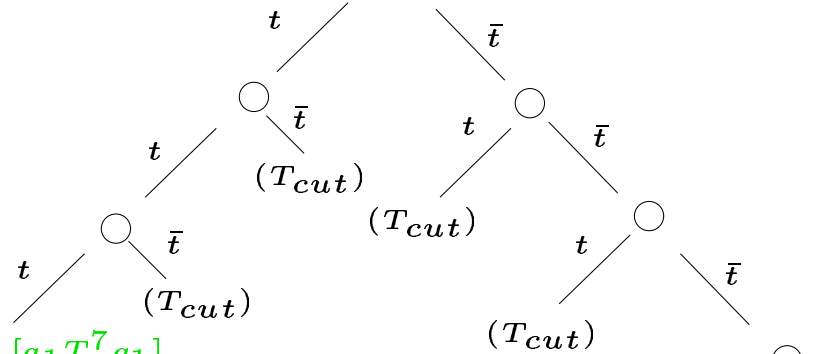
$$[q_0 T^6 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^6 A q_2] \equiv [q_1 T^6 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_1 T^6 A q_2]$$

$$\parallel T_B^+(2)$$

$$[q_0 T^6 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^6 A q_2] \equiv [q_1 T^4 q_1][q_1 e q_1]([q_0 T^2 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^2 A q_2]) \quad (3)$$

$$\parallel T_C(3)$$

$$[q_0 T^4 q_0] \equiv [q_1 T^4 q_1] \quad (4)$$



$$[q_0 T^7 q_0] \equiv [q_1 T^7 q_1]$$

$$\parallel T_B^+(4)$$

$$[q_0 T^7 q_0] \equiv [q_1 T^4 q_1][q_1 e q_1][q_0 T^3 q_0] \quad (5)$$

$$\parallel T_C(5)$$

$$[q_0 T^4 q_0] \equiv [q_1 T^4 q_1]$$

$$T_{cut}(4)$$

$$(T_{cut}) \quad \epsilon \equiv \epsilon \quad (T_\epsilon)$$

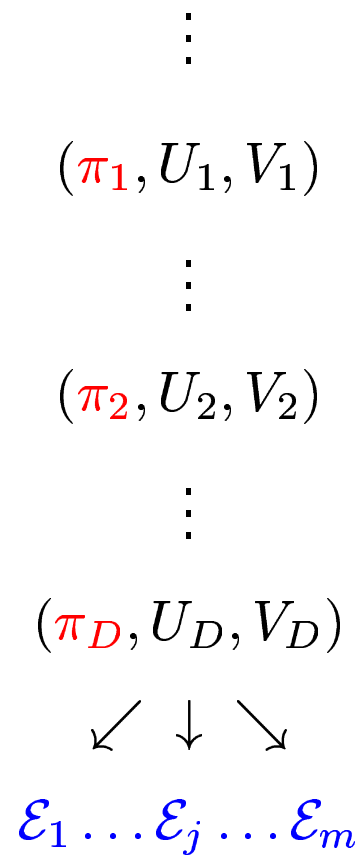
T_C : an example.

$$\begin{array}{rcl}
 \pi, \alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 & \equiv & \beta_1 S_1 + \beta_2 S_2 + \beta_3 S_3 \\
 & & \vdots \\
 \pi + 3, \gamma_1 S_1 + \gamma_2 S_2 + \gamma_3 S_3 & \equiv & \delta_1 S_1 + \delta_2 S_2 + \delta_3 S_3 \\
 & & \downarrow \searrow \\
 \pi + 2, \gamma_1 \hat{\beta}_3 + \gamma_3 & \equiv & \delta_1 \hat{\beta}_3 + \delta_3 \quad \pi + 2, \gamma_2 \equiv \delta_2
 \end{array}$$

We know that first equation $\vdash \pi + 2, S_1 \equiv \hat{\beta}_3 S_3$, hence
 {first equation, result of T_C } \vdash second equation.

4.4- THE SYSTEM \mathcal{D}_0 - STRATEGIES

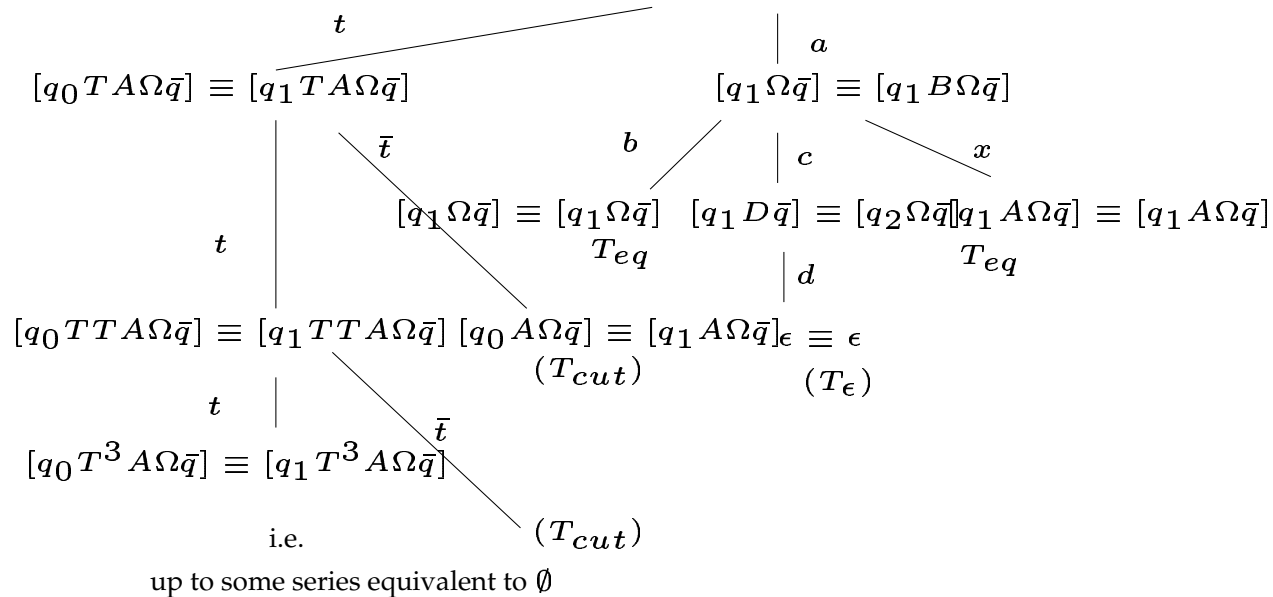
T_C : general case.



$$[q_0 A \Omega \bar{q}] \equiv [q_1 A \Omega \bar{q}] \quad (1)$$

i.e., up to some series equivalent to \emptyset

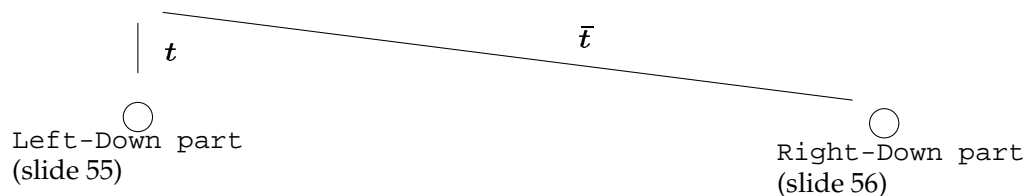
$$[q_0 A q_1][q_1 \Omega \bar{q}] + [q_0 A q_2][q_2 \Omega \bar{q}] \equiv [q_1 A q_1][q_1 \Omega \bar{q}] + [q_1 A q_2][q_2 \Omega \bar{q}]$$



$$[q_0 T^3 A q_1][q_1 \Omega \bar{q}] + [q_0 T^3 A q_2][q_2 \Omega \bar{q}] \equiv [q_1 T^3 A q_1][q_1 \Omega \bar{q}] + [q_1 T^3 A q_2][q_2 \Omega \bar{q}]$$

||| (T_C(1))

$$[q_0 T^3 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_0 T^3 A q_2] \equiv [q_1 T^3 A q_1][q_1 B q_1]^* [q_1 B q_2] + [q_1 T^3 A q_2] \quad (2)$$



5.1 - DECIDABILITY- TERMINATING STRATEGY

The strategy \mathcal{S}_{ABC} is a deterministic strategy, choosing for every branch to stop the construction of that branch or to add some new assertions (by means of T_A, T_B, T_C). Transformation T_C has highest priority.

idea 1: Successive applications of T_A, T_B on a branch make it **smooth**: from some point, many assertions have both sides on a given deterministic space

$$V(S_1, \dots, S_n)$$

with n uniformly bounded (bound d_0 : **exponential**)

idea 2: on a smooth (hypothetically) infinite branch, T_C must be applicable, with some uniform bound on the size of the result (bound $\Sigma_0 + s_0$: **not known to be p.r.**)

5.2- DECIDABILITY- SYSTEM \mathcal{D}_0 IS COMPLETE

Theorem 23 *The system \mathcal{D}_0 is complete.*

Sketch of proof: From every true assertion $(0, S, T)$, \mathcal{S}_{ABC} constructs a proof-tree. If the tree is infinite, then it has some infinite branch. T_C cannot apply i.o. on that branch (because its results are bounded). Hence an infinite suffix is smooth. But T_C is always applicable on such a path, hence \mathcal{S}_{ABC} should apply T_C : contradiction ! Hence the tree (hence the proof) is finite. \diamond

5.3- DECIDABILITY- TWO SEMI-DECISION ALGORITHMS

Theorem 24 *The equivalence problem for deterministic pushdown automata is **decidable**.*

Sketch of proof: Let $S, T \in \text{DRB}\langle\langle V \rangle\rangle$.

1- $S \equiv T$ is semi-decidable: just enumerate the finite \mathcal{D}_0 -proofs, until you find a proof P containing $(0, S, T)$.

2- $S \not\equiv T$ is semi-decidable: just enumerate the words $w \in X^*$, until you find a witness w such that $S \odot w = \varepsilon \Leftrightarrow T \odot w \neq \varepsilon$.

◇

6.1 - REFINEMENTS-EXTENSIONS-COMPLEXITY

Theorem 25 (*Stirling, ICALP'02*) *The equivalence problem for dpda's is primitive recursive.*

6.2- REFINEMENTS-EXTENSIONS-BISIMULATION

Theorem 26 (*Sénizergues, FOCS'98*) *The bisimulation problem for non-deterministic pushdown automata, with deterministic, decreasing ϵ -moves, is decidable.*

To appear in (SIAM, 2005).

6.3- REFINEMENTS-EXTENSIONS-TRANSDUCERS

Theorem 27 (Sénizergues, ICALP'99) *The equivalence problem for deterministic pushdown transducers, from X^* , into a **free group**, is decidable.*

Theorem 28 (Sénizergues, ICWL'00, Kyoto) *The equivalence problem for deterministic pushdown transducers, from X^* , into $(\mathbb{Q}, +, \times)$, over two rational series S, T , which are supposed*

- 1- \smile -deterministic
 - 2- but might be **non-V-deterministic** rational
- is decidable.*

Work in progress.

7- PERSPECTIVES

Open question 0:

Is linear independence a **decidable** property ?.

Open problem 1:

Develop the theory of prefix (resp. deterministic) matrices.

(Monoid of square matrices, deterministic spaces)

Open problem 2:

Find a non-trivial **lower-bound** for the complexity of the equivalence problem for dpda.

Open problem 3:

Find a better **upper-bound** for the complexity of the equivalence problem for dpda.

7- PERSPECTIVES

Open problem 4:

Show that the following problem is decidable:

INSTANCE: a deterministic pushdown transducer $X^* \rightarrow \mathbb{H}$ (the quaternions over \mathbb{Q}), and two rational \smile -deterministic, non- V -deterministic series $S, T \in R\mathbb{H}\langle\langle V \rangle\rangle$.

QUESTION: Are the series S, T bisimilar?

7- PERSPECTIVES

i.e. does there exist a binary relation \mathcal{R} over $\mathbb{R}\mathbb{H}\langle\langle V \rangle\rangle$ such that:

1- $\forall (U, U') \in \mathcal{R}, \forall x \in X$, and for every choice of $U_1 \in U \odot x$, there **exists** a choice $U'_1 \in U' \odot x$ such that $(U_1, U'_1) \in \mathcal{R}$

1'- $\forall (U, U') \in \mathcal{R}, \forall x \in X$, and for every choice of $U'_1 \in U' \odot x$, there **exists** a choice $U_1 \in U \odot x$ such that $(U_1, U'_1) \in \mathcal{R}$

2- $\forall (U, U') \in \mathcal{R}, (U \in \mathbb{H} \Rightarrow U = U')$.

2'- $\forall (U, U') \in \mathcal{R}, (U' \in \mathbb{H} \Rightarrow U = U')$.