The equivalence problem
for Deterministic PushDown Automata

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0- Main result

Decision problem:
INSTANCE: $A$, $B$, two deterministic pushdown automata;
QUESTION: $L(A) = L(B)$?

Raised in:
S. Ginsburg and S. Greibach, Deterministic Context-free languages, Information and Control, pages 620-648, 1966
0- MAIN RESULT


0- MAIN RESULT

REFERENCE:

SUBMISSION PROCESS:

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   pages 9-82
2- Simplifications of the logical system:
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3- Generalization to transducers $X^* \rightarrow (\mathbb{Z}, +)$; pages 104-141
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0 Main result

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1.1- Prefix matrices- Operations

Let us remark:

\[ \langle \mathcal{P}(X^*), \cup, \cdot, \emptyset, \{\varepsilon\} \rangle \cong \langle \mathbb{B}\langle \langle X \rangle \rangle, +, \cdot, 0, 1 \rangle. \]

We rather use the second notation.

We call a series \( S = \sum_{w \in X^*} S_w \cdot w \) prefix-free iff: for every \( u, v \in X^* \),

\[ (S_{uv} = 1 \text{ and } S_u = 1) \Rightarrow v = \varepsilon. \]
1.1- Prefix matrices- Operations

**Definition 1** A row-vector \((A_1, \ldots, A_i, \ldots, A_m) \in (\mathbb{B}(X))^{n}\) is said **prefix-free** iff

1-the supports of the \(A_i\)'s are disjoint
2-\(\sum_{i=1}^{n} A_i\) is prefix-free.

**Definition 2** A matrix \(A \in (\mathbb{B}(X))^{n \times m}\) is said **prefix-free** iff every row-vector \((A_{i,1}, \ldots, A_{i,j}, \ldots, A_{i,m})\) is prefix-free.

**Example 3** Some prefix matrices:

\[
\begin{pmatrix}
ab & 0 & a^*c \\
1 & 0 & 0 \\
abc & abd & a^*cd
\end{pmatrix},
\begin{pmatrix}
a \\
ab^*c \\
abc + ac
\end{pmatrix}
\]
1.1- Prefix matrices- Operations

Product: defined as usual.
Let $A, B \in (\mathbb{B}\langle\langle X \rangle\rangle)^{1 \times m}$ and $1 \leq j_0 \leq m$.

If $A = (a_1, \ldots, a_j, \ldots, a_m), B = (b_1, \ldots, b_j, \ldots, b_m)$ then

$$A \nabla_{j_0} B = (a_1 + a_{j_0} \cdot b_1, \ldots, a_{j_0-1} + a_{j_0} \cdot b_{j_0-1}, 0, a_{j_0+1} + a_{j_0} \cdot b_{j_0+1}, \ldots, a_m + a_{j_0} \cdot b_m)$$

$$\nabla^*_j (A) = (a^*_{j_0} \cdot a_1, \ldots, a^*_{j_0} \cdot a_{j_0-1}, 0, a^*_{j_0} \cdot a_{j_0+1}, \ldots, a^*_{j_0} \cdot a_m)$$
1.1- Prefix matrices - Operations

We denote by $\mathbb{DB}_{n,m}\langle\langle X \rangle\rangle$ the set of prefix matrices (which will turn out to be a particular case of Deterministic matrices).

**Lemma 4** Let $A \in \mathbb{DB}_{n,m}\langle\langle X \rangle\rangle$, $B \in \mathbb{DB}_{m,s}\langle\langle X \rangle\rangle$. Then $A \cdot B \in \mathbb{DB}_{n,s}\langle\langle X \rangle\rangle$

**Lemma 5** Let $A, B \in \mathbb{DB}_{1,m}\langle\langle X \rangle\rangle$ and $1 \leq j_0 \leq m$. Then
1. $A \nabla_{j_0} B \in \mathbb{DB}_{1,m}\langle\langle X \rangle\rangle$
2. $\nabla_{j_0}^* (A) \in \mathbb{DB}_{1,m}\langle\langle X \rangle\rangle$
1.2- Prefix matrices- Right-action

**Definition 6** Let \( S = \sum_{w \in X^*} S_w \cdot w \) and \( u \in X^* \). Then

\[
S \cdot u = \sum_{w \in X^*} S_{u \cdot w} \cdot w,
\]

(i.e. \( S \cdot u \) is the left-quotient of \( S \) by \( u \), or the *residual* of \( S \) by \( u \)).

This right-action is extended componentwise to matrices.

**Lemma 7** Let \( A \in \text{DB}_{n,m} \langle \langle X \rangle \rangle \) and \( u \in X^* \). Then \( A \cdot u \in \text{DB}_{n,m} \langle \langle X \rangle \rangle \)
1.2- Prefix matrices- Right-action

Lemma 8  Let $S \in DB_{1,m}^{W}, T \in B_{m,s}^{W}$ and $u \in W^*$. Exactly one of the following cases is true:

(1) \( \exists j, S_j \circ u \notin \{\emptyset, \epsilon\} \)
    in this case \( (S \cdot T) \circ u = (S \circ u) \cdot T \).

(2) \( \exists j_0, \exists u', u'', u = u' \cdot u'', S_{j_0} \circ u' = \epsilon; \)
    in this case \( (S \cdot T) \circ u = T_{j_0,*} \circ u'' \).

(3) \( \forall j, \forall u' \leq u, S_j \circ u = \emptyset, S_j \circ u' \neq \epsilon; \)
    in this case \( (S \cdot T) \circ u = \emptyset = (S \circ u) \cdot T \).
1.3- Prefix matrices- Linear identities

Fundamental remark (Meitus 1989):
Let $S_1, \ldots, S_j, \ldots, S_m \in DB\langle X \rangle$ and let $\vec{\alpha}, \vec{\beta} \in DB_{1,m}\langle X \rangle$ such that

\[(0) \sum_{j=1}^{m} \alpha_j \cdot S_j = \sum_{j=1}^{m} \beta_j \cdot S_j\]

Then one of cases (1), (2) must occur:

(1) $\vec{\alpha} = \vec{\beta}$

(2) $\exists j_0 \in [1, m], \exists \vec{\gamma} \in DB_{1,m}\langle X \rangle, \gamma_{j_0} = 0$, such that $S_{j_0} = \sum_{1 \leq j \leq m} \gamma_j \cdot S_j$. 

\[\]
1.3- PREFIX MATRICES - LINEAR IDENTITIES

Sketch of proof: Let $S = (S_j)_{1 \leq j \leq m} \in \mathbb{D} \mathbb{B}_m, \langle \langle X \rangle \rangle$. Suppose that $\vec{\alpha} \neq \vec{\beta}$. Let us consider

$$u = \min \{\vec{\alpha} \Delta \vec{\beta}\}.$$

Then there exists, $j_0 \in [1, m]$, such that

$$\vec{\alpha} \cdot u = \epsilon_{j_0}^m \Leftrightarrow \beta_{j_0} \cdot u \neq \epsilon_{j_0}^m.$$

Let us suppose, for example, that $\vec{\alpha} \cdot u = \epsilon_{j_0}^m$ while $\vec{\beta} \cdot u \neq \epsilon_{j_0}^m$ and let $\vec{\gamma} = \vec{\beta} \cdot u$. Let $u$ act (by $\cdot$) on both sides of hypothesis (0):

$$(\vec{\alpha} \cdot S) \cdot u = (\vec{\beta} \cdot S) \cdot u. \quad (1)$$

Using lemma 8 we obtain:

$$(\vec{\alpha} \cdot S) \cdot u = S_{j_0}. \quad (2)$$
Let us examine now the righthand-side of equality (1). Let $u' < u$. By minimality of $u$, $\vec{\beta} \bullet u'$ is a unit iff $\vec{\alpha} \bullet u'$ is a unit. But if $\vec{\alpha} \bullet u'$ is a unit, then $\vec{\alpha} \bullet u = \emptyset^m$, which is false. Hence $\vec{\beta} \bullet u'$ is not a unit. By lemma 8

$$(\vec{\beta} \cdot S) \bullet u = (\vec{\beta} \bullet u) \cdot S.$$  

(3)

Let us plug equalities (2) and (3) in equivalence (1) and let us define $\vec{\gamma} = \vec{\beta} \bullet u$. We obtain:

$$S_{j_0} = \vec{\gamma} \bullet S, \text{ where } \vec{\gamma} \neq \epsilon_{j_0}^m.$$ 

Using Arden’s lemma we get:

$$S_{j_0} = \sum_{j \neq j_0} \gamma^*_j \gamma_j \cdot S_j \text{ i.e. } S_{j_0} = \vec{\gamma}' \bullet S$$

where $\vec{\gamma}' = \nabla_{j_0}^*(\vec{\gamma})$, hence $\gamma'_{j_0} = 0$. ◇
1.4.1 - Prefix matrices - Bases

A subset $V \subseteq \mathbb{D}_{\langle X \rangle}$ is said to be a prefix space iff it is closed under linear combinations (with prefix vector of coefficients).

A set $\mathcal{F} = \{S_1, \ldots, S_m\}$ (where $m = \text{Card} \mathcal{F}$) is called:
- a generating subset of $V$ iff

$$V = \left\{ \sum_{j=1}^{m} \alpha_j \cdot S_j \mid \vec{\alpha} \in \mathbb{D}_{1,m} \langle X \rangle \right\}.$$ 

- a linearly independent set if, for every $\vec{\alpha}, \vec{\beta} \in \mathbb{D}_{1,m} \langle X \rangle$,

$$\vec{\alpha} \cdot \vec{S} = \vec{\beta} \cdot \vec{S} \Leftrightarrow \vec{\alpha} = \vec{\beta}$$

- a base of $V$ iff, it is both a generating set of $V$ and it is linearly independent.
1.4.1- Prefix matrices- Bases

Proposition 9  Every finitely generated prefix space has a base. Moreover this base is unique.

Question: For what kind of series (i.e. languages) is the base of \( V(\mathcal{F}) \) computable from the set \( \mathcal{F} \) ?
1.4.2- **Prefix matrices- Most general unifier**

Let $\vec{\alpha}, \vec{\beta} \in DB_{1,m} \langle \langle X \rangle \rangle$. We call **unifier** for the pair $(\vec{\alpha}, \vec{\beta})$ any matrix $U \in DB_{m,n} \langle \langle X \rangle \rangle$ ($n \geq 1$) such that:

$$\vec{\alpha} \cdot U = \vec{\beta} \cdot U.$$

It is said to be a **Most General Unifier** iff, every unifier has the form $U \cdot T$ for some matrix $T \in DB_{n,s}$ (where $s \geq 1$).
1.4.2- Prefix matrices- Most general unifier

**Proposition 10**  Every pair of prefix row-vectors \((\vec{\alpha}, \vec{\beta})\) has a MGU. This MGU is unique (except in the case where \(0_{m,1}\) is a MGU).

**Question**: For what kind of series (i.e. languages) is the MGU computable from the pair of prefix row-vectors?
1.5- Prefix matrices- Example

A linear equation:

\[ ab \cdot U_1 + 0 \cdot U_2 + aab \cdot U_3, 0 \cdot U_1 + abc \cdot U_2 + (aab + abd) \cdot U_3. \]

Computation of a MGU:
applying \( ab \) on both sides:

\[ U_1, c \cdot U_2 + d \cdot U_3. \]

substituting \( U_1 \) by the r.h.s:

\[ abc \cdot U_2 + (aab + abd) \cdot U_3, abc \cdot U_2 + (abd + aab) \cdot U_3, \]
hence:

\[ MGU\left((ab, 0, aab), (0, abc, (aab + abd))\right) = \begin{pmatrix} c & d \\ \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \]
1.5- Prefix matrices- Example

A prefix column vector:

\[
S = \begin{pmatrix}
(ca^*b + d)a^*c \\
a^*ba^*c \\
a^*c
\end{pmatrix}
\]

This vector is a solution of:

\[
ab \cdot S_1 + 0 \cdot S_2 + aab \cdot S_3, 0 \cdot S_1 + abc \cdot S_2 + (aab + abd) \cdot S_3.
\]

Hence it is obtained by substitution from the MGU:

\[
S_1 = cS_2 + dS_3
\]

\[
V(\{S_1, S_2, S_3\}) = V(\{S_2, S_3\})
\]

A base of \(V(\{S_1, S_2, S_3\})\) is \(\{S_3\}\).
2.0- Deterministic Matrices - Deterministic Grammars

(Harrison-Havel, JCSS 1973):

**Definition 11** Let \( G = \langle X, V, P \rangle \) be some context-free grammar.

\( G \) is said **strict deterministic** iff there exists some equivalence relation \( \sim \) over \( X \cup V \) such that

1- \( X \) is some class mod \( \sim \)

2- \( \forall v_1, v'_1 \in V, \forall \alpha \in (X \cup V)^*, \forall w, w' \in V^* \), if

\[
    v_1 \sim v'_1, v_1 \rightarrow \alpha w_1, v'_1 \rightarrow \alpha w'_1
\]

then \( (w_1 = w'_1 = \epsilon \text{ and } v_1 = v'_1) \text{ or } \exists v_2, v'_2 \in (X \cup V), w_2, w'_2 \in (X \cup V)^* \),

\[
    v_2 \sim v'_2, w_1 = v_2 w_2, w'_1 = v'_2 w'_2.
\]
Theorem 12  (Harrison-Havel,73) A language $L \subseteq X^*$ is recognized (by empty stack) by some dpda iff it is generated by some strict-deterministic context-free grammar.
2.1- Deterministic Matrices- Operations

**Definition 13** A series $S \in B\langle\langle V^* \rangle\rangle$ is said **deterministic** iff, for every words $\alpha, w_1, w'_1 \in V^*$ such that $\alpha \cdot w_1, \alpha \cdot w'_1 \in \text{supp}(S)$

- either $w_1 = w'_1 = \epsilon$
- or $\exists v_2, v'_2 \in V, w_2, w'_2 \in V^*, \text{such that}$

$$v_2 \sim v'_2, w_1 = v_2 w_2, w'_1 = v'_2 w'_2.$$ 

Intuition: a deterministic series is a generalized configuration of the initial pushdown automaton.

**Definition 14** A row-vector $(S_1, \ldots, S_i, \ldots, S_m)$ is said **deterministic** iff

1- The supports of the $S_i$'s are disjoint
2- $\sum_{j=1}^{m} S_i$ is deterministic.
Definition 15  A matrix of dimension $(n, m)$, $(S_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$ is said deterministic iff every one of its row-vectors is deterministic.

We denote by $\mathbb{D\mathbb{B}}_{n,m} \langle \langle V \rangle \rangle$ the set of deterministic matrices.
2.1- Deterministic Matrices- Operations

Lemma 16 Let $A \in DB_{n,m} \langle\langle V \rangle\rangle$, $B \in DB_{m,s} \langle\langle V \rangle\rangle$. Then $A \cdot B \in DB_{n,s} \langle\langle V \rangle\rangle$

Lemma 17 Let $A, B \in DB_{1,m} \langle\langle V \rangle\rangle$ and $1 \leq j_0 \leq m$. Then
1- $A \nabla_{j_0} B \in DB_{1,m} \langle\langle V \rangle\rangle$
2- $\nabla_{j_0}^* (A) \in DB_{1,m} \langle\langle V \rangle\rangle$
2.2- Deterministic matrices - Right-action

We define a $\sigma$-right-action $\odot$ of the monoid $X^*$ over the semi-ring $\mathbb{B}\langle\langle V^* \rangle\rangle$ by:

for every $w \in V^*, u \in X^*$,

$$w \odot u = \sum_{\substack{w' \in V^* \atop w \rightarrow^*_m u \cdot w'}} w'$$

for every $S \in \mathbb{B}\langle\langle V^* \rangle\rangle, u \in X^*$,

$$(\sum_{w \in V^*} S_w \cdot w) \odot u = \sum_{w \in V^*} S_w \cdot (w \odot u).$$

This right-action is extended componentwise to matrices.
2.2- Deterministic matrices - Right-action

Lemma 18 Let $A \in \mathbb{DB}_{n,m} \langle \langle V \rangle \rangle$ and $u \in X^*$. Then $A \odot u \in \mathbb{DB}_{n,m} \langle \langle V \rangle \rangle$

Lemma 19 Let $S \in \mathbb{DB}_{1,m} \langle \langle W \rangle \rangle, T \in \mathbb{B}_{m,s} \langle \langle W \rangle \rangle$ and $u \in W^*$. Exactly one of the following cases is true:

1. $\exists j, S_j \odot u \notin \{\emptyset, \epsilon\}$
   In this case $(S \cdot T) \odot u = (S \odot u) \cdot T$.

2. $\exists j_0, \exists u', u''$, $u = u' \cdot u''$, $S_{j_0} \odot u' = \epsilon$;
   In this case $(S \cdot T) \odot u = T_{j_0,*} \odot u''$.

3. $\forall j, \forall u' \preceq u$, $S_j \odot u = \emptyset$, $S_j \odot u' \neq \epsilon$;
   In this case $(S \cdot T) \odot u = \emptyset = (S \odot u) \cdot T$. 
2.2- Deterministic matrices- Right-action

A map \( \varphi : V \to \mathbb{B}\langle\langle X^* \rangle\rangle \) is defined by:

\[
\varphi(v) = \sum_{v \odot u = e} u
\]

This map is extended as a \( \sigma \)-additive semi-ring homorphism \( \varphi : \mathbb{B}\langle\langle V \rangle\rangle \to \mathbb{B}\langle\langle X \rangle\rangle \) by

\[
\varphi(\sum_{w \in V^*} S_w \cdot w) = \sum_{w \in V^*} S_w \cdot \varphi(w).
\]

We set

\[
S \equiv T \text{ iff } \varphi(S) = \varphi(T).
\]
Lemma 20  For every
\[ S \in \mathbb{D}\mathbb{B}_{n,m}(V),\ T \in \mathbb{D}\mathbb{B}_{m,s}(V),\ u \in X^*, \]
1- \[ \varphi(S \cdot T) = \varphi(S) \cdot \varphi(T) \]
2- \[ \varphi(S \odot u) = \varphi(S) \bullet u \]
2.3- Deterministic matrices - Linear identities

Fundamental remark:
Let $S_1, \ldots, S_j, \ldots, S_m \in \mathbb{D}\mathbb{B}\langle X \rangle$ and let $\alpha, \beta \in \mathbb{D}\mathbb{B}_{1,m}\langle X \rangle$ such that

(0) $\sum_{j=1}^{m} \alpha_j \cdot S_j \equiv \sum_{j=1}^{m} \beta_j \cdot S_j S$

Then one of cases (1),(2) must occur:

(1) $\alpha \equiv \beta$

(2) $\exists j_0 \in [1, m], \exists \gamma \in \mathbb{D}\mathbb{B}_{1,m}\langle X \rangle, \gamma_{j_0} = 0$, such that

$S_{j_0} \equiv \sum_{1 \leq j \leq m} \gamma_j \cdot S_j$. 
2.4.1 - Deterministic matrices- Bases

A subset $V \subseteq \mathbb{D} \langle \langle V \rangle \rangle$ is said to be a deterministic space iff it is closed under linear combinations (with deterministic vector of coefficients).

A set $\mathcal{F} = \{S_1, \ldots, S_m\}$ (where $m = \text{Card} \mathcal{F}$) is called :
- a generating subset of $V$ iff

$$V = \left\{ \sum_{j=1}^{m} \alpha_j \cdot S_j \mid \alpha \in \mathbb{D} \langle \langle V \rangle \rangle \right\}.$$  

- a linearly independent set iff, for every $\alpha, \beta \in \mathbb{D} \langle \langle V \rangle \rangle$,

$$\alpha \cdot \vec{S} = \beta \cdot \vec{S} \iff \alpha = \beta$$

- a base of $V$ iff, it is both a generating set of $V$ and it is linearly independent.
2.4.1- Deterministic matrices- Bases

Proposition 21  Every finitely generated deterministic space has a base. Moreover this base is unique (up to the equivalence $\equiv$).

It is unknown whether linear independence is a decidable property. It is unknown whether the base of $V(\mathcal{F})$ is computable from the set $\mathcal{F}$.
2.4.2- Deterministic matrices- Most general unifier

Let $\vec{\alpha}, \vec{\beta} \in \mathbb{DB}_{1,m} \langle \langle V \rangle \rangle$. We call unifier for the pair $(\vec{\alpha}, \vec{\beta})$ any matrix $U \in \mathbb{DB}_{m,n} \langle \langle V \rangle \rangle$ ($n \geq 1$) such that:

$$\vec{\alpha} \cdot U \equiv \vec{\beta} \cdot U.$$ 

It is said to be a Most General Unifier iff, every unifier is equivalent (mod $\equiv$) to $U \cdot T$ for some matrix $T \in \mathbb{DB}_{n,s}$ (where $s \geq 1$).
2.4.2- Deterministic matrices- Most general unifier

**Proposition 22** Every pair of prefix row-vectors \((\vec{a}, \vec{b})\) has a MGU. This MGU is unique (up to equivalence \((\mod \equiv)\), and with the exception of the case where \(0_{m,1}\) is a MGU).

- The MGU is computable from the pair of prefix row-vectors.
- This will be a corollary of the main result ... hence *not* directly usable in its proof! We shall merely adapt the underlying idea.
3.1 - Deduction systems - Motivation

Let $P$ be a set of equations such that, every equation is implied by the others. When can we conclude that they are all true?
3.1 - Deduction Systems - Motivation

\[
\alpha_1 S_1 + \alpha_2 S_2 \equiv \beta_2 S_2 + \beta_3 S_3 \? \quad S_1 \equiv \beta_3 S_3 \?
\]

\[
\gamma_1 S_1 + \gamma_2 S_2 \equiv \delta_2 S_2 + \delta_3 S_3 \?
\]

\[
\gamma_1 \beta_3 \equiv \delta_3 \quad \gamma_2 \equiv \delta_2
\]
3.2- Deduction Systems- General Definitions

General philosophy of (Harrison-Havel-Yehudai79), (Courcelle 83).
Let us call deduction system any triple $\mathcal{D} = (\mathcal{A}, H, \vdash)$ where
- $\mathcal{A}$ is a denumerable set called the set of assertions
- $H : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is the cost-function
- $\vdash$, the deduction relation is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$
such that the following axioms are satisfied:

(A1) $\forall (P, A) \in \vdash$ , $(\inf \{H(p), p \in P\} < H(A)) \text{ or } (H(A) = \infty)$.

(A2) $\vdash$ is recursively enumerable.

We call proof in the system $\mathcal{D}$, any subset $P \subseteq \mathcal{A}$ fulfilling:

$$\forall p \in P, \exists Q \subseteq P, Q \vdash p.$$
3.2- Deduction systems- General definitions

If $P$ is a proof, then $\forall A \in P, H(A) = \infty$. 

3.3- Deduction systems- Strategies

Informal definition:
A strategy is a “method” allowing to enlarge every “partial proof” $P$ into a partial proof $Q$ with $P \subset Q$.

Notice: this method needs not be computable.
4.1 - The system $\mathcal{D}_0$ - Assertions and cost function

Set of assertions:

$$\mathcal{A} = \mathbb{N} \times \text{DRB}\langle\langle V \rangle\rangle \times \text{DRB}\langle\langle V \rangle\rangle$$

The “cost-function”:

$$H(n, S, S') = n + 2 \cdot \text{Div}(S, S').$$

Where the divergence between $S$ and $S'$, is defined by:

$$\text{Div}(S, S') = \inf \{| u | | u \in \varphi(S) \triangle \varphi(S')\}.$$
4.2-The system $D_0$- The rules

(R0)

$$\{ (p, S, T) \} \vdash (p + 1, S, T)$$

for $p \in \mathbb{N}, S, T \in \text{DRB} \langle \langle V \rangle \rangle$

(R1)

$$\{ (p, S, T) \} \vdash (p, T, S)$$

for $p \in \mathbb{N}, S, T \in \text{DRB} \langle \langle V \rangle \rangle$.

(R2)

$$\{ (p, S, S'), (p, S', S'') \} \vdash (p, S, S'')$$

for $p \in \mathbb{N}, S, S', S'' \in \text{DRB} \langle \langle V \rangle \rangle$.

(R3)

$$\emptyset \vdash (0, S, S)$$

for $S \in \text{DRB} \langle \langle V \rangle \rangle$. 
4.2-The system $\mathcal{D}_0$- The rules

(R0)

$$\{(p, S, T)\} \vdash (p + 1, S, T)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}(\langle V \rangle)$.

(R1)

$$\{(p, S, T)\} \vdash (p, T, S)$$

for $p \in \mathbb{N}, S, T \in \text{DRB}(\langle V \rangle)$.

(R2)

$$\{(p, S, S'), (p, S', S'')\} \vdash (p, S, S'')$$

for $p \in \mathbb{N}, S, S', S'' \in \text{DRB}(\langle V \rangle)$.

(R3)

$$\emptyset \vdash (0, S, S)$$

for $S \in \text{DRB}(\langle V \rangle)$. 
4.2-The system $\mathcal{D}_0$- The rules

(R’3)

$$\emptyset \vdash (0, S, T)$$

for $S \in \mathbb{DRB} \langle \langle V \rangle \rangle, T \in \{\emptyset, \epsilon\}, S \equiv T$.

(R4)

$$\{(p + 1, S \odot x, T \odot x) \mid x \in X\} \vdash (p, S, T)$$

for $p \in \mathbb{N}, S, T \in \mathbb{DRB} \langle \langle V \rangle \rangle, (S \neq \epsilon \land T \neq \epsilon)$.

(R5)

$$\{(p, S, S')\} \vdash (p + 2, S \odot x, S' \odot x)$$

for $p \in \mathbb{N}, S, T \in \mathbb{DRB} \langle \langle V \rangle \rangle, x \in X$.

(R6)

$$\{(p, S \cdot T' + S', T')\} \vdash (p, S^* \cdot S', T')$$

for $p \in \mathbb{N}, (S, S') \in \mathbb{DRB}_{1,2} \langle \langle V \rangle \rangle, T' \in \mathbb{DRB} \langle \langle V \rangle \rangle, S \neq \epsilon$. 
4.2-The system $\mathcal{D}_0$- The rules

(R’3)\[
\emptyset \vdash (0, S, T)
\]
for $S \in \mathcal{DRB}\langle \langle V \rangle \rangle$, $T \in \{\emptyset, \epsilon\}$, $S \equiv T$.

(R4)\[
\{(p + 1, S \odot x, T \odot x) \mid x \in X\} \vdash (p, S, T)
\]
for $p \in \mathbb{N}$, $S, T \in \mathcal{DRB}\langle \langle V \rangle \rangle$, $(S \not\equiv \epsilon \land T \not\equiv \epsilon)$.

(R5)\[
\{(p, S, S')\} \vdash (p + 2, S \odot x, S' \odot x)
\]
for $p \in \mathbb{N}$, $S, T \in \mathcal{DRB}\langle \langle V \rangle \rangle$, $x \in X$.

(R6)\[
\{(p, S \cdot T' + S', T')\} \vdash (p, S^* \cdot S', T')
\]
for $p \in \mathbb{N}$, $(S, S') \in \mathcal{DRB}_{1,2}\langle \langle V \rangle \rangle$, $T' \in \mathcal{DRB}\langle \langle V \rangle \rangle$, $S \not\equiv \epsilon$. 
4.2-The system $\mathcal{D}_0$-The rules

(R’3)

\[
\emptyset \vdash (0, S, T)
\]

for $S \in \text{DRB} \{ V \}, T \in \{ \emptyset, \epsilon \}, S \equiv T$.

(R4)

\[
\{(p + 1, S \diamond x, T \diamond x) \mid x \in X\} \vdash (p, S, T)
\]

for $p \in \mathbb{N}, S, T \in \text{DRB} \{ V \}, (S \not= \epsilon \land T \not= \epsilon)$.

(R5)

\[
\{(p, S, S')\} \vdash (p + 2, S \diamond x, S' \diamond x)
\]

for $p \in \mathbb{N}, S, T \in \text{DRB} \{ V \}, x \in X$.

(R6)

\[
\{(p, S \cdot T' + S', T')\} \vdash (p, S^* \cdot S', T')
\]

for $p \in \mathbb{N}, (S, S') \in \text{DRB}_{1,2} \{ V \}, T' \in \text{DRB} \{ V \}, S \not= \epsilon, T' \not= \epsilon$. 
4.2-The system $\mathcal{D}_0$-The rules

(R’3)

\[ \emptyset \vdash (0, S, T) \]

for $S \in \text{DRIB} \ll V \gg, T \in \emptyset, \varepsilon$, $S \equiv T$.

(R4)

\[ \{(p + 1, S \odot x, T \odot x) \mid x \in X\} \vdash (p, S, T) \]

for $p \in \mathbb{N}$, $S, T \in \text{DRIB} \ll V \gg$, $(S \not\equiv \varepsilon \land T \not\equiv \varepsilon)$.

(R5)

\[ \{(p, S, S')\} \vdash (p + 2, S \odot x, S' \odot x) \]

for $p \in \mathbb{N}$, $S, T \in \text{DRIB} \ll V \gg$, $x \in X$.

(R6)

\[ \{(p, S \cdot T' + S', T')\} \vdash (p, S^* \cdot S', T') \]

for $p \in \mathbb{N}$, $(S, S') \in \text{DRIB}_{1,2} \ll V \gg$, $T' \in \text{DRIB} \ll V \gg$, $S \not\equiv \varepsilon$. 
4.2 - The system $\mathcal{D}_0$ - The rules

(R7)

$$\{(p, S, S') \}, \{p, T, T'\} \rightarrow (p, S + T, S' + T')$$

for $p \in \mathbb{N}, (S, T), (S', T') \in \text{DRB}_1, \langle \langle V \rangle \rangle$.

(R8)

$$\{(p, S, S')\} \rightarrow (p, S \cdot T, S' \cdot T)$$

for $p \in \mathbb{N}, S, S', T \in \text{DRB}_1 \langle \langle V \rangle \rangle$.

(R9)

$$\{(p, T, T')\} \rightarrow (p, S \cdot T, S \cdot T')$$

for $p \in \mathbb{N}, S, T, T' \in \text{DRB}_1 \langle \langle V \rangle \rangle$. 
4.3- The system $\mathcal{D}_0$ - Example

\[
\pi, \alpha_1 S_1 + \alpha_2 S_2 \equiv \beta_2 S_2 + \beta_3 S_3 \quad \pi + 2, S_1 \equiv \hat{\beta}_3 S_3
\]

\[
\pi + 3, \gamma_1 S_1 + \gamma_2 S_2 \equiv \delta_2 S_2 + \delta_3 S_3
\]

\[
\pi + 2, \gamma_1 \hat{\beta}_3 \equiv \delta_3 \quad \pi + 2, \gamma_2 \equiv \delta_2
\]
\[ [q_0 \Lambda q] \equiv [q_1 \Lambda q] \quad (1) \]

i.e., up to some series equivalent to \( \emptyset \)

\[ [q_0 Aq_1][q_1 \Omega q] + [q_0 Aq_2][q_2 \Omega q] \equiv [q_1 Aq_1][q_1 \Omega q] + [q_1 Aq_2][q_2 \Omega q] \]

\[ [q_0 T \Lambda q] \equiv [q_1 T \Lambda q] \]

\[ [q_1 \Omega q] \equiv [q_1 B \Omega q] \]

\[ [q_1 T \Lambda q] \equiv [q_1 T \Lambda q] \]

\[ [q_0 T \Lambda q] \equiv [q_1 T \Lambda q] \]

i.e., up to some series equivalent to \( \emptyset \)

\[ [q_0 T^3 \Lambda q_1][q_1 \Omega q] + [q_0 T^3 \Lambda q_2][q_2 \Omega q] \equiv [q_1 T^3 \Lambda q_1][q_1 \Omega q] + [q_1 T^3 \Lambda q_2][q_2 \Omega q] \]

\[ [q_0 T^3 \Lambda q_1][q_1 Bq_1 \ast q_1 Bq_2] + [q_0 T^3 \Lambda q_2] \equiv [q_1 T^3 \Lambda q_1][q_1 Bq_1 \ast q_1 Bq_2] + [q_1 T^3 \Lambda q_2] \quad (2) \]
\[ [q_0 T^4 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_0 T^4 Aq_2] \equiv [q_1 T^4 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_1 T^4 Aq_2] \]

\[ (T_{\text{cut}}) \]

\[ [q_0 T^6 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_0 T^6 Aq_2] \equiv [q_1 T^6 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_1 T^6 Aq_2] \]

\[ (T_{\text{cut}}) \]

\[ [q_0 T^6 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_0 T^6 Aq_2] \equiv [q_1 T^4 q_1][q_1 e q_1][[q_0 T^2 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_0 T^2 Aq_2]] \]

\[ (T_B^\dagger(2)) \]

\[ (T_{\text{cut}}) \]

\[ (T_C(3)) \]

\[ [q_0 T^4 q_0] \equiv [q_1 T^4 q_1] \]

\[ (T_{\text{cut}}) \]

\[ [q_0 T^7 q_0] \equiv [q_1 T^7 q_1] \]

\[ (T_{\text{cut}}) \]

\[ T_B^\dagger(4) \]

\[ [q_0 T^7 q_0] \equiv [q_1 T^4 q_1][q_1 e q_1][q_0 T^3 q_0] \]

\[ (T_{\text{cut}}) \]

\[ T_C(5) \]

\[ [q_0 T^4 q_0] \equiv [q_1 T^4 q_1] \]

\[ (T_{\text{cut}}) \]

\[ (T_{\epsilon}) \]

\[ \varepsilon \equiv \varepsilon \]
\[ [q_0 A q_1] [q_1 B q_1]^* [q_1 B q_2] + [q_0 A q_2] \equiv [q_1 A q_1] [q_1 B q_1]^* [q_1 B q_2] + [q_1 A q_2] \]

\[ [q_1 B q_1]^* [q_1 B q_2] \equiv [q_1 B q_1] [q_1 B q_1]^* [q_1 B q_2] + [q_1 B q_2] \]

\[ (T_{eq}) \]

\[ \epsilon \equiv \epsilon \]

\[ (T_{\epsilon}) \]
4.4- The system $D_0$ - Strategies

We suppose $X = \{x_1, x_2\}$.

$$\begin{align*}
T_A \\
A_1 \\
A_2 \\
\vdots \\
A_{n-1} \\
(\pi, S, T) \\
\uparrow \quad \downarrow \\
(\pi + 1, S \otimes x_1, T \otimes x_1) \quad (\pi + 1, S \otimes x_2, T \otimes x_2)
\end{align*}$$
\[ T^+_B \]

\[ (\pi, S, \sum_{q \in Q} [\bar{p}zq] \cdot V_q) \]

\[ \downarrow \]

\[ (\pi + k_1 - 1, S \odot u, \sum_{q \in Q'} ([\bar{p}zq] \odot u) \cdot (S \odot u_q) \]

where \( Q' = \{ q \in Q \mid [\bar{p}zq] \neq \emptyset \} \) and \( u_q = \min(\varphi([\bar{p}zq])) \)
\[
[q_0 T^4 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_0 T^4 Aq_2] \equiv [q_1 T^4 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_1 T^4 Aq_2] \\
\downarrow t \\
T_{cut}(4)
\]

\[
[q_0 T^6 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_0 T^6 Aq_2] \equiv [q_1 T^6 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_1 T^6 Aq_2] \\
\downarrow t \\
\downarrow \bar{t} \\
(T_{cut})
\]

\[
[q_0 T^6 Aq_1][q_1 Bq_1]^* [q_1 Bq_2] + [q_0 T^6 Aq_2] \equiv [q_1 T^4 q_1]\left(\{q_0 T^2 Aq_1\}[q_1 Bq_1]^* [q_1 Bq_2] + [q_0 T^2 Aq_2]\right) (3) \\
\downarrow T_B^+(2) \\
\downarrow T_C(3) \\
[q_0 T^4 q_0] \equiv [q_1 T^4 q_1] (4) \\
\downarrow t \\
\downarrow \bar{t} \\
(T_{cut})
\]

\[
[q_0 T^7 q_0] \equiv [q_1 T^7 q_1] \\
\downarrow T_B^+(4) \\
[q_0 T^7 q_0] \equiv [q_1 T^4 q_1][q_1 \epsilon q_1][q_0 T^3 q_0] (5) \\
\downarrow T_C(5) \\
[q_0 T^4 q_0] \equiv [q_1 T^4 q_1] \\
T_{cut}(4) \\
\downarrow \bar{t} \\
(T_{cut}) \epsilon \equiv \epsilon (T_{\epsilon})
$T_C$: an example.

\[
\begin{align*}
\pi, \alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 & \equiv \beta_1 S_1 + \beta_2 S_2 + \beta_3 S_3 \\
\vdots \\
\pi + 3, \gamma_1 S_1 + \gamma_2 S_2 + \gamma_3 S_3 & \equiv \delta_1 S_1 + \delta_2 S_2 + \delta_3 S_3 \\
\downarrow & \\
\pi + 2, \gamma_1 \hat{\beta}_3 + \gamma_3 & \equiv \delta_1 \hat{\beta}_3 + \delta_3 \pi + 2, \gamma_2 \equiv \delta_2 
\end{align*}
\]

We know that first equation $\vdash \pi + 2, S_1 \equiv \hat{\beta}_3 S_3$, hence 
\{first equation, result of $T_C$\} $\vdash$ second equation.
4.4- The system $\mathcal{D}_0$ - Strategies

$T_C$: general case.

\[ \vdots \]
\[ (\pi_1, U_1, V_1) \]
\[ \vdots \]
\[ (\pi_2, U_2, V_2) \]
\[ \vdots \]
\[ (\pi_D, U_D, V_D) \]
\[ \downarrow \downarrow \downarrow \]
\[ \mathcal{E}_1 \ldots \mathcal{E}_j \ldots \mathcal{E}_m \]
\[ [q_0 A \Omega \varphi] \equiv [q_1 A \Omega \varphi] \] (1)

i.e., up to some series equivalent to \( \emptyset \)

\[ [q_0 A q_1] [q_1 \Omega \varphi] + [q_0 A q_2] [q_2 \Omega \varphi] \equiv [q_1 A q_1] [q_1 \Omega \varphi] + [q_1 A q_2] [q_2 \Omega \varphi] \]

\[ [q_0 T A \Omega \varphi] \equiv [q_1 T A \Omega \varphi] \]

\[ [q_1 A \Omega \varphi] \equiv [q_1 B \Omega \varphi] \]

\[ [q_1 \Omega \varphi] \equiv [q_1 \Omega \varphi] \]

\[ [q_1 D \varphi] \equiv [q_2 \Omega \varphi] [q_1 A \Omega \varphi] \equiv [q_1 A \Omega \varphi] \]

\[ [q_0 T T A \Omega \varphi] \equiv [q_1 T T A \Omega \varphi] \]

\[ [q_0 A \Omega \varphi] \equiv [q_1 A \Omega \varphi] e \equiv e \]

\( (T_{cut}) \)

\( (T_e) \)

\[ [q_0 T^3 A \Omega \varphi] \equiv [q_1 T^3 A \Omega \varphi] \]

\( (T_{cut}) \)

\[ [q_0 T^3 A q_1] [q_1 \Omega \varphi] + [q_0 T^3 A q_2] [q_2 \Omega \varphi] \equiv [q_1 T^3 A q_1] [q_1 \Omega \varphi] + [q_1 T^3 A q_2] [q_2 \Omega \varphi] \]

\( (T_C(1)) \)

\[ [q_0 T^3 A q_1] [q_1 B q_1]^* [q_1 B q_2] + [q_0 T^3 A q_2] \equiv [q_1 T^3 A q_1] [q_1 B q_1]^* [q_1 B q_2] + [q_1 T^3 A q_2] \] (2)

\[ [q_0 T^3 A q_1] [q_1 B q_2] + [q_0 T^3 A q_2] \equiv [q_1 T^3 A q_1] [q_1 B q_1]^* [q_1 B q_2] + [q_1 T^3 A q_2] \] (2)
5.1 - Decidability- Terminating strategy

The strategy $S_{ABC}$ is a deterministic strategy, choosing for every branch to stop the construction of that branch or to add some new assertions (by means of $T_A, T_B, T_C$). Transformation $T_C$ has highest priority.

**idea 1**: Successive applications of $T_A, T_B$ on a branch make it smooth: from some point, many assertions have both sides on a given deterministic space

$$V(S_1, \ldots, S_n)$$

with $n$ uniformly bounded (bound $d_0$: exponential)

**idea 2**: on a smooth (hypothetically) infinite branch, $T_C$ must be applicable, with some uniform bound on the size of the result (bound $\Sigma_0 + s_0$: not known to be p.r.)
5.2- Decidability- System $\mathcal{D}_0$ is complete

**Theorem 23** *The system $\mathcal{D}_0$ is complete.*

**Sketch of proof:** From every true assertion $(0, S, T)$, $S_{ABC}$ constructs a proof-tree. If the tree is infinite, then it has some infinite branch. $T_C$ cannot apply i.o. on that branch (because its results are bounded). Hence an infinite suffix is smooth. But $T_C$ is always applicable on such a path, hence $S_{ABC}$ should apply $T_C$: contradiction! Hence the tree (hence the proof) is finite. ☐
Theorem 24 The equivalence problem for deterministic pushdown automata is decidable.

Sketch of proof: Let $S, T \in \mathbb{D}_V$.

1- $S \equiv T$ is semi-decidable: just enumerate the finite $D_0$-proofs, until you find a proof $P$ containing $(0, S, T)$.

2- $S \not\equiv T$ is semi-decidable: just enumerate the words $w \in X^*$, until you find a witness $w$ such that $S \circ w = \varepsilon \Leftrightarrow T \circ w \neq \varepsilon$.

\diamond
6.1 - REFINEMENTS-EXTENSIONS-COMPLEXITY

Theorem 25  *(Stirling, ICALP’02)* The equivalence problem for dpda’s is *primitive recursive.*
Theorem 26  (Sénizergues, FOCS’98) The bisimulation problem for non-deterministic pushdown automata, with deterministic, decreasing \( \epsilon \)-moves, is decidable.

To appear in (SIAM, 2005).
6.3- REFINEMENTS-EXTENSIONS-TRANSDUCERS

Theorem 27 (Sénizergues, ICALP’99) The equivalence problem for deterministic pushdown transducers, from $X^*$, into a free group, is decidable.

Theorem 28 (Sénizergues, ICWL’00, Kyoto) The equivalence problem for deterministic pushdown transducers, from $X^*$, into $(\mathbb{Q}, +, \times)$, over two rational series $S, T$, which are supposed
1- $\sim$-deterministic
2- but might be non-$V$-deterministic rational
is decidable.

Work in progress.
7- PERSPECTIVES

Open question 0:
Is linear independence a **decidable** property ?.

Open problem 1:
Develop the theory of prefix (resp. deterministic) matrices.
(Monoid of square matrices, deterministic spaces)

Open problem 2:
Find a non-trivial **lower-bound** for the complexity of the equivalence problem for dpda.

Open problem 3:
Find a better **upper-bound** for the complexity of the equivalence problem for dpda.
7- PERSPECTIVES

Open problem 4:
Show that the following problem is decidable:
INSTANCE: a deterministic pushdown transducer \( X^* \rightarrow H \) (the quaternions over \( \mathbb{Q} \)), and two rational \( \sim \)-deterministic, non-\( V \)-deterministic series \( S, T \in RH\langle V \rangle \).
QUESTION: Are the series \( S, T \) bisimilar?
7- PERSPECTIVES

i.e. does there exist a binary relation \( \mathcal{R} \) over \( \mathbb{R}^{\langle \mathbb{V} \rangle} \) such that:

1- \( \forall (U, U') \in \mathcal{R}, \forall x \in X \), and for every choice of \( U_1 \in U \circ x \), there exists a choice \( U'_1 \in U' \circ x \) such that \( (U_1, U'_1) \in \mathcal{R} \)

1' - \( \forall (U, U') \in \mathcal{R}, \forall x \in X \), and for every choice of \( U'_1 \in U' \circ x \), there exists a choice \( U_1 \in U \circ x \) such that \( (U_1, U'_1) \in \mathcal{R} \)

2- \( \forall (U, U') \in \mathcal{R}, (U \in \mathbb{I} \Rightarrow U = U') \)

2' - \( \forall (U, U') \in \mathcal{R}, (U' \in \mathbb{I} \Rightarrow U = U') \).