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# Sequences of level $1, 2, 3, \dots, k, \dots$

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## 0-SUBJECT

What is the subject of this talk?

Current work about:

**Pushdown** Automata of Level  $k$  ( $k \geq 0$ ).  
and their **links with**  
**Recurrent** sequences of integers/words.

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## 0-SUBJECT

What are the results of this talk?

### Theorem

Let us consider a mapping  $f : A^* \rightarrow B^*$ . The following properties are equivalent:

1-  $f$  is computed by some **pushdown** automaton of level  $k \geq 2$ .

2-  $f$  is a composition of  $k - 1$  **HDTOL sequences**

$g_1 : A^* \rightarrow C_1^*, \dots, g_i : C_{i-1}^* \rightarrow C_i^*, \dots, g_{k-1} : C_{k-2}^* \rightarrow B^*$ .

Moreover the  $g_1, \dots, g_{k-2}$  can be chosen to be DTOL's.

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## 0-SUBJECT

What are the results of this talk?

### Corollary 1

Let us consider a mapping  $f : A^* \rightarrow \mathbb{N}$ . The following properties are equivalent:

- 1-  $f$  is computed by some **pushdown** automaton of level 3.
- 2-  $f$  is composition of a DTOL sequence  $g : A^* \rightarrow C^*$  by a rational series  $h : C^* \rightarrow \mathbb{N}$ .
- 3- There exists a finite family  $(f_i)_{i \in [1, n]}$  of mappings  $A^* \rightarrow \mathbb{N}$  fulfilling a system of **polynomial recurrent** relations and such that  $f = f_1$ .

### Corollary 2

The **equality problem** is decidable for sequences of rational numbers computed by pushdown automata of level 3.

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## 1.1-INTRODUCTION- AUTOMATA

Some references on pushdown automata of level  $k$ :

**Early works:** (A. Aho JACM 68), (S. Greibach 70), (T. Ayashi 73), (A. Maslov 74)

**Complexity, hierarchy results:** (W. Damm 82), (J. Engelfriet and E.M. Schmidt 77)

**Muchnik's theorem:** (Semenov 84), (Muchnik and Semenov 92), (Walukiewicz 02), (D. Berwanger and A. Blumensath 02)

**Logical properties of  $k$ -dpda:** (D. Caucal 02), (T. Knapik and D. Niwinski and P. Urzyczyn 02), (A. Carayol and S. Wöhrle 03),

**Links with sequences:** (L. Lisovik and T. Karnaukh 03), (S. Fratani and G. Sénizergues 06), (N. Marin 07).

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## 1.1-INTRODUCTION- AUTOMATA

**Definition 1 (*k*-iterated pushdown store)** Let  $\Gamma$  be a set. We define inductively the set of *k*-iterated pushdown-stores over  $\Gamma$  by:

$$0\text{-pds}(\Gamma) = \{\varepsilon\} \quad (k+1)\text{-pds}(\Gamma) = (\Gamma[k\text{-pds}(\Gamma)])^* \quad \text{it-pds}(\Gamma) = \bigcup_{k \geq 0} k\text{-pds}(\Gamma).$$

The elementary operations that a *k*-pda can perform are:

- **pop** of level  $j$  (where  $1 \leq j \leq k$ ), which consists of popping the leftmost letter of level  $j$  and all the structure which is “above” this letter
- **push** of level  $j$  (where  $1 \leq j \leq k$ ), which consists of pushing a new letter  $C$  (or word  $CC'$ ) in place of the leftmost letter  $D$  of level  $j$  and copying above this new letter  $C$  (or letters  $C, C'$ ), all the structure which was “above” the letter  $D$ .



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## 1.1-INTRODUCTION- AUTOMATA

Examples:

$$\omega := S_1[T_1[a_1 a_2 a_3]T_2[a_2 a_1]]S_2[T_2[a_3]T_2[a_2 a_1]] \xrightarrow{\text{push}_1(SS')} S[T_1[a_1 a_2 a_3]T_2[a_2 a_1]]S'[T_1[a_1 a_2 a_3]T_2[a_2 a_1]]S_2[T_2[a_3]T_2[a_2 a_1]].$$

$$S_1[T_1[a_1 a_2 a_3]T_2[a_2 a_1]]S_2[T_2[a_3]T_2[a_2 a_1]] \xrightarrow{\text{push}_2(TT')} S_1[T[a_1 a_2 a_3]T'[a_1 a_2 a_3]T_2[a_2 a_1]]S'[T_1[a_1 a_2 a_3]T_2[a_2 a_1]]S_2[T_2[a_3]T_2[a_2 a_1]].$$

$$S_1[T_1[a_1 a_2 a_3]T_2[a_2 a_1]]S_2[T_2[a_3]T_2[a_2 a_1]] \xrightarrow{\text{push}_3(aa')} S_1[T_1[aa'a_2 a_3]T_2[a_2 a_1]]S_2[T_2[a_3]T_2[a_2 a_1]]$$

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## 1.1-INTRODUCTION- AUTOMATA

$$S_1[T_1[a_1 a_2 a_3] T_2[a_2 a_1]] S_2[T_2[a_3] T_2[a_2 a_1]] \xrightarrow{\text{pop}_1} S_2[T_2[a_3] T_2[a_2 a_1]].$$

$$S_1[T_1[a_1 a_2 a_3] T_2[a_2 a_1]] S_2[T_2[a_3] T_2[a_2 a_1]] \xrightarrow{\text{pop}_2} S_1[T_2[a_2 a_1]] S_2[T_2[a_3] T_2[a_2 a_1]].$$

$$S_1[T_1[a_1 a_2 a_3] T_2[a_2 a_1]] S_2[T_2[a_3] T_2[a_2 a_1]] \xrightarrow{\text{pop}_3} S_1[T_1[a_2 a_3] T_2[a_2 a_1]] S_2[T_2[a_3] T_2[a_2 a_1]]$$

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## 1.1-INTRODUCTION- AUTOMATA

**Definition 2 (*k*-pdas)** Let  $k \geq 1$ , let  $POP = \{\text{pop}_j \mid j \in [k]\}$ ,

$PUSH(\Gamma) = \{\text{push}_j(\gamma) \mid \gamma \in \Gamma^+, j \in [k]\}$ , and

$TOPSYMS(\Gamma) = \Gamma^{\leq k} - \{\varepsilon\}$ .

A *k*-iterated pushdown automaton over a terminal alphabet  $B$  is a 6-tuple  $\mathcal{A} = (Q, B, \Gamma, \delta, q_0, Z_0)$  where

- $Q$  is a finite set of states,  $q_0 \in Q$  denoting the initial state,
- $\Gamma$  is a finite set of pushdown-symbols,  $Z_0 \in \Gamma$  denoting the initial symbol,
- the transition function  $\delta$  is a map from  $Q \times (B \cup \{\varepsilon\}) \times TOPSYMS(\Gamma)$  into the set of finite subsets of  $Q \times (PUSH(\Gamma) \cup POP)$ .

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## 1.1-INTRODUCTION- AUTOMATA

The automaton  $\mathcal{A}$  is said *deterministic* iff, for every  $q \in Q, \gamma \in \Gamma^{\leq k} - \{\varepsilon\}, b \in B$

$$\text{Card}(\delta(q, \varepsilon, \gamma)) \leq 1 \text{ and } \text{Card}(\delta(q, b, \gamma)) \leq 1, \quad (1)$$

$$\text{Card}(\delta(q, \varepsilon, \gamma)) = 1 \Rightarrow \text{Card}(\delta(q, b, \gamma)) = 0. \quad (2)$$

$\mathcal{A}$  is called *strongly deterministic* iff, for every  $q \in Q, \gamma \in \Gamma^{\leq k} - \{\varepsilon\}$

$$\sum_{\bar{b} \in \{\varepsilon\} \cup B} \text{Card}(\delta(q, \bar{b}, \gamma)) \leq 1 \quad (3)$$

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## 1.1-INTRODUCTION- AUTOMATA

**Definition 3 (*k*-computable mapping)** *A mapping  $f : A^* \mapsto B^*$  is called *k*-computable iff there exists a strongly deterministic *k*-pda  $\mathcal{A}$ , over a pushdown-alphabet  $\Gamma$ , such that  $\Gamma$  contains  $k - 1$  symbols  $\gamma_1, \gamma_2, \dots, \gamma_{k-1}$ , the alphabet  $A$  is a subset of  $\Gamma_k$  and for all  $w \in A^*$ :*

$$(q_0, f(w), \gamma_1[\gamma_2 \dots [\gamma_{k-1}[w]] \dots]) \vdash_{\mathcal{A}}^* (q_0, \varepsilon, \varepsilon).$$

*One denotes by  $\mathbb{S}_k(A^*, B^*)$  the set of *k*-computable mappings from  $A^*$  to  $B^*$ .*

The particular case where  $\text{Card}(A) = \text{Card}(B) = 1$  was studied in (Fratani-Senizergues 2006).

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## 1.1-INTRODUCTION- AUTOMATA

**Example 4** *Fibonacci sequence.*

$$F(n + 2) = F(n + 1) + F(n), \quad F(1) = F(0) = 1.$$

*Suppose  $\mathcal{A}$  fulfils the rules:*

$$S_1[\varepsilon] \rightarrow_{\mathcal{A}} b; \quad S_2[\varepsilon] \rightarrow_{\mathcal{A}} \varepsilon;$$

$$S_1[F\Omega] \rightarrow_{\mathcal{A}} S_1[\Omega]S_2[\Omega]; \quad S_2[F\Omega] \rightarrow_{\mathcal{A}} S_1[\Omega].$$

*Then we have*

$$S_1[\varepsilon] \rightarrow_{\mathcal{A}} b = b^{F_0}.$$

$$S_1[F] \rightarrow_{\mathcal{A}} S_1[\varepsilon]S_2[\varepsilon] \xrightarrow{2}_{\mathcal{A}} b\varepsilon = b^{F_1}.$$

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$$\begin{aligned} S_1[F^{n+2}] &\rightarrow_{\mathcal{A}} S_1[F^{n+1}]S_2[F^{n+1}] \\ &\rightarrow_{\mathcal{A}} S_1[F^{n+1}]S_1[F^n] \\ &\rightarrow_{\mathcal{A}}^* b^{F_{n+1}}b^{F_n} \\ &= b^{F_{n+2}}. \end{aligned}$$

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## 1.1-INTRODUCTION- AUTOMATA

**Example 5** *Factorial sequence.*

$u(n + 1) = (n + 1) \cdot u(n)$ ,  $u(0) = 1$ . Suppose  $\mathcal{A}$  fulfils the rules:

$$\bar{S}[\varepsilon] \rightarrow_{\mathcal{A}} b; \quad S[T_1[\varepsilon]\Omega] \rightarrow_{\mathcal{A}} \bar{S}[\Omega]; \quad S[T_2[\varepsilon]\Omega] \rightarrow_{\mathcal{A}} \varepsilon$$

$$S[T_2[F\Omega']\Omega] \rightarrow_{\mathcal{A}} S[\Omega]S[T_2[\Omega']\Omega]; \quad S[T_1[F\Omega']\Omega] \rightarrow_{\mathcal{A}} S[T_2[F\Omega']T_1[\Omega']\Omega]$$

Then we have

$$S[T_1[\varepsilon]] \rightarrow_{\mathcal{A}}^2 b = b^{u(0)}.$$

$$S[T_2[F^n]\Omega] \rightarrow_{\mathcal{A}}^* (S[\Omega])^n$$

$$\begin{aligned} S[T_1[F^{n+1}]] &\rightarrow_{\mathcal{A}}^* S[T_2[F^{n+1}]T_1[F^n]] \\ &\rightarrow_{\mathcal{A}}^* (S[T_1[F^n]])^{n+1} \\ &\rightarrow_{\mathcal{A}}^* (b^{u(n)})^{n+1} \\ &= b^{u(n+1)}. \end{aligned}$$



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## 1.2-INTRODUCTION- RECURRENCES

Some references on recurrent sequences of integers/words:

[Linear recurrences](#): (J. Berstel and C. Reutenauer 88)

[P-recurrences](#):(R.P. Stanley 80), (M. Petkovšek, H.S. Wilf and D. Zeilberger 96)

[Sequences and finite automata](#):(J.P. Allouche and J. Shallit 03)

[L-systems](#):(L. Kari and G. Rozenberg and A. Salomaa 97)

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## 1.2-INTRODUCTION- RECURRENCES

**Definition 6 ( $\mathbb{N}$ -rational formal power series)** *A mapping  $f : A^* \rightarrow \mathbb{N}$  is  $\mathbb{N}$ -rational iff there is an homomorphism  $M : A^* \rightarrow \mathbb{N}^{d \times d}$  and two vectors  $L$  in  $\mathbb{B}^{1 \times d}$  and  $T$  in  $\mathbb{B}^{d \times 1}$  such that, for every  $w \in A^*$*

$$f(w) = L \cdot M(w) \cdot T.$$

**Definition 7 (Polynomial recurrent relations)** *Given a finite index set  $I = [1, n]$  and a family of mappings  $f_i : A^* \rightarrow \mathbb{N}$  (for  $i \in I$ ), we call  $\mathbb{N}$ -system of polynomial recurrent relations a system of the form*

$$f_i(aw) = P_{i,a}(f_1(w), f_2(w), \dots, f_n(w)) \text{ for all } i \in I, a \in A, w \in A^*$$

where  $P_i \in \mathbb{N}[X_1, X_2, \dots, X_n]$ .

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## 1.2-INTRODUCTION- RECURRENCES

**Definition 8 (catenative recurrent relations)** *Given a finite set  $I$  and a family of mappings indexed by  $I$ ,  $f_i : A^* \rightarrow B^*$  (for  $i \in I$ ), we call **system of catenative recurrent relations** over the family  $(f_i)_{i \in I}$  a system of the form*

$$f_i(aw) = \prod_{j=1}^{\ell(i,a)} f_{\alpha(i,a,j)}(w) \text{ for all } i \in I, a \in A, w \in A^*$$

where  $\ell(a, i) \in \mathbb{N}$ ,  $\alpha(i, a, j) \in I$ .

When  $\text{Card}(B) = 1$ ,  $f : A^* \rightarrow B^*$  fulfils a system of catenative recurrent relations iff  $f$  is a rational series.

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## 1.2-INTRODUCTION- RECURRENCES

**Definition 9 (recurrent relations in  $M$ )** Given a finite set  $I$  and a family of mappings indexed by  $I$ ,  $f_i : A^* \rightarrow M$  (for  $i \in I$ ), we call **system of recurrent relations in  $M$**  over the family  $(f_i)_{i \in I}$ , a system of the form

$$f_i(aw) = \prod_{j=1}^{\ell(i,a)} f_{\alpha(i,a,j)}(w) \text{ for all } i \in I, a \in A, w \in A^*$$

where  $\ell(a, i) \in \mathbb{N}$ ,  $\alpha(i, a, j) \in I$  and the symbol  $\prod$  stands for the extension of the binary product in  $M$  to an arbitrary finite number of arguments.

The monoid  $(Hom(C^*, C^*), \circ, Id)$  will be of particular interest for studying mappings of level  $k \geq 3$ .

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## 1.2-INTRODUCTION- RECURRENCES

The following notion turns out to be crucial for describing all  $k$ -computable mappings as compositions of simpler mappings.

**Definition 10 (Kari-Rozenberg-Salomaa 1997)** *Let  $f : A^* \rightarrow B^*$ . The mapping  $f$  is called a **HDTOL sequence** iff there exists a finite alphabet  $C$ , a homomorphism  $H : A^* \rightarrow \text{Hom}(C^*, C^*)$ , an homomorphism  $h \in \text{Hom}(C^*, B^*)$  and a letter  $c \in C$  such that, for every  $w \in A^*$*

$$f(w) = h(H^w(c)).$$

(here we denote by  $H^w$  the image of  $w$  by  $H$ ).

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## 1.3-INTRODUCTION- PROBLEMS

**Problem 1:** Find **characterisations** of  $k$ -computable sequences.

**Problem 2:** Solve the **equality** problem for  $k$ -computable sequences. i.e. the following decision problem:

INPUT: two s-deterministic  $k$ -automata  $\mathcal{A}, \mathcal{B}$ .

QUESTION: Do these automata compute the same mapping  $A^* \rightarrow B^*$  ?

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## 2.1-CLOSURE PROPERTIES-RECURRENCES

### Theorem 11 (Fratani-Sénizergues 06)

- 0- For every  $f \in \mathbb{S}_{k+1}$ ,  $k \geq 1$ , and every integer  $c \in \mathbb{N}$ , the sequences  $Ef$  (the shift of  $f$ ),  $f + \frac{c}{1-X}$ , belong to  $\mathbb{S}_{k+1}$ ; if  $\forall n \in \mathbb{N}$ ,  $f(n) \geq c$  then  $f - \frac{c}{1-X}$  belongs to  $\mathbb{S}_{k+1}$ ; the sequence defined by  $0 \mapsto c, n + 1 \mapsto f(n)$  belongs to  $\mathbb{S}_{k+1}$ .
- 1- For every  $f, g \in \mathbb{S}_{k+1}$ ,  $k \geq 1$ , the sequence  $f + g$  belongs to  $\mathbb{S}_{k+1}$ .
- 2- For every  $f, g \in \mathbb{S}_{k+1}$ ,  $k \geq 2$ ,  $f \odot g$  (the *ordinary product*), belongs to  $\mathbb{S}_{k+1}$  and for every  $f' \in \mathbb{S}_{k+2}$ ,  $f'^g$  belong to  $\mathbb{S}_{k+2}$ .
- 3- For every  $f \in \mathbb{S}_{k+1}$ ,  $g \in \mathbb{S}_k$ ,  $k \geq 2$ ,  $f \times g$  (the convolution product) belongs to  $\mathbb{S}_{k+1}$ .
- 4- For every  $g \in \mathbb{S}_k$ ,  $k \geq 2$ , the sequence  $f$  defined by:  $f(0) = 1$  and  $f(n + 1) = \sum_{m=0}^n f(m) \cdot g(n - m)$  (the convolution inverse of  $1 - X \times f$ ) belongs to  $\mathbb{S}_{k+1}$ .

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5- For every  $k \geq 2$  and for every *system of recurrent equations expressed by polynomials* in  $\mathbb{S}_{k+1}[X_1, \dots, X_p]$ , with initial conditions in  $\mathbb{N}$ , every solution belongs to  $\mathbb{S}_{k+1}$ .



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## 2.2-CLOSURE PROPERTIES-COMPOSITION

### Theorem 12

For every  $f \in \mathbb{S}_k(A^*, B^*)g \in \mathbb{S}_\ell(B^*, C^*)$ ,  $k, l \geq 2$ ,  $f \circ g$  (the sequence *composition*) belongs to  $\mathbb{S}_{k+\ell-1}(A^*, C^*)$ .

Proved in (Fra-Sen06) in the case where  $\text{Card}(A) = \text{Card}(B) = \text{Card}(C) = 1$ . But extension is straightforward.

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## 3.1-DECOMPOSITION PROPERTIES-LEVEL 1

**Theorem 13** *The elements of  $\mathbb{S}_1(A^*, B^*)$  are exactly the generalized sequential mappings from  $A^*$  to  $B^*$ .*

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## 3.2-DECOMPOSITION PROPERTIES-LEVEL 2

**Theorem 14 (N. Marin 2007)** *Let us consider a mapping  $f : A^* \rightarrow B^*$ . The following properties are equivalent:*

1-  $f \in \mathbb{S}_2(A^*, B^*)$

2- *There exists a finite family  $(f_i)_{i \in [1, n]}$  of mappings  $A^* \rightarrow B^*$  which fulfils a system of *catenative recurrent* relations and such that  $f = f_1$*

3-  $f$  is a *HDTOL* sequence.

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

**Theorem 15** *Let us consider a mapping  $f : A^* \rightarrow B^*$ . The following properties are equivalent:*

- 1-  $f \in \mathbb{S}_3(A^*, B^*)$
- 2-  $f$  is a *composition* of a DTOL sequence  $g : A^* \rightarrow C^*$  by a HDTOL sequence  $h : C^* \rightarrow B^*$ .

**Corollary 16** *Let us consider a mapping  $f : A^* \rightarrow \mathbb{N}$ . The following properties are equivalent:*

- 1-  $f \in \mathbb{S}_3(A^*, \mathbb{N})$
- 2-  $f$  is composition of a DTOL sequence  $g : A^* \rightarrow C^*$  by a rational series  $h : C^* \rightarrow \mathbb{N}$ .
- 3- There exists a finite family  $(f_i)_{i \in [1, n]}$  of mappings  $A^* \rightarrow \mathbb{N}$  fulfilling a system of *polynomial recurrent* relations and such that  $f = f_1$ .

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

**Definition 17** Let  $\mathcal{S}$  be a set of mappings  $A^* \rightarrow \mathbb{N}$ . We denote by  $\mathcal{D}(\mathcal{S})$  the set of mappings of the form:

$$f(w) = g(w) - h(w) \quad \text{for all } w \in A^*,$$

for some mappings  $g, h \in \mathcal{S}$ . We denote by  $\mathcal{F}(\mathcal{S})$  the set of mappings of the form:

$$f(w) = \frac{g(w) - h(w)}{f'(w) - g'(w)} \quad \text{for all } w \in A^*,$$

for some mappings  $f, g, f', g' \in \mathcal{S}$ .

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

Using point 3 of corollary 16 we can prove the following

**Theorem 18** *The **equality problem** is decidable for sequences in  $\mathcal{F}(\mathbb{S}_3(A^*, \mathbb{N}))$ .*

The method consists, in a way similar to (Senizergues, ICALP'99) or (Honkala, 2000), in reducing such an equality problem to deciding whether some polynomial belongs to the ideal generated by a finite set of other polynomials.

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

Main [ideas](#) in the proof of theorem 15.

Idea 1: for every  $u \in 2 - \text{pds}(\Gamma)$

$$\Phi(u) : (pSq) \mapsto \prod_{i=1}^n (p_i S_i q_i)$$

iff

$$pS[u\Omega]q \rightarrow_{\mathcal{A}}^* \prod_{i=1}^n (p_i S_i [\Omega] q_i).$$

If :

$$p_i S[v\Omega]q_i \rightarrow_{\mathcal{A}}^* \prod_{j=1}^{n_i} (r_{i,j} S'_{i,j} [\Omega] s_{i,j}),$$

then

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$$pS[u \cdot v \cdot \Omega]q \xrightarrow{*} \mathcal{A} \prod_{i=1}^n \prod_{j=1}^{n_i} (r_{i,j} S'_{i,j}[\Omega] s_{i,j}).$$

i.e.

$$\Phi(u \cdot v) = \Phi(u) \circ \Phi(v).$$

Based on this observation: define, for  $w \in \Gamma^*$ ,

$$H_T^w := \Phi(T[w]).$$

Family  $(H_T^w)_{T \in \Gamma}$  should fulfill a **compositional** recurrence.



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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

$$\text{push}_2: pS[T[w]]q \rightarrow_{\mathcal{A}} p_1S[T_1[w]T_2[w]]q$$

$$\Phi(T[w])(pSq) = \Phi(T_1[w]T_2[w])(p_1Sq)$$

$$\Phi(T[w])(pSq) = (\Phi(T_1[w]) \circ \Phi(T_2[w]))(p_1Sq)$$

i.e.

$$H_T^w(pSq) = (H_{T_1}^w \circ H_{T_2}^w)(pSq)$$

$$\text{push}_1: pS[T[w]]q \rightarrow_{\mathcal{A}} p_1S_1[T[w]]r \cdot rS_2[T[w]]q$$

$$H_T^w(pSq) = H_{T_1}^w(p_1S_1r) \cdot H_{T_2}^w(rS_2q)$$

Mixed recurrence rules .

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

Let us make definition of  $H_T^w$  correct.

**Definition 19 (equivalences associated with a  $k$ -pda)** *Let  $\mathcal{A} = (Q, B, \Gamma, \delta, q_0, Z_0)$  be a  $k$ -iterated pushdown automaton, with  $k \geq 3$ . We define binary relations  $\equiv_{\mathcal{A},i}$  by:*

1- for every  $w, w' \in k - \text{pds}(\Gamma)$ ,

$$w \equiv_{\mathcal{A},1} w' \Leftrightarrow (\forall p, q \in Q, L(\mathcal{A}, pwq) \neq \emptyset \Leftrightarrow L(\mathcal{A}, pw'q) \neq \emptyset)$$

2- for every  $w, w' \in (k - 1) - \text{pds}(\Gamma)$ ,

$$w \equiv_{\mathcal{A},2} w' \Leftrightarrow (\forall S \in \Gamma, \forall v \in (k - 1) - \text{pds}(\Gamma), S[w \cdot v] \equiv_{\mathcal{A},1} S[w' \cdot v])$$

3- for every  $w, w' \in (k - 2) - \text{pds}(\Gamma)$ ,

$$w \equiv_{\mathcal{A},3} w' \Leftrightarrow (\forall T \in \Gamma, T[w] \equiv_{\mathcal{A},2} T[w'])$$

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

**Lemma 20** *Let  $\mathcal{A} = (Q, B, \Gamma, \delta, q_0, Z_0)$  be a  $k$ -iterated pushdown automaton, with  $k \geq 3$ .*

- 1- *For every  $j \in \{1, 2, 3\}$ ,  $\equiv_{\mathcal{A},j}$  is a **regular** equivalence relation.*
- 2- *For every  $j \in \{1, 2\}$ ,  $\equiv_{\mathcal{A},j}$  is a **congruence**.*

1- Each class is definable by a MSO formula  $\Phi(w)$ , over  $(k - j + 1) - \text{pds}(\Gamma)$ . We then use the fact that the **definable** subsets of  $k - \text{pds}(\Gamma)$  are exactly the  **$k$ -recognizable** subsets of  $k - \text{pds}(\Gamma)$  ((Fratani, PHD, 2005)).

2- Easy.

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

Correct definition of  $H_t^w$ :

$$\mathcal{W} := \{(pS[e]q) \mid p, q \in Q, S \in \Gamma_1, e \in 2 - \text{pds}(\Gamma) / \equiv_{\mathcal{A},2}\}.$$

If  $W = (pS[e]q)$  and

$$(pS[T[w]\Omega]q) \rightarrow_{\mathcal{A}}^+ \prod_{j=1}^{\ell(i,W)} p_{i,j} S_{i,j}[\Omega] q_{i,j} \quad (4)$$

and

$$\forall j \in [1, \ell(i, W)], \forall t \in e \quad L(p_{i,j} S_{i,j}[t] q_{i,j}) \neq \emptyset \quad (5)$$

then

$$(pS[e]q) \mapsto \prod_{j=1}^{\ell(i,W)} p_{i,j} S_{i,j}[e] q_{i,j}. \quad (6)$$

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otherwise

$$(pS[e]q) \mapsto (pS[e]q). \quad (7)$$

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

We obtain a recurrence of the form:

Let  $T \in \Gamma$ ,  $d \in 1 - \text{pds}(\Gamma) / \equiv_3$ ,  $a \in \Gamma$ ,  $w \in 1 - \text{pds}(\Gamma)$  and  $W = pS[e]q$ . There exists an integer  $\ell(i, a, d, W) \in [1, 2]$  and for every  $j \in [1, \ell(i, a, d, W)]$  there exist indices  $\alpha(i, a, d, W, j), \beta(i, a, d, W, j) \in \Gamma$ , words  $\omega_{i,a,d,W,j} \in \Gamma^{\leq 2}$ , letters  $V_{i,a,d,W,j} \in \mathcal{W}$  and an alphabetic homomorphism  $\Phi_{i,a,d,W,j} \in \text{Hom}(\mathcal{W}^*, \mathcal{W}^*)$  such that

$$w \in d \Rightarrow H_i^{aw}(W) = \prod_{j=1}^{\ell(i,a,d,W)} H_{\alpha(i,a,d,W,j)}^{\omega_{i,a,d,W,j} \cdot w} \circ \Phi_{i,a,d,W,j} \circ H_{\beta(i,a,d,W,j)}^{\omega_{i,a,d,W,j} \cdot w} (V_{i,a,d,W,j}) \quad (8)$$

Idea 2: transform this mixed recurrence into a **compositional** recurrence

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

Here is a **mixed** recurrence for  $f(n) := n!$

$$\Phi(n) := \lambda u.(n! \times u); \quad P(n) := \lambda u.(n \times u)$$

$$\Phi(n + 1) = P(n) \circ \Phi(n)$$

$$P(n + 1) = P(n) + I$$

$$\Phi(0) = I$$

$$P(0) = Z$$

where  $I := \lambda u.u$  and  $Z := \lambda u.0$ .

$(\Phi, P, I, Z \in HOM(\{F_1\}^*, \{F_1\}^*)).$

$$f(n) = \Phi(n)(1).$$

---

### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

Let  $\mathcal{W} := \{F_1, F_2\}$ .

$$H : F_1 \mapsto F_1F_2; \quad F_2 \mapsto \varepsilon, \quad \Pi_{2,1} : F_1 \mapsto F_1; \quad F_2 \mapsto F_1$$

$$\Phi_1(n) : F_1 \mapsto F_1^{n!}, F_2 \mapsto F_2; \quad P_1(n) := F_1 \mapsto F_1^n, F_2 \mapsto F_2.$$

Here is a [compositional](#) recurrence for  $n!$

$$\Phi_1(n+1) = P_1(n) \circ \Phi_1(n)$$

$$P_1(n+1) = \pi_{2,1} \circ P_1(n) \circ H(n)$$

$$\Phi_1(0) = I(n)$$

$$P_1(0) = Z(n)$$



---

### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

By this kind of trick we obtain a recurrence of the form:

$w \in d \Rightarrow$

$$H_T^{aw} = \psi_{i,a,d} \circ \prod_{W \in \mathcal{W}} \left( \prod_{j=1}^{\ell(i,a,d,W)} \theta_{i,a,d,W,j} \circ H_{\alpha(i,a,d,W,j)}^{\omega_{i,a,d,W,j} \cdot w} \circ \Phi_{i,a,d,W,j} \circ H_{\beta(i,a,d,W,j)}^{\omega_{i,a,d,W,j}} \right) \circ \hat{\psi}_{i,a,d}.$$

Difficult point: is this compositional recurrence **terminating** ?

In fact **no**.

idea 3: new index  $i = (\mathcal{V}, \mathcal{T})$  where  $\mathcal{V} \subseteq \mathcal{W}$ .

$$H_i^w(W) := H_T(W) \text{ if } W \in \mathcal{V}, \quad H_i^w(W) := W \text{ if } W \notin \mathcal{V}.$$

Termination proof is technical.

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

Last technical point: eliminate the congruence  $\equiv_{\mathcal{A},3}$  over  $\Gamma^*$ , in the compositional recurrence?

Solution: (standard direct) **product of  $\mathcal{A}$**  (which computes the recurrence) **by  $\mathcal{B}$**  which computes the finite index congruence.

Finally: we have proved that (1)  $\Rightarrow$  (2).

Converse: follows from composition theorem 12.

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### 3.3-DECOMPOSITION PROPERTIES-LEVEL 3

Main **idea** in the proof of corollary 16.

Example:  $u(n) = F(F(n))$

$F(n) : 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 \dots$

$u(n) : 1, 1, 2, 3, 8, 34, 377, \dots$

$$\begin{pmatrix} F(n+1) \\ F(n) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that

$$\begin{pmatrix} F(F(n)+1) \\ u(n) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{F(n)} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

---

Let us set  $\begin{pmatrix} e(n) & f(n) \\ g(n) & h(n) \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{F(n)}$

then

$$u(n) = g(n)$$

Since

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{F(n+1)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{F(n)} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{F(n-1)}$$

we obtain the relation:

$$\begin{pmatrix} e(n+1) & f(n+1) \\ g(n+1) & h(n+1) \end{pmatrix} = \begin{pmatrix} e(n) & f(n) \\ g(n) & h(n) \end{pmatrix} \cdot \begin{pmatrix} e(n-1) & f(n-1) \\ g(n-1) & h(n-1) \end{pmatrix}$$

which gives a **polynomial** recurrence system defining  $u(n)$ .

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### 3.4-DECOMPOSITION PROPERTIES-LEVEL $k \geq 4$

**Theorem 21** *Let us consider a mapping  $f : A^* \rightarrow B^*$ . The following properties are equivalent:*

1-  $f \in \mathbb{S}_k(A^*, B^*)$

2-  $f$  is a *composition* of  $k - 1$  HDTOL sequences

$g_1 : A^* \rightarrow C_1^*, \dots, g_i : C_{i-1}^* \rightarrow C_i^*, \dots, g_{k-1} : C_{k-2}^* \rightarrow B^*$ .

Moreover the  $g_1, \dots, g_{k-2}$  can be chosen to be DTOL's.

---

### 3.4-DECOMPOSITION PROPERTIES-LEVEL $k \geq 4$

Main [idea](#) in the proof of theorem 21.

**Definition 22 (regular recurrent relations of order  $k$ )** *Let us consider a monoid  $(M, \cdot, 1)$ , a finite alphabet  $\Gamma$ , a finite set  $I$  and a family of mappings indexed by  $I$ ,  $f_i : k - \text{pds}(\Gamma) \rightarrow M$  (for  $i \in I$ ). We call [system of regular recurrent relations of order  \$k\$  in  \$M\$](#)  over the family  $(f_i)_{i \in I}$ , a system of equations of the form:*

$$\forall i \in I, c \in k - \text{pds}(\Gamma) / \equiv, w \in k - \text{pds}(\Gamma),$$

$$(w \neq \varepsilon \ \& \ w \in c) \Rightarrow f_i(w) = \prod_{j=1}^{\ell(i,c)} f_{\alpha(i,c,j)}(\omega_{i,c,j}(w)) \quad (10)$$

where  $\ell(i, c) \in \mathbb{N}$ ,  $\alpha(i, c, j) \in I$ ,  $\omega_{i,c,j} \in \text{PUSH}(\Gamma) \cup \text{POP}$  and  $\equiv$  is a regular equivalence relation over  $k - \text{pds}(\Gamma)$ .

---

## 3.4-DECOMPOSITION PROPERTIES-LEVEL $k \geq 4$

Definition of set of indices  $I$  and homomorphisms:

$$H_i^w : \hat{\mathcal{W}}^* \rightarrow \hat{\mathcal{W}}^*$$

(as for level 3). The family  $(H_i^w)_{i \in I}$  fulfils a *system of regular recurrent relations of order  $k$  in  $Hom(\hat{\mathcal{W}}^*, \hat{\mathcal{W}}^*)$* .

New difficulty: how to eliminate the equivalence  $\equiv_{\mathcal{A},3}$  over  $(k-2) - \text{pds}(\Gamma)$  ?

Solution: **product of  $\mathcal{A}$**  (which computes the recurrence) **by  $\mathcal{B}$**  which computes the finite index regular equivalence over  $(k-2) - \text{pds}(\Gamma)$  (Fratani 2005, PHD).

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## 4.1-PERSPECTIVES-AUTOMATA

P1: Study the equivalence problem for [deterministic](#) automata of level 2,3.



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## 4.2-PERSPECTIVES-RECURRENCES

P2: Study the equivalence problem for **strongly deterministic** automata of level 3

P3: Study the decision problem:

INPUT: one sequence  $u \in \mathcal{D}(\mathbb{S}_k(A^*, \mathbb{N}))$ .

QUESTION: Do there **exist** a word  $w \in A^*$  such that  $u(w) = 0$ ?

For  $k = 2$ ,  $\text{Card}(A) = 1$ , known as the Pisot problem and open.

For  $k = 3$ ,  $\text{Card}(A) > 1$ , undecidable.

P4: Study the decision problem:

INPUT: two sequences  $u, v \in \mathbb{S}_k(A^*, \mathbb{N})$ .

QUESTION: Are they **almost** equal ?

i.e.  $\text{Card}\{w \in A^* \mid u(w) \neq v(w)\} < \infty$ ?

For  $k = 2$ ,  $\text{Card}(A) = 1$ , easily seen decidable.

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P5: Is  $\mathcal{S}_3(\mathbb{N}, \mathbb{N})$  closed under **convolution** ?  
(true for  $\mathcal{S}_2(\mathbb{N}, \mathbb{N})$  ).

P6: Study the **closure properties** of  $\mathcal{S}_k(\mathbb{N}, \mathbb{N}), \mathcal{S}_k(A^*, \mathbb{N}),$   
 $\mathcal{D}(\mathcal{S}_k(\mathbb{N}, \mathbb{N})), \mathcal{F}(\mathcal{S}_k(\mathbb{N}, \mathbb{N})), \dots$