

## LOGICS

### Exercises on Chapter 2: Sequent Calculus

#### Symmetries in the system LK

##### Exercice 2.1 Reversibility

An inference rule  $\frac{S_1}{S_2}$  in a formal system  $\mathcal{S}$  is called *reversible* iff  $\frac{S_2}{S_1}$  is a derived rule of the system  $\mathcal{S}$ . Which rules of system LK are reversible ?

##### Exercice 2.2 Duality

Let us add a connector  $\top$  to the language of system LK i.e. the set of connectors is now  $\{\perp, \wedge, \vee, \rightarrow, \neg\}$ , the set of quantifiers is  $\{\forall, \exists\}$ , we have a denumerable set of relation symbols  $\mathcal{R}$  and a denumerable set of fonction symbols  $\mathcal{F}$ .

To every well-formed formula  $F$ , we associate a *dual* formula  $D(F)$  by structural induction:

$$\begin{aligned} D(\perp) &:= \top, & D(\top) &:= \perp \\ D(R(t_1, \dots, t_k)) &:= R(t_1, \dots, t_k) & \text{pour tout } R \in \mathcal{R} \\ D(F \wedge G) &:= D(F) \vee D(G), & D(F \vee G) &:= D(F) \wedge D(G) \\ D(F \rightarrow G) &:= \neg(D(G) \rightarrow D(F)), & D(\neg F) &:= \neg D(F) \\ D(\forall x F) &:= \exists x D(F), & D(\exists x F) &:= \forall x D(F) \end{aligned}$$

1- Show that, for every formula  $F$ , with atomic subformulas  $A_1, \dots, A_n$ , the following equivalence holds:

$$F(\neg A_1, \dots, \neg A_n) \models \neg D(F)(A_1, \dots, A_n).$$

We extend the map  $F \mapsto D(F)$  to sequents by:

$$D(F_1, \dots, F_n \vdash G_1, \dots, G_m) := D(G_1), \dots, D(G_m) \vdash D(F_1), \dots, D(F_n).$$

2- Show that, if for every structure  $\mathcal{T}$  we have:

$$\mathcal{T} \models S_1 \text{ implies } \mathcal{T} \models S_2$$

then, for every structure  $\mathcal{T}$  we also have:

$$\mathcal{T} \models D(S_1) \text{ implies } \mathcal{T} \models D(S_2).$$

3- What are the rules of LK,  $\frac{S_1}{S_2}$  (where  $S_1, S_2$  are sequents) such that  $\frac{D(S_1)}{D(S_2)}$  is still a rule of LK?

4- Could you manage to add a rule to LK in such a way that it takes into account the extension to connector  $\top$  ? We would like that the sequent  $\vdash (\neg \perp \rightarrow \top) \wedge (\top \rightarrow \neg \perp)$  be derivable in this extension of LK and we would like to keep (or even improve) the symmetries of system LK.

### Examples of derivations

#### Exercise 2.3 Propositional LK

Find a derivation in LK for the sequents:

$$\begin{aligned} &\vdash A \rightarrow (B \rightarrow A) \\ &\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \\ &\vdash ((P \rightarrow Q) \rightarrow P) \rightarrow P \end{aligned}$$

#### Exercise 2.4 LK

Find a derivation in LK for the sequents:

$$\begin{aligned} &\neg \exists x R(x) \vdash \forall x \neg R(x) \\ &\neg \forall x R(x) \vdash \exists x \neg R(x) \\ &\vdash \forall x (Q \vee R(x)) \rightarrow Q \vee \forall x R(x) \\ &\vdash \exists x \forall y (R(y) \rightarrow R(x)) \end{aligned}$$

#### Exercise 2.5 Propositional LJ

Find a derivation in LJ for the sequents:

$$\begin{aligned} &\neg(A \vee B) \vdash \neg A \wedge \neg B \\ &\neg A \wedge \neg B \vdash \neg(A \vee B) \\ &A \vdash \neg \neg A \\ &\neg \neg \neg A \vdash \neg A \end{aligned}$$

#### Exercise 2.6 LJ

1- What do you think about the following “derivation”?

$$\begin{array}{ll} R(x) \vdash R(x) & \mathbf{ax} \\ \exists x R(x) \vdash R(x) & \exists_g \\ \neg R(x), \exists x R(x) \vdash & \neg_g \\ \forall x \neg R(x), \exists x R(x) \vdash & \forall_g \\ \forall x \neg R(x) \vdash \neg \exists x R(x) & \neg_d \end{array}$$

2- Does the sequent  $\forall x \neg R(x) \vdash \neg \exists x R(x)$  admit a derivation in LJ ?

### Properties of derivations

#### Exercise 2.7 Right-elimination

Let us consider the following right-elimination rule:

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$$

(where  $\Gamma$  is a multi-set of formulas and  $A, B$  are formulas).

1- Is this rule a derived rule of LK ?

2- Is this rule a derived rule of  $\text{LK} \setminus \{\text{cut}\}$  ?

#### Exercise 2.8 Contractions

Let us consider the sequent  $S := \exists x (R(a) \vee R(b) \rightarrow R(x))$ .

1- Find a derivation of  $S$  in LK.

2- Find a cut-free derivation of  $S$  in LK.

3- Prove that there does not exist any derivation, without neither cut nor contraction, of  $S$  in LK.

4- Does there exist some derivation, without contraction, of  $S$  in LK ?