

LOGICS

TD 3 : Cut elimination

Exercice 3.1 Harrop

1- Let us consider the theories (i.e. sets of formulas) : EG, P₀, PA, MO, GR. Which ones are Harrop theories ? What (interesting) conclusions can we draw from this ?

Exercice 3.2 PA versus P₀

1- Let us denote by P'₀ (resp. PA') the set of formulas of P₀ (resp. PA) the axiom A2 excepted. Is PA' a Harrop theory ?

2- Give a derivation, within LK, of PA' ⊢ A2.

3- Can you give a derivation within LJ of PA' ⊢ A2 ? does there exist a term *t*, and a proof within LK of PA' ⊢ $x = 0 \vee x = S(t)$?

Exercice 3.3 A non-standard model for P'₀.

Let $\mathcal{M} =_{\text{d\u00e9f.}} \langle \mathcal{N}, 0_{\mathcal{N}}, S_{\mathcal{N}}, +_{\mathcal{N}}, \times_{\mathcal{N}} \rangle$ the following « arithmetical » structure :

$$\mathcal{N} =_{\text{d\u00e9f.}} (\mathbb{N} \times \{\bullet\}) \cup (\mathbb{N} \times \{\circ\}) \text{ where } \bullet \neq \circ$$

$$0_{\mathcal{N}} =_{\text{d\u00e9f.}} \langle 0, \bullet \rangle$$

$$S_{\mathcal{N}} \langle p, \alpha \rangle =_{\text{d\u00e9f.}} \langle Sp, \alpha \rangle \text{ where } \alpha \in \{\bullet, \circ\}$$

$$\langle p, \alpha \rangle +_{\mathcal{N}} \langle q, \beta \rangle =_{\text{d\u00e9f.}} \langle p + q, \alpha \rangle \text{ where } \alpha, \beta \in \{\bullet, \circ\}$$

$$\langle p, \alpha \rangle \times_{\mathcal{N}} \langle q, \beta \rangle =_{\text{d\u00e9f.}} \langle p \times q, \beta \rangle \text{ where } \alpha, \beta \in \{\bullet, \circ\}$$

The domain of \mathcal{N} consists of two copies of \mathbb{N} , the « black » integers $\langle p, \bullet \rangle$ and the « white » integers $\langle p, \circ \rangle$. The zero constant is interpreted by the black zero $\langle 0, \bullet \rangle$; the operations successor, addition and multiplication are interpreted in such a way that :

- the successor keeps the color of its argument,
- the addition takes the la color of its *first* argument,
- the multiplication takes the color of its *second* argument.

1. Show that \mathcal{M} is a model for P'₀. is it a model for P₀ ?
2. Show that none of the following properties is a logical consequence of P'₀ :

$$\forall x (0 + x = x) \tag{1}$$

$$\forall x, y (x + y = y + x) \tag{2}$$

$$\forall x, y, z (x + y = x + z \rightarrow y = z) \tag{3}$$

$$\forall x (x \times S0 = x) \tag{4}$$

$$\forall x, y (x \times y = y \times x) \tag{5}$$

Exercice 3.4 Heyting arithmetics

Let Φ be some first-order formula over the signature of arithmetics.

- 1- Show that, if $P'_0 \vdash \forall x \Phi$ is derivable within LJ, then $P'_0 \vdash \Phi$ is derivable within LJ.
- 2- does the property shown in question 1 remain true if we take PA as left-part of the sequent ? or if we still take the same sequent but consider the formal system LK ?
- 3- Show that, if $P'_0 \vdash \forall x, \exists y \Phi(x, y)$ is derivable within LJ, then, there exists some terme t , such that $P'_0 \vdash \Phi(x, t)$ is derivable within LJ.
- 4- Does the property of question 3 hold true if we take, as left-hand side of the sequent, PA ? or if we still take the same sequent but consider the formal system LK ?
- 5- Assume that $PA' \vdash \forall x, \exists y \Phi(x, y)$ is derivable within LJ, by “ only one recurrence” i.e.

$$P'_0 \vdash_{LJ} \exists y \Phi(0, y); \quad P'_0 \vdash_{LJ} (\exists y \Phi(x, y)) \rightarrow (\exists y \Phi(S(x), y))$$

- 5.1- Check that, under these assumptions, there does exist in LJ a derivation of $PA' \vdash \forall x, \exists y \Phi(x, y)$. For every $n \in \mathbb{N}$, we denote by \underline{n} the term $S(S(\dots(S(0))\dots))$ which represents the integer n within the language of Peano arithmetics.
- 5.2- Show that, for every integer $n \in \mathbb{N}$, there edists a term t_n such that $P'_0 \vdash_{LJ} \Phi(\underline{n}, t_n)$.
- 5.3- Give an algorithm that computes the function $n \mapsto t_n$ and which relies on the cut elimination algorithm.

Let us now admit that the property shown in exercice 4, question 5, is still valid for every formula of the form $\forall x, \exists y \Phi(x, y)$ (whether the derivation uses one recurrence or more).

Exercice 3.5 Recursive functions

We are thinking about the possibility of a converse of the above-admitted statement. Let us make the assumption(**ASSUMP**) for every computable, total, function $f : \mathbb{N} \rightarrow \mathbb{N}$, there exists a formula $\Phi(x, y)$ such that

- (P1) $PA \vdash \forall x, \exists_1 y \Phi(x, y)$ is derivable within LJ
 where $\exists_1 x F(x)$ abbreviates $\exists x F(x) \wedge (\forall y, z F(y) \wedge F(z) \rightarrow y = z)$
 (P2) for every integers n, m , $\mathbb{N} \models \Phi(\underline{n}, \underline{m})$ if and only if $f(n) = m$.

- 1- Can we deduce from this assumption an effective enumeration of all computable, total, functions ?
- 2- Find a diagonal argument showing that **ASSUMP** is false.

Let us admit the theorem [Matiyasevich, 1971] : a subset $M \subseteq \mathbb{N}$ is recursively enumerable iff, there exists an integer $q \in \mathbb{N}$ and a polynomial $P \in \mathbb{Z}[X, Y_1, \dots, Y_q]$ such that, for every $x \in \mathbb{N}$,

$$x \in M \Leftrightarrow \exists \vec{y} \in \mathbb{N}^q, P(x, \vec{y}) = 0.$$

Exercice 3.6 PA is undecidable

Let q be some natural integer and P be a polynomial in $\mathbb{Z}[X, Y_1, \dots, Y_q]$.

- 1- Show that, if $\mathbb{N} \models \exists \vec{y} P(\underline{n}, \vec{y}) = 0$.
 then, there exists some natural integers vector \vec{m} such that $PA \vdash_{LK} P(\underline{n}, \vec{m}) = 0$.
- 2- Show that, if $PA \vdash_{LK} \exists \vec{y} P(\underline{n}, \vec{y}) = 0$, then $\mathbb{N} \models \exists \vec{y} P(\underline{n}, \vec{y}) = 0$.
- 3- Show that there exists a polynomial P such that the problem
 Instance : $n \in \mathbb{N}$; Question : does there exist a vector $\vec{m} \in \mathbb{N}^q$ such that $P(n, \vec{m}) = 0$?
 is undecidable.
- 4- Show that the following problem is undecidable too :
 Instance : a formula Φ ; Question : $PA \vdash_{LK} \Phi$?