

Preferential Attachment as a Unique Equilibrium

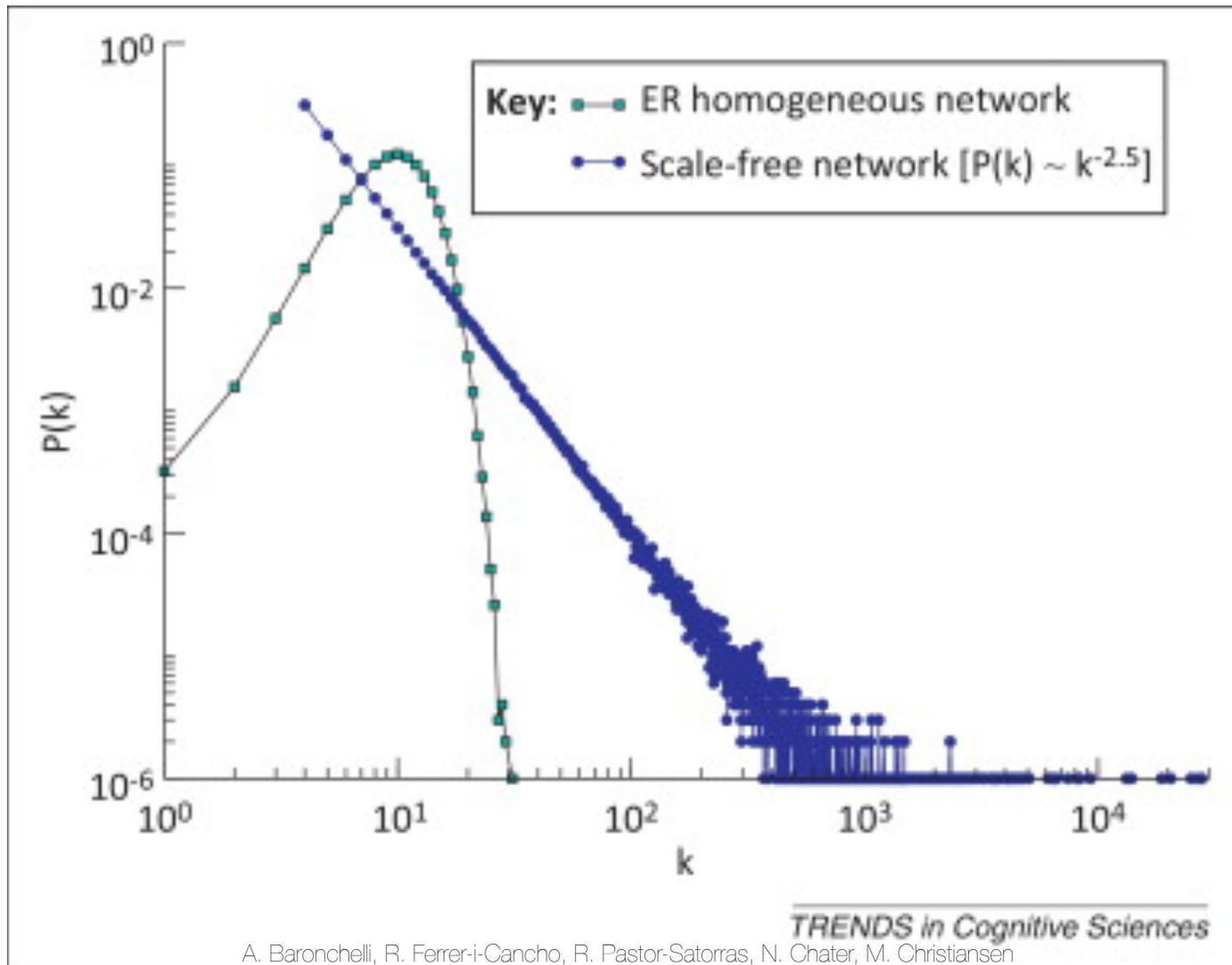
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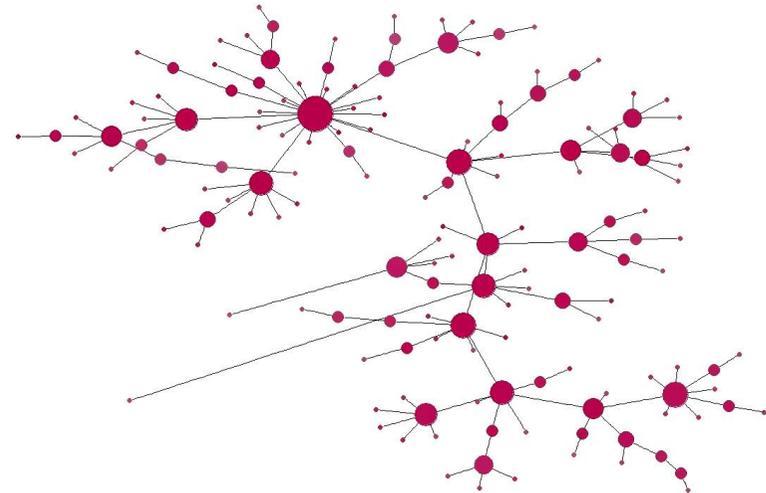
Common Knowledge



Social Network Model

Preferential Attachment (Barabási–Albert)

- Nodes arrive one after the other
- A new node u connects to $k \geq 1$ existing nodes
- $\Pr[u \rightarrow v] \approx \deg_G(v)$
- For $k=1$, PA yields a tree



Rationals for Preferential Attachment

Empirical

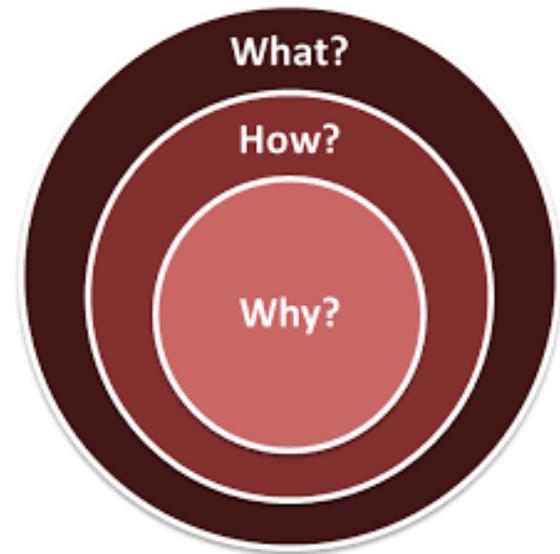
- Rich get richer aphorism (a.k.a. Matthew effect)
- Special case of Price's model

Analytical

- Generate graphs “similar to” real networks
- Has desirable properties (degree sequence, short paths, clustering, etc.)

Why Social Networks are PA Graphs?

- **The what:** PA
- **The how:** Random graph theory
- **The why:** Game theory



A Hint why Social Networks are PA Graphs

PA is the unique Nash equilibrium of a natural network formation game

The Network Formation Game: Framework

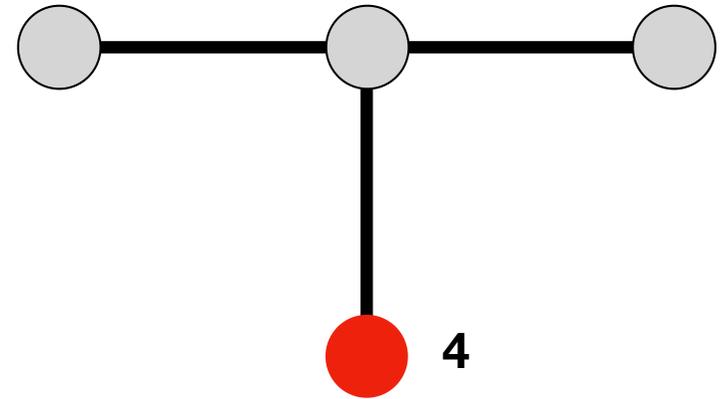
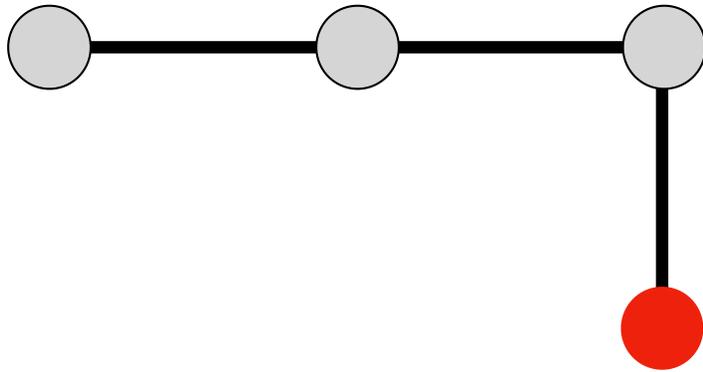
- Society = **graph**
- Social capital of a node = **degree**
- Wealth of society = **$\alpha \in [0,1]$**
- Formation process = new connections are:
 - accepted with prob **α**
 - rejected with prob **$1-\alpha$** , and pushed to a neighbor chosen u.a.r.

The Network Formation Game: Strategy & Utility

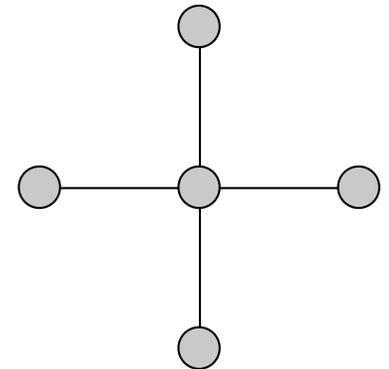
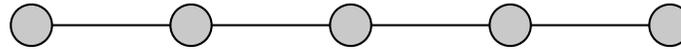
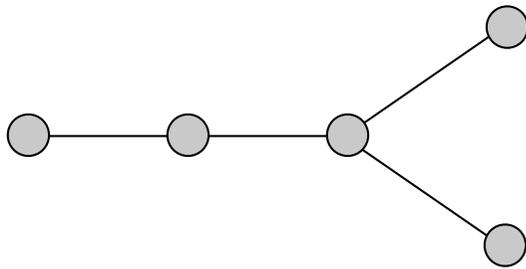
- Nodes arrive one after the other
- A new node u arriving at time t connects to one of the existing nodes
- $\Pr[u \rightarrow v] = \pi_u(v)$ where π_u is distributed over degree sequences — this is the **strategy** of node u .
- Connections accepted according to probabilities $(\alpha_t)_{t \geq 1}$
- **Utility**(v) at time $t = \mathbb{E}[\text{deg}(v) \text{ at time } t]$

Example

Step 4:



Step 5:



Universal Nash Equilibrium

Remark There is a game for each stopping times $\tau \geq 1$ and each wealth sequences $(\alpha_t)_{t \geq 1}$

Definition A strategy profile $(\pi_t)_{t \geq 1}$ is a **universal** NE if it is a NE for **all** stopping times $\tau \geq 1$, and **all** wealth sequences $(\alpha_t)_{t \geq 1}$

Universal NE Exist

Definition $\pi_{PA}(v) = \deg(v) / \sum_z \deg(z) = \deg(v) / 2m$

Theorem PA is a universal NE

Lemma $\Pr[u \text{ connect to } v \mid T] = \pi_{PA}(v)$

Proof: $\Pr[u \text{ connect to } v \mid T] = \alpha \pi(v) + \sum_{w \in N(v)} \pi(w)(1-\alpha)/\deg(w)$
 $= \alpha \deg(v) / \sum_z \deg(z) + \sum_{w \in N(v)} \left((1-\alpha) / \sum_z \deg(z) \right)$
 $= \deg(v) / \sum_z \deg(z) = \pi_{PA}(v) \quad \square$

PA is a universal NE (proof)

Assume PA is used.

Assume that there exists a sequence $(\alpha_t)_{t \geq 1}$ and some player v_t for $t \geq 4$ who could increase her utility by deviating from PA to $\pi'_t \neq \text{PA}$.

X_s = degree of player v_t at time $s \geq t$.

$X_t = 1$, and, for $s > t$, by the lemma, independently from π'_t :

- $X_s = X_{s-1} + 1$ with probability $X_{s-1}/2(s-2)$
- $X_s = X_{s-1}$ with probability $1 - X_{s-1}/2(s-2)$

□

Main Result

Theorem PA is the **unique** universal NE

Lemma Let $\Pi = (\pi_t)_{t \geq 1}$ be a strategy profile that is not PA. There exists a wealth sequence $(\alpha_t)_{t \geq 1}$ such that Π is not a NE for $(\alpha_t)_{t \geq 1}$.

Remark The result holds for only two different values $\alpha_t \neq \alpha_{t'}$.

Time-Invariant Games

- The wealth remains constant over time
- **Definition** $\alpha_t = \alpha \in [0, 1]$ for every $t \geq 1$.
- **Theorem** If a strategy profile $\Pi = (\pi_t)_{t \geq 1}$ is a universal Nash equilibrium for the time-invariant game, then each player plays PA on every graph that is not a star (and if player t plays PA on the star S_{t-1} then all subsequent players $t' > t$ play PA on all graphs).

Degree-Consistent Strategies

Definition :

- A strategy π_t is degree- k consistent if, for every degree- k node, the probability of selecting that node is independent of the degree sequence.
- A strategy π_t is degree consistent if it is degree- k consistent for every $k \geq 0$.
- A strategy profile $\Pi = (\pi_t)_{t \geq 1}$ is degree consistent if π_t is degree consistent for every $t \geq 1$.

Remark : PA is a degree consistent strategy.

Static Games

- Systematically connect to the host
- **Definition** $\alpha_t = 1$ for every $t \geq 1$.

Theorem Let $\Pi = (\pi_t)_{t \geq 1}$ be a universal Nash equilibrium for the static game. If the strategy $\pi_{t'}$ is degree consistent for every $t' \in \{1, 2, \dots, t - 1\}$, and $\pi_{t'}(k) > 0$ for every $k \in \{1, \dots, t - 1\}$, then π_t is a degree consistent strategy. In particular, if every player $t' \in \{1, 2, \dots, t - 1\}$ played PA, then π_t is a degree consistent strategy.

Conclusion

- What if the recommendation proceeds recursively? (By same arguments PA remains a universal Nash equilibrium in this case too).
- What if each new node connects to $m > 1$ existing nodes?
- In addition to node-events, considering edge-events
- What if the players have more knowledge about the actual structure than just its degree sequence?

Thank you!