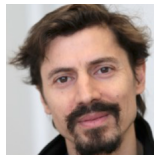


# Graph temporalisation

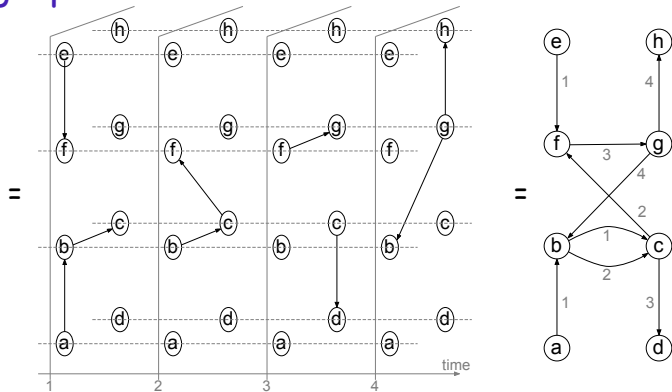
Filippo Brunelli, Pierluigi Crescenzi and Laurent Viennot



Inria Paris - GSSI - Irif (Université de Paris and CNRS)

Descartes - Fontainebleau 2021

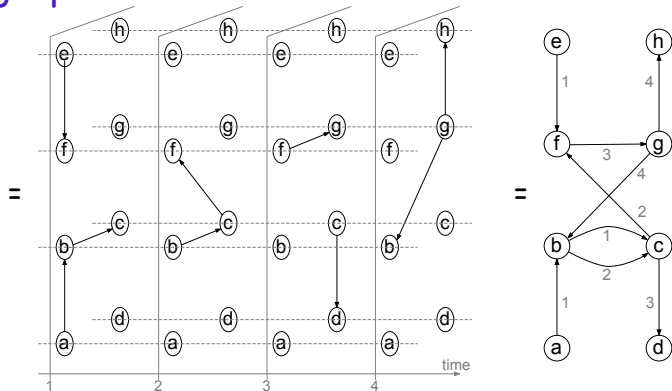
# Temporal graphs



Also known as :

- time-dependent networks [Cooke, Halsey 1966],
- edge scheduled networks [Berman 1996], dynamic graphs [Harary, Gupta 1997], temporal networks [Kempe, Kleinberg, Kumar 2002], evolving graphs [Bhadra, Ferreira 2003],
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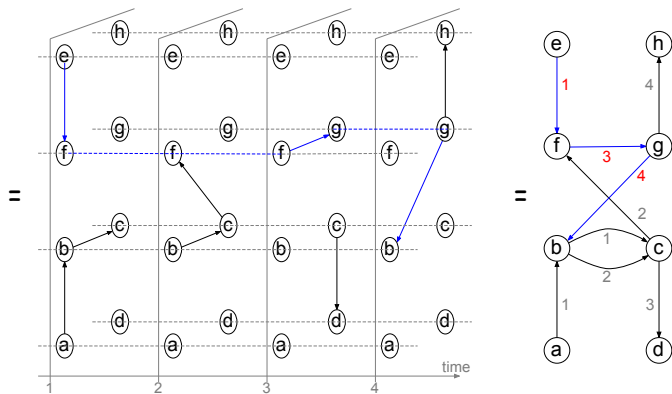
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# Temporal path



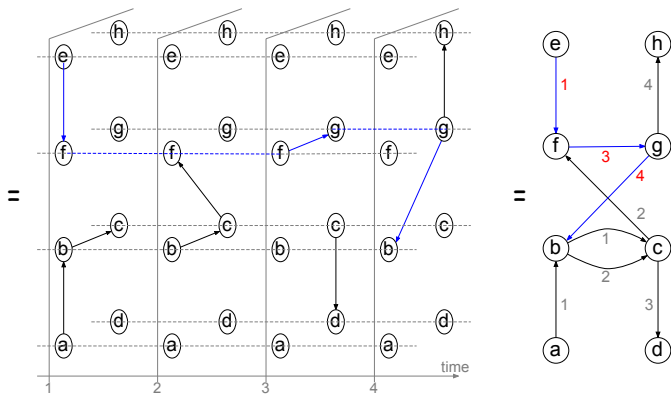
A path with increasing time labels.

Temporal paths are strict and waiting is allowed.

Reachable set  $\mathcal{R}(u)$ : the set of nodes  $v$  that  $u$  can reach by a temporal path (we let  $u\mathcal{R}v$  denote  $v \in \mathcal{R}(u)$ ).

Above, we have:  $e\mathcal{R}b$ ,  $b\mathcal{R}h$ ,  $\neg a\mathcal{R}f$ ,  $\neg a\mathcal{R}h$

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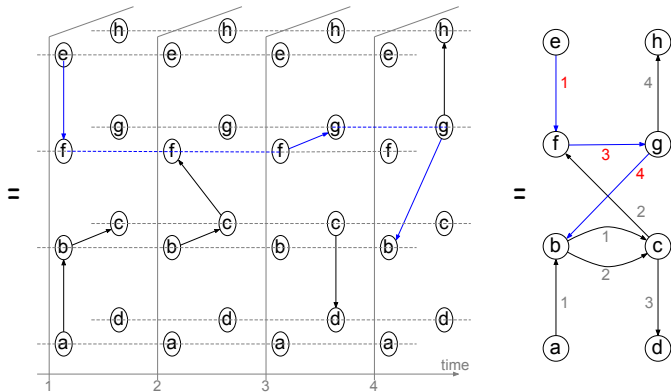
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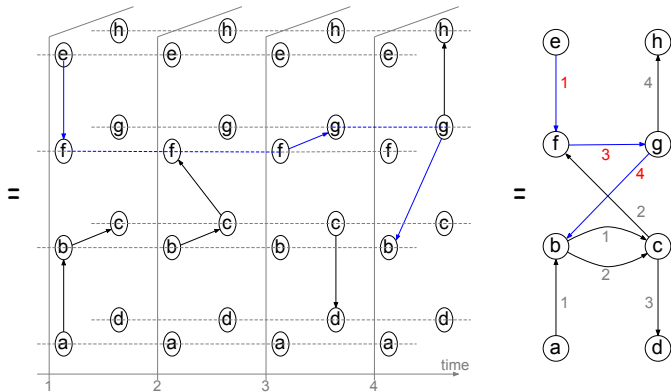
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Maximum reachability goal (**MRTT**): maximise total reachability  $\sum_u |\mathcal{R}(u)|$ .



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# Undirected temporalisation and label connectivity

**Label-connected graph** : assign time labels to an undirected graph  $(V, E)$  such that  $\mathcal{R}(u) = V$  for all  $u \in V$ .

**Undirected setting** : if  $uv$  is available at time  $t$ , both  $(u, v)$  and  $(v, u)$  are available at time  $t$ .

Known results [Göbel, Cerdeira, Veldman 1991] :

- Deciding label-connectivity is **NP-hard**.
- A graph is **label-connected** iff it contains **two edge-disjoint spanning tree-pairs** and some 4-cycle condition.

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# Public transport



**Temporal edge** : a vehicle on a **line** going from one stop to the next one.

The **trip** of a vehicle produces a sequence of temporal edges.

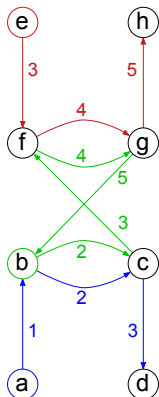
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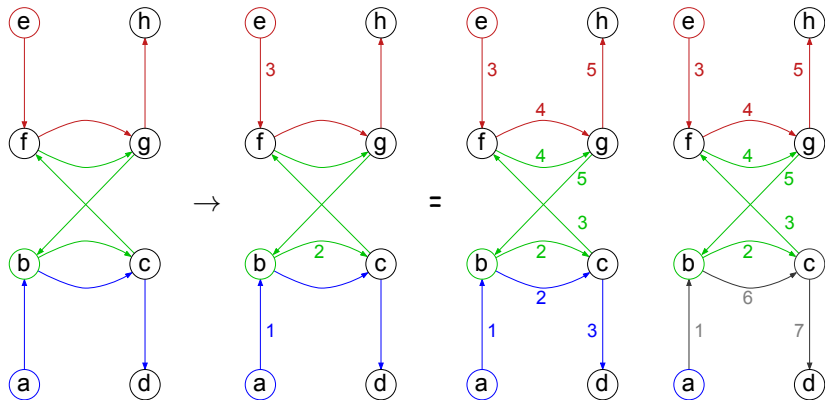


Temporal edges are grouped into walks called **trips**.



# Trip temporalisation

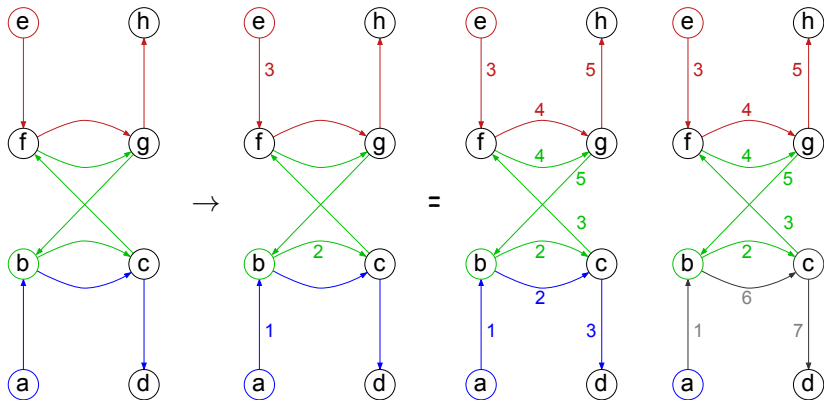
**Trip temporalisation** : Given a multidigraph  $M$  whose edges are partitioned into  $t$  trips, set a **starting time** for each trip (we note  $n = |V(M)|$  and  $m = |E(M)|$ ).



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# One-to-one reachability is hard

**O2O-RTT** : given  $u, v \in V$  decide if there exists a **trip temporalisation** such that  $uRv$ .

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**Theorem** For any  $\varepsilon > 0$ , **MRTT** cannot be approximated within  $O(n^{1-\varepsilon})$  unless  $P=NP$ .

**Idea** : connect many sources to  $v_1$  and many sinks to  $w_{m+1}$ .

**Theorem** **O2O-RTT** using at most  $k$  trips can be solved in  $2^{O(k)} m \log T$  where  $m = |E(M)|$  and  $t$  is the number of trips.

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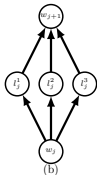
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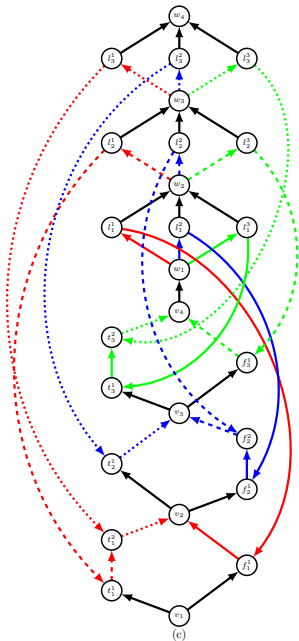
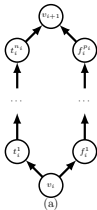
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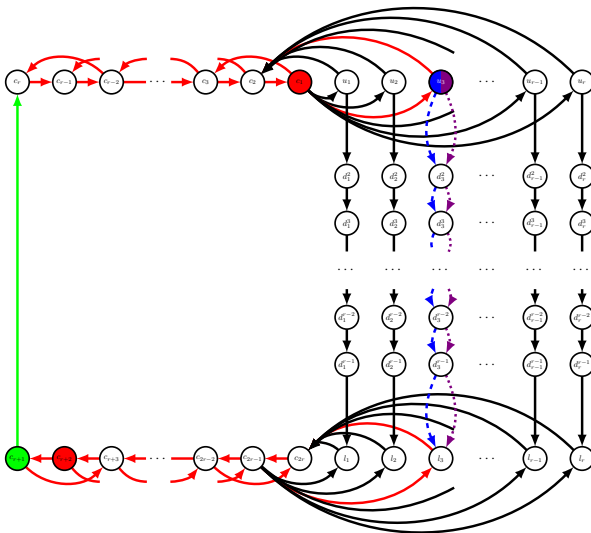
3-SAT reduction :



# Strong temporalisabilty

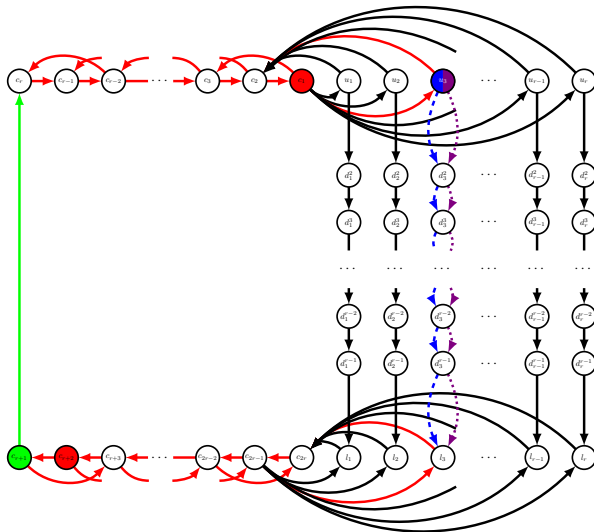
A trip network is **strongly temporisable** when for each pair  $u, v \in V$ , there exists a temporalisation such that  $uRv$ .

# A difficult strongly temporisable network



Bounded reachability :  $\sum_u |\mathcal{R}(u)| = O(n^{1.5})$  ( $\sqrt{n}$  gap).

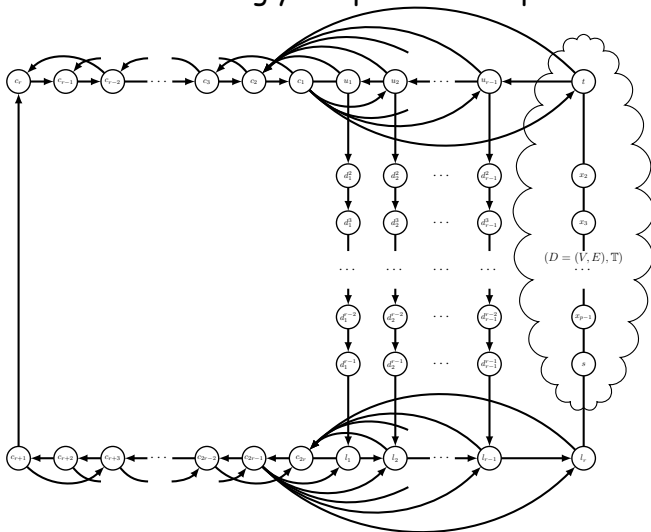
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# Unapproximability of MRTT

**Theorem** MRTT cannot be approximated within  $\sqrt{n}/12$  unless  $P=NP$  even in strongly temporisable trip networks.



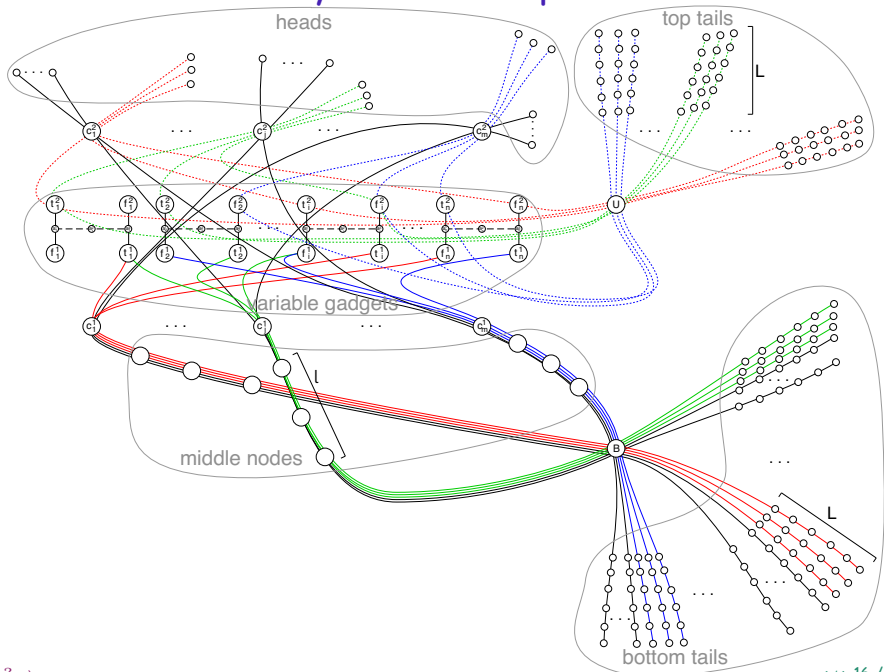
Reduction from O2O-MRTT.

# Symmetry



For every trip  $\langle v_1, \dots, v_k \rangle$ , there is a **reverse** trip  $\langle v_k, \dots, v_1 \rangle$ .

# MRTT is hard in symmetric trip networks



# Approximability of symmetric trip networks

**Fact** A symmetric trip network is **strongly temporisable** iff  $M$  is (strongly) **connected**.

**Theorem** MRTT can be approximated within  $9/2$  in strongly temporisable symmetric trip networks.

**Idea :**

- group trips and reverse trips into pairs,
- compute a **centroid pair**  $C$ , and a partition  $A, \{C\}, B$  of trip pairs,
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All results hold with arbitrary positive **delays** on edges,

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