Spanner problems in temporal graphs

Arnaud Casteigts

LaBRI, Université de Bordeaux

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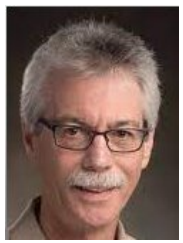
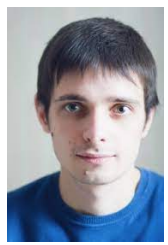
Based on joint works with:


Jason Schoeters


Joseph Peters


Michael Raskin


Malte Renken


Viktor Zamaraev

Temporal graphs (*a.k.a.* time-varying, time-dependent, evolving, dynamic,...)

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns *presence times* to edges (here, discrete)

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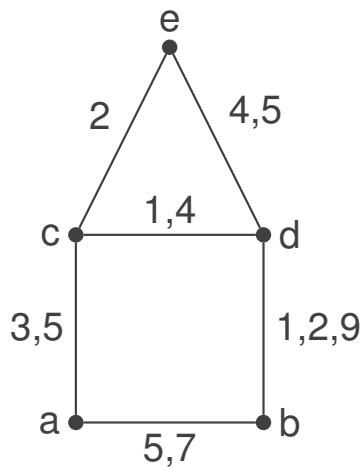

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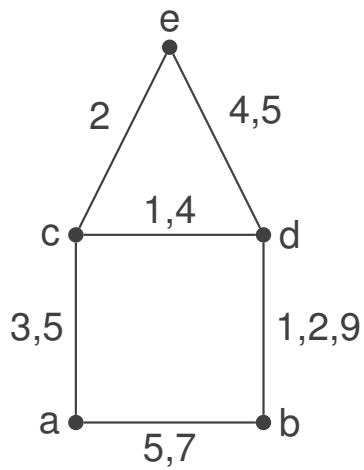

Temporal paths

- Ex: $\langle (a, c, 3), (c, d, 4), (d, e, 4) \rangle$ (*non-decreasing*)
- Ex: $\langle (a, c, 3), (c, d, 4), (d, e, 5) \rangle$ **(*increasing*)**

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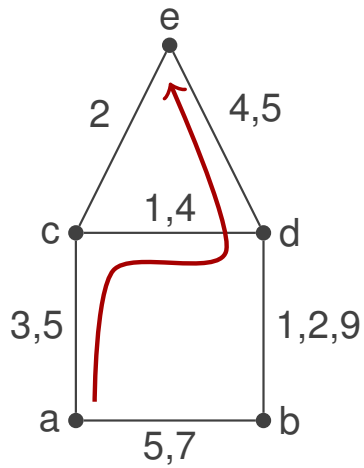

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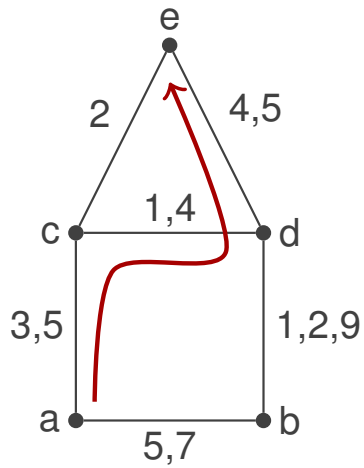

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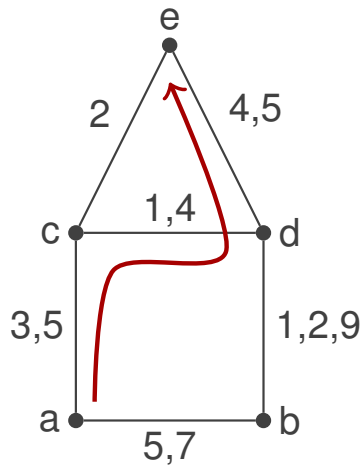

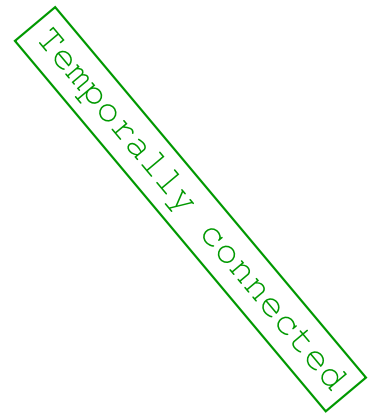

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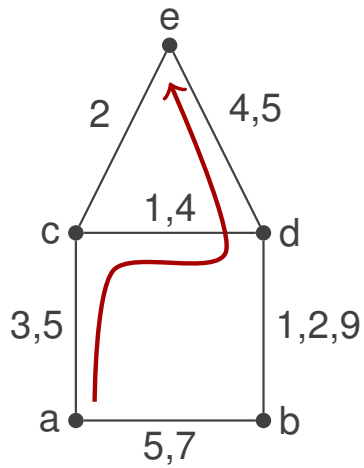

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Temporal connectivity: all vertices can reach each other through temporal paths

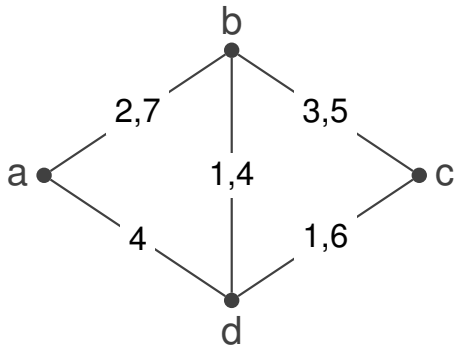
Remark: reachability is non-transitive in general!

Temporal spanners

Input: a graph $\mathcal{G}$ that is temporally connected ($\mathcal{G} \in TC$)

Output: a graph $\mathcal{G}' \subseteq \mathcal{G}$ that preserves temporal connectivity ($\mathcal{G}' \in TC$)

Cost measure: size of the spanner (in number of time labels)



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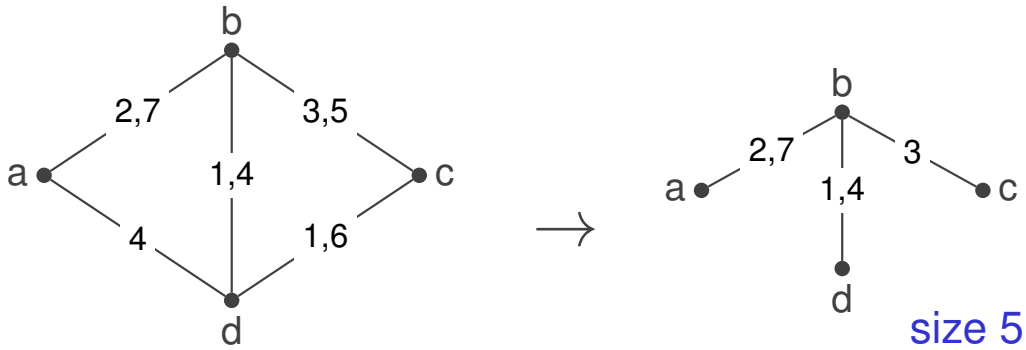

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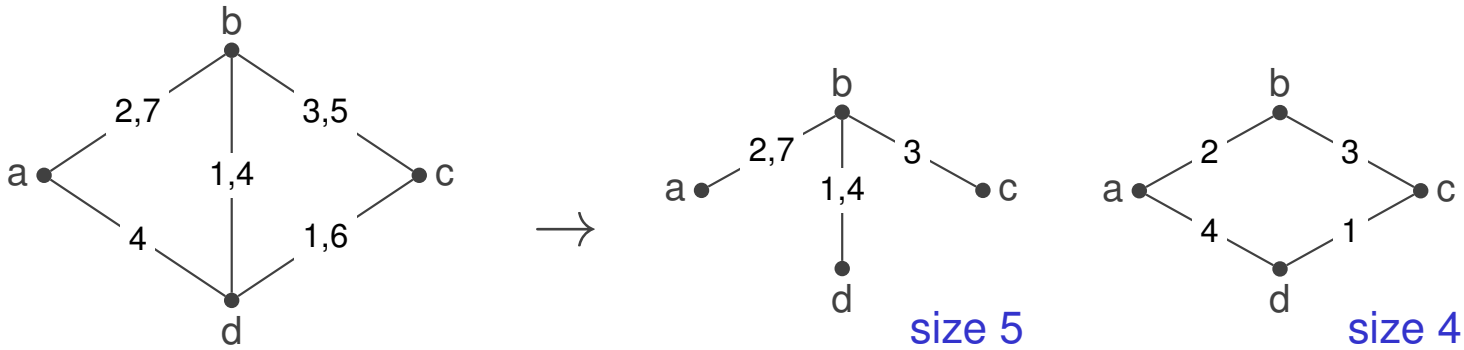

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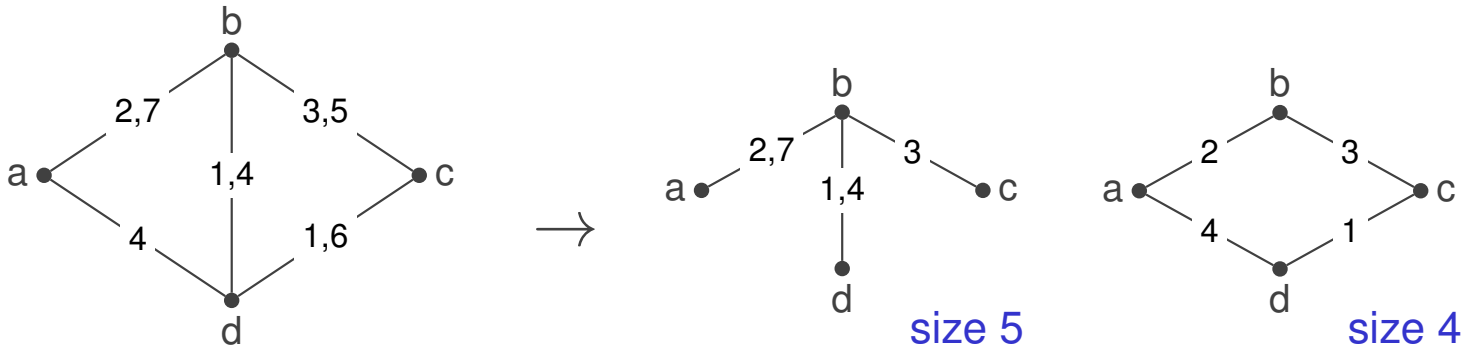

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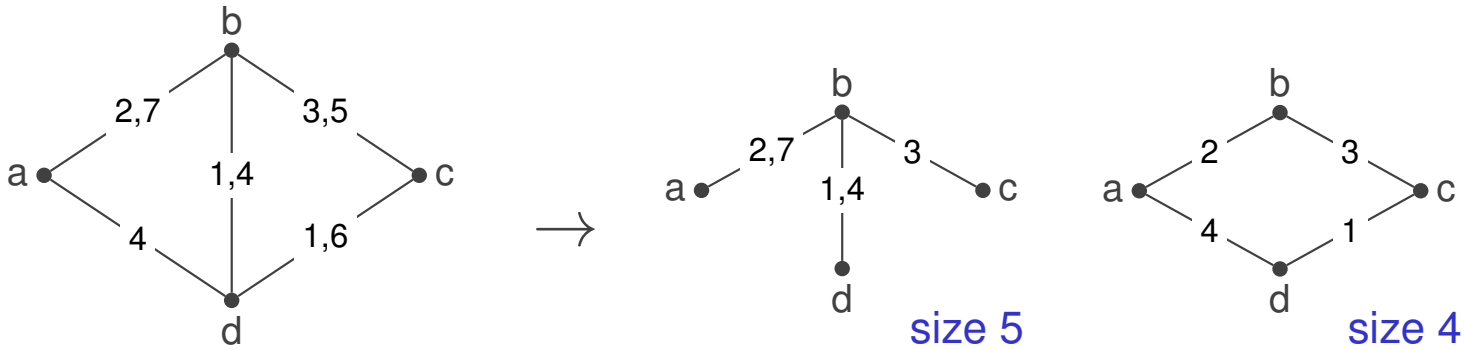

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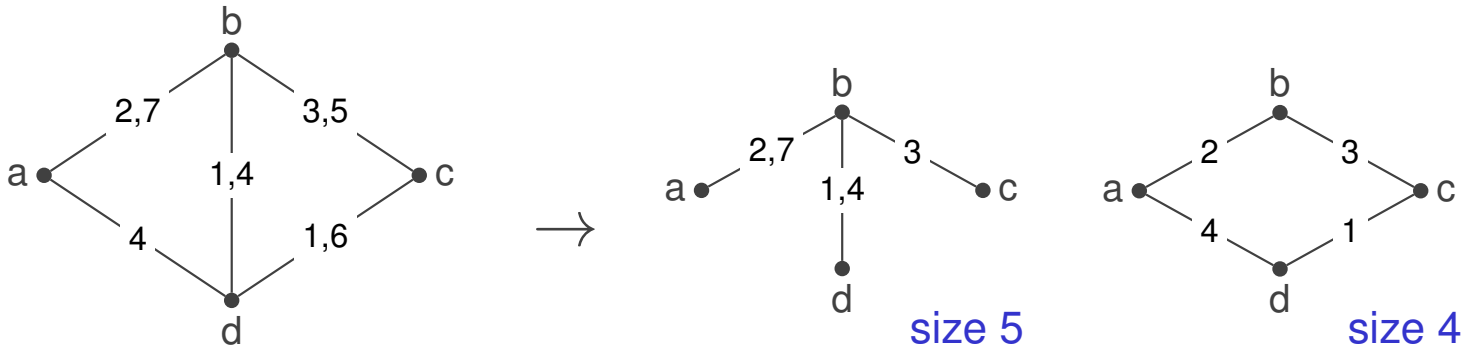

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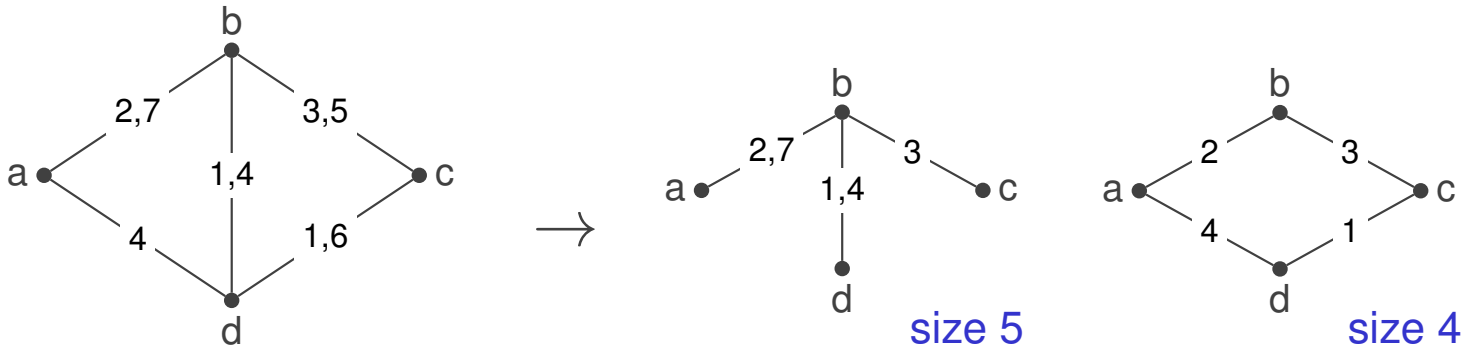

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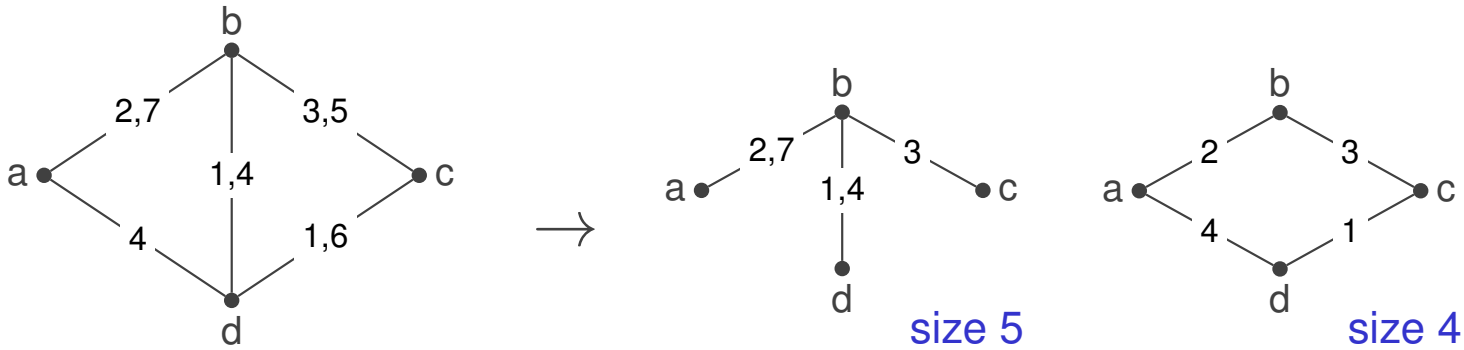

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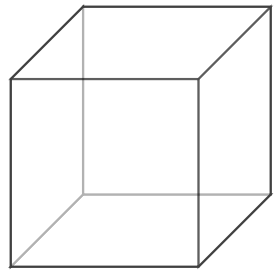
How about complexity?

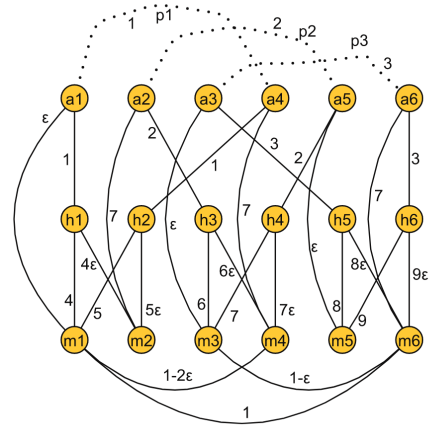
- Computing minimum-size spanner is APX-hard (Akrida et al., 2017)

What about the positive side?

Recall the bad news:

- $\Omega(n \log n)$ - easy
- $\Omega(n^2)$ - rather unexpected





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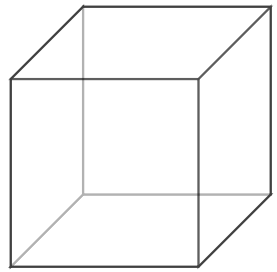
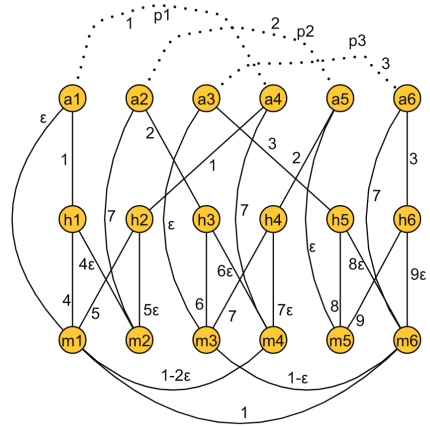
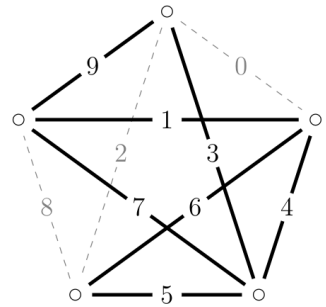




Good news 1: (C., Peters, Schoeters, ICALP 2019):

- Spanners of size $O(n \log n)$ always exist in **complete** temporal graphs



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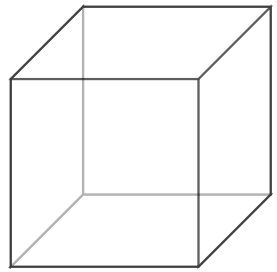

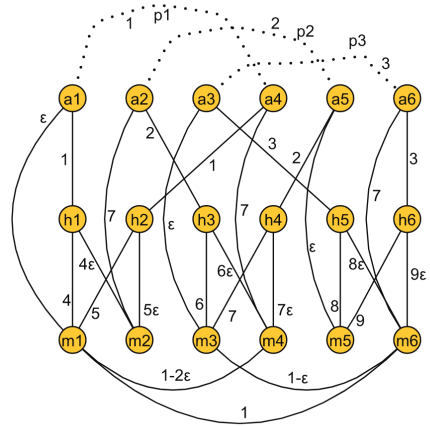

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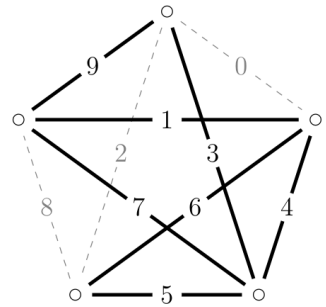

Good news 2: (C., Raskin, Renken, Zamaraev, FOCS 2021):

- Nearly optimal spanners (of size $2n + o(n)$) almost surely exist in **random** temporal graphs, as soon as the graph is temporally connected

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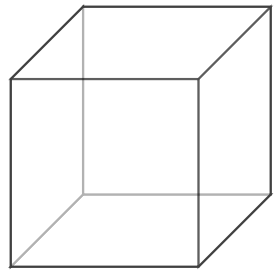

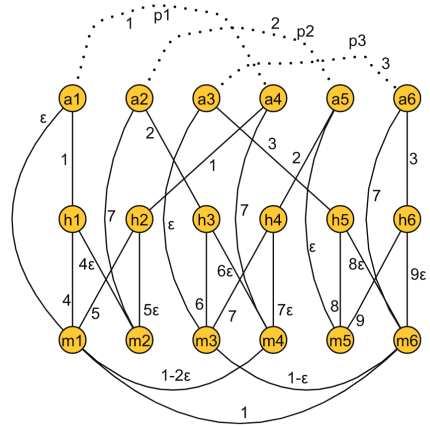

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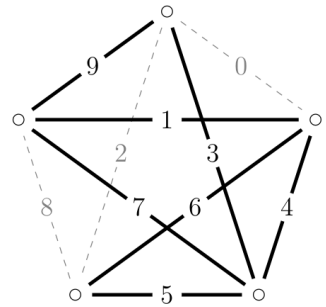

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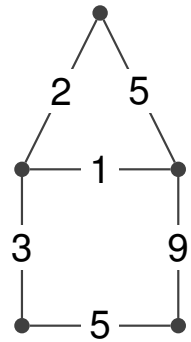
- (Beyond spanners) Interesting thresholds on temporal reachability

Before we start... an easier model

Simple Temporal Graphs (STGs):

1. A single presence time per edge ($\lambda : E \to \mathbb{N}$)
2. Adjacent edges have different times (λ is locally injective)



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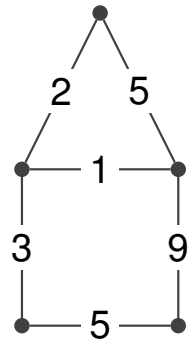

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- Positive results extend to general case
- No distinction between strict and non-strict temporal paths

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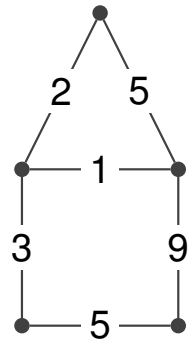

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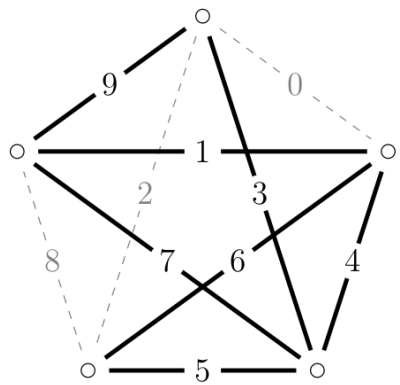
Further motivations:

- Distributed models by pairwise interactions,
  e.g. population protocols or gossip models (without repetition)
- Topics in edge-ordered graphs

Good news 1:

Temporal cliques admit sparse spanners



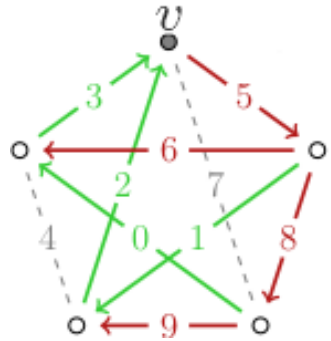
Two promising techniques...

Pivotability

Node v and time t such that:

- all nodes can reach v before t
- v can reach all nodes after t

Then in-tree $\cup$ out-tree is a spanner of size $2n - 2$



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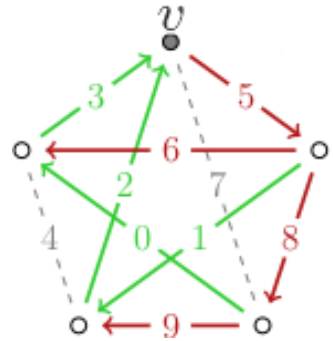
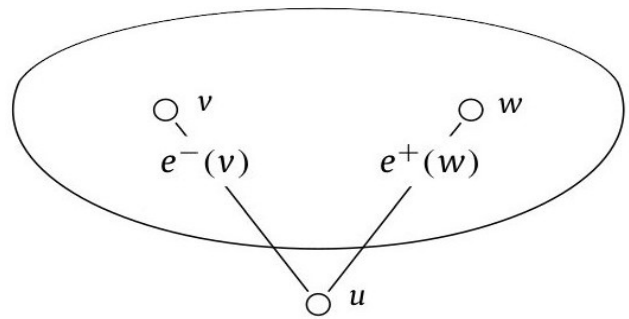


Dismountability

Three nodes u, v, w such that:

- uv =`min-edge(`v`)`
- uw =`max-edge(`w`)`

Then `spanner(`$\mathcal{G}$`)` := `spanner(`$\mathcal{G}[V \setminus u]$`)` + uv + uw



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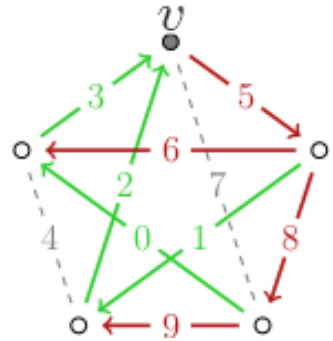

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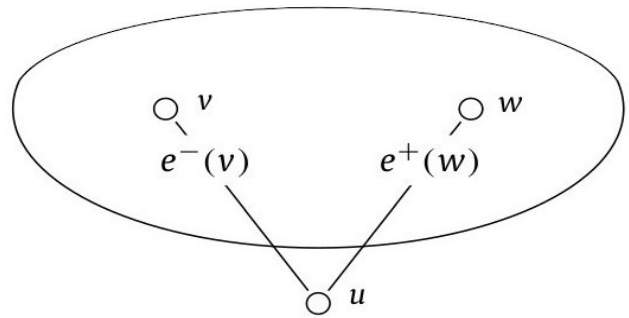
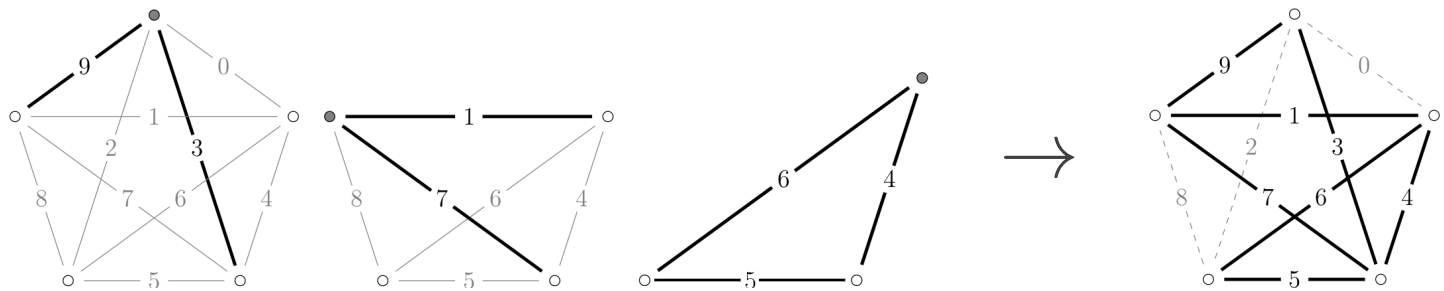


Recursively,



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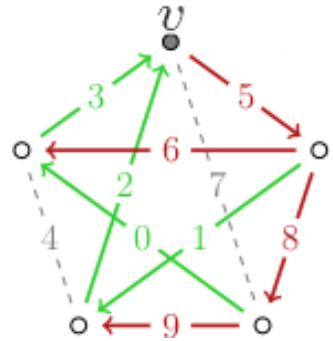
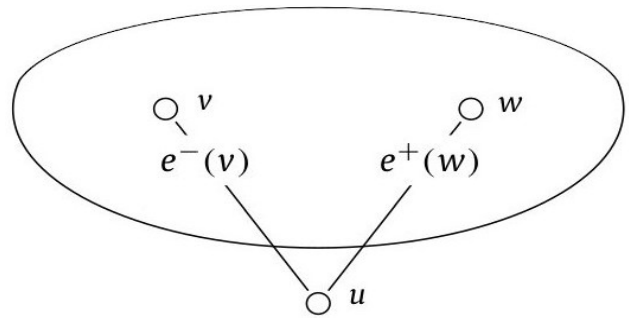


Dismountability

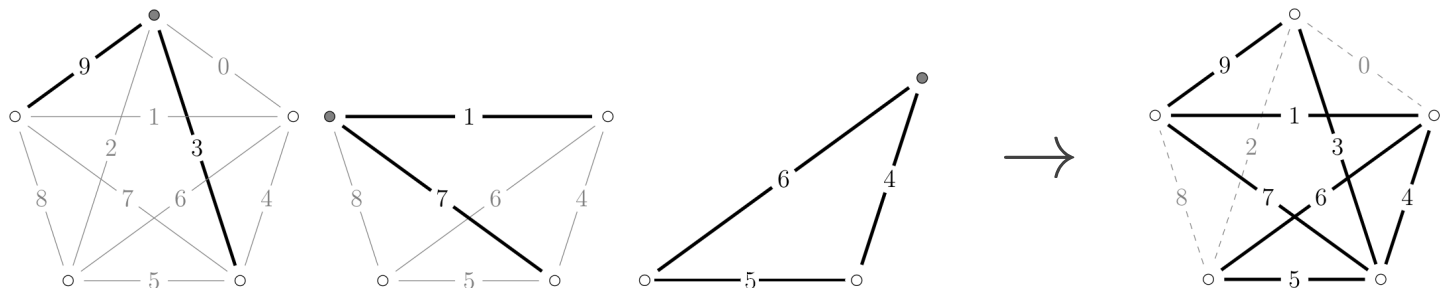
Three nodes u, v, w such that:

- uv =`min-edge(`v`)`
- uw =`max-edge(`w`)`

Then `spanner(`$\mathcal{G}$`)` := `spanner(`$\mathcal{G}[V \setminus u]$`)` + uv + uw



Recursively,



spanner of size $2n - 3$.

... but unfortunately

Both techniques fail in some cliques!

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Principle:

- Min edges $\rightarrow$ “directed” forest
- Transitive delegations towards emitters (sinks)
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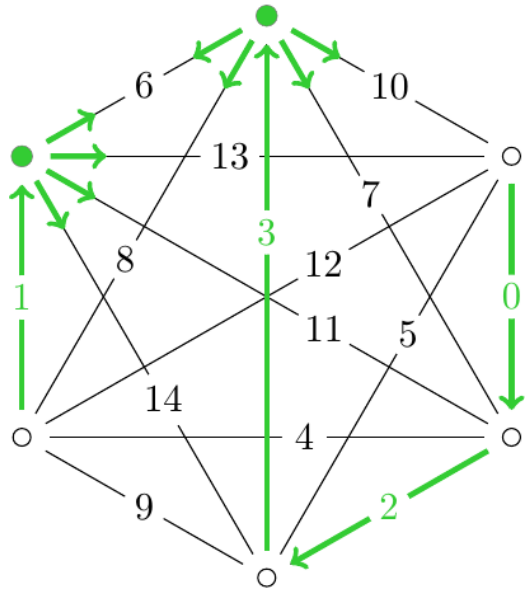

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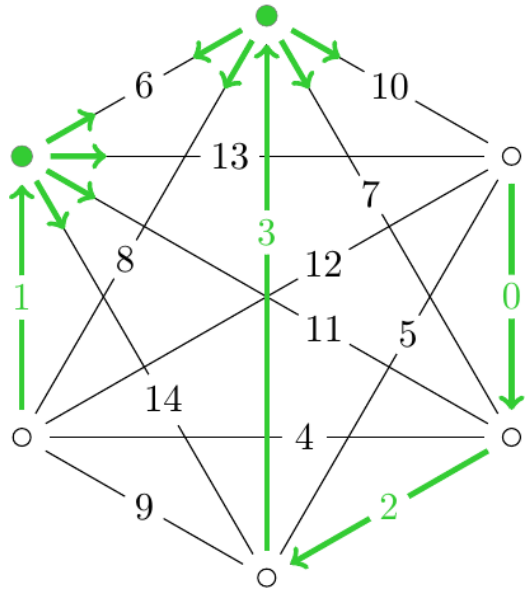

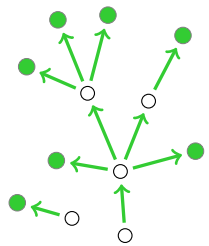

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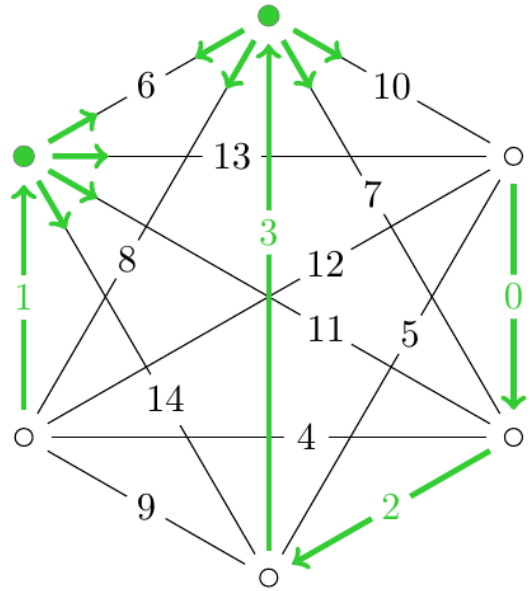

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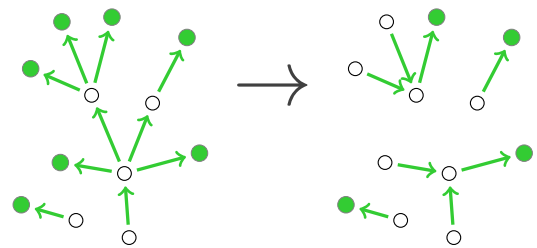
$\rightarrow$ *Transformation of the forest*:

- At most $n/2$ emitters



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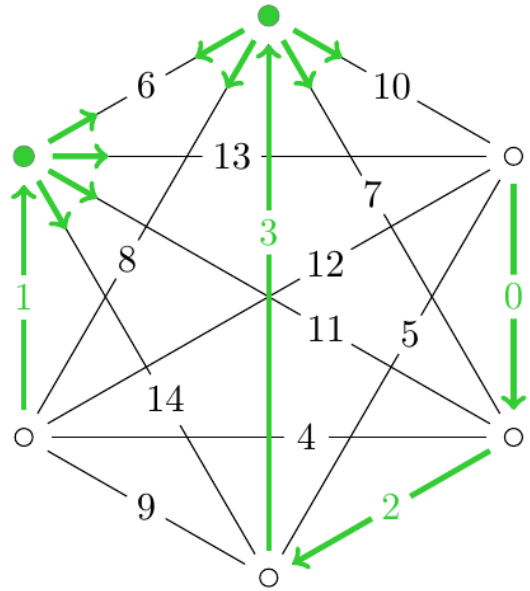

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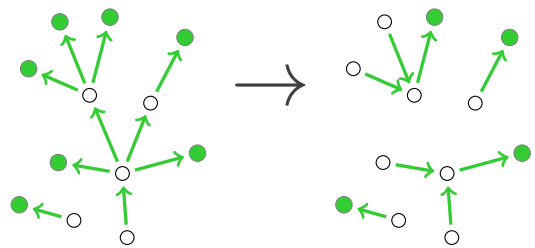
Theorem: $\exists$ spanners of size $\frac{3}{4}\binom{n}{2} + O(n)$



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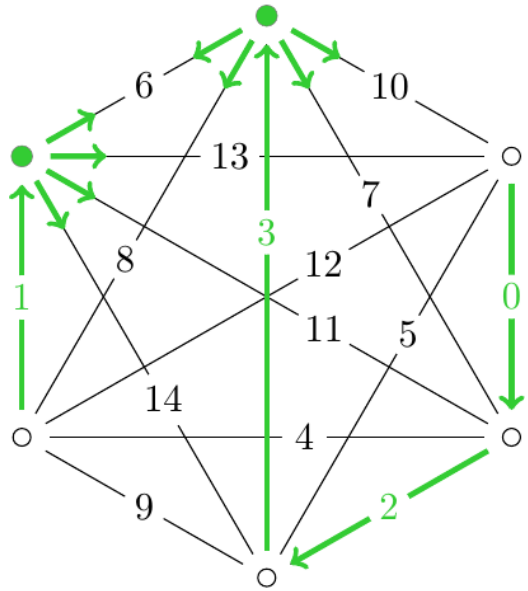

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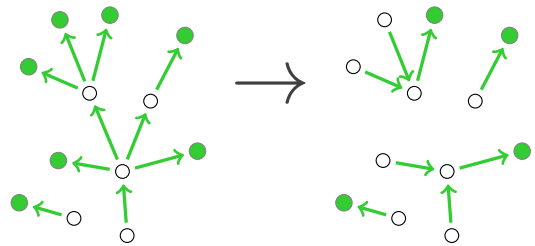
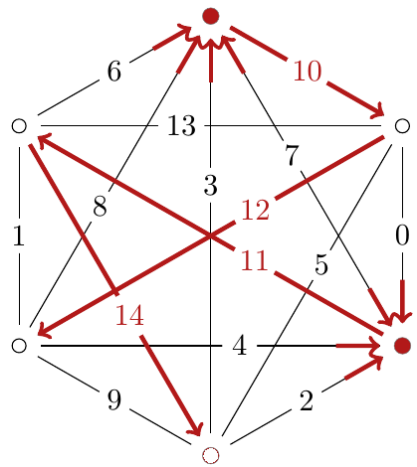


Note: also works for receptions (“backward fireworks”):

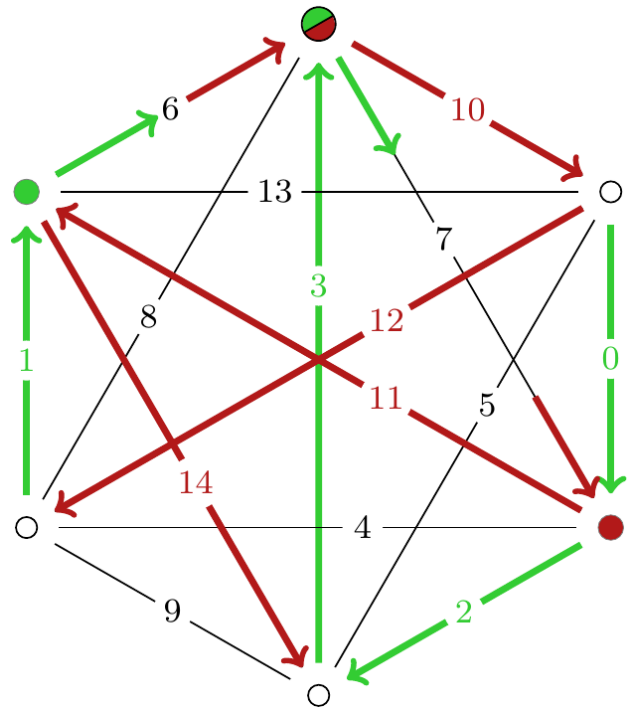
$\rightarrow$ Spanner = max edges + all edges of collectors



Combining both directions

Principle

- Every vertex can reach at least one emitter u through u's min edge
- Every vertex can be reached by a collector v through v's max edge
- Every emitter can reach all collectors through direct edges



$\rightarrow$ Spanner = min edges + max edges

+ edges between emitters and collectors

Theorem:

At most $n/2$ emitters and $n/2$ collectors $\Rightarrow$ $\boxed{\exists \text{ Spanners of size } \binom{n}{2}/2 + O(n)}$

$\approx$ half of the edges

Recurse or sparsify?

Two options:

- Case 1: emitters $\cup$ collectors $\subsetneq V$
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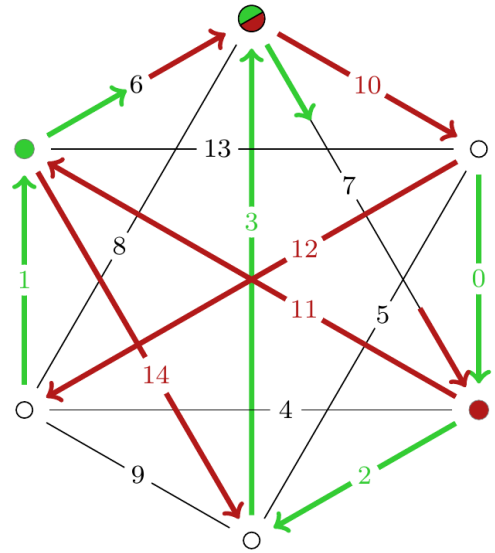

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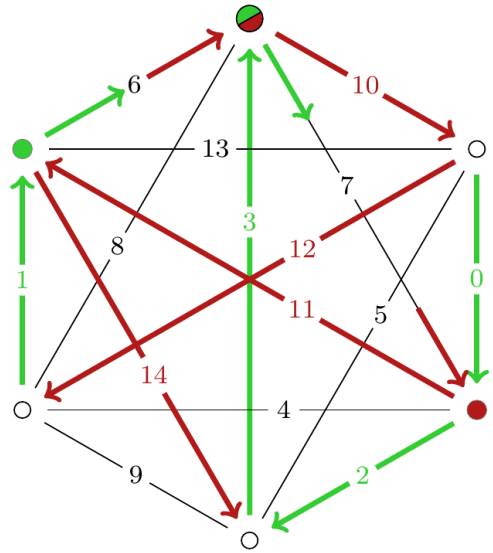

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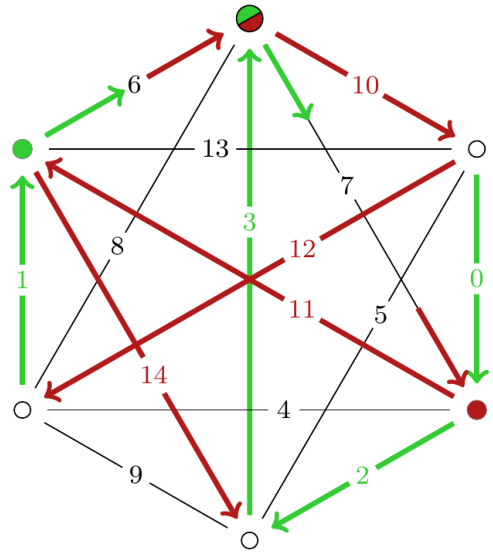

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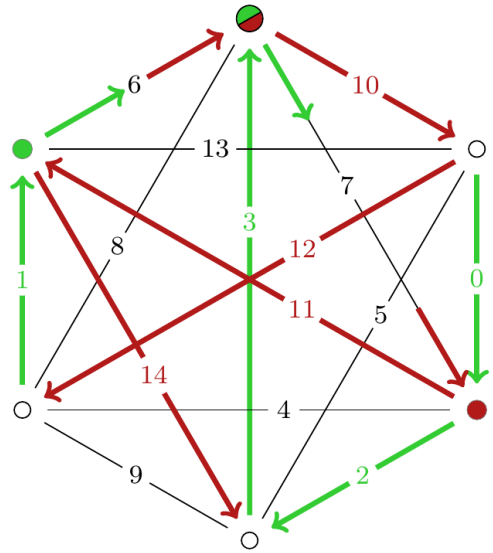

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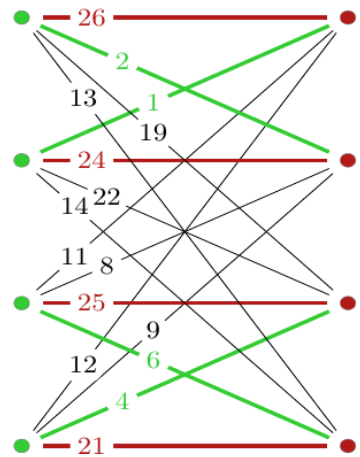

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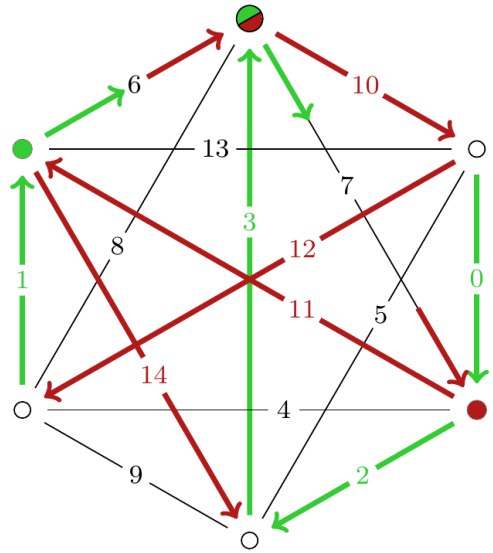

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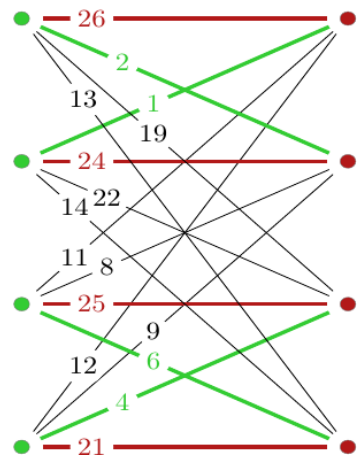
A lot of structure to work with:

- Complete bipartite graph $\mathcal{H}$ between emitters and collectors
- Min edges and max edges form two perfect matchings
- W.l.o.g. min edges (max edges) are *reciprocal* in $\mathcal{H}$



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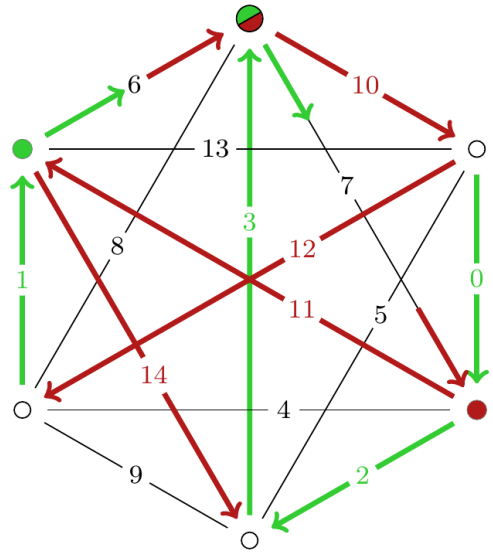

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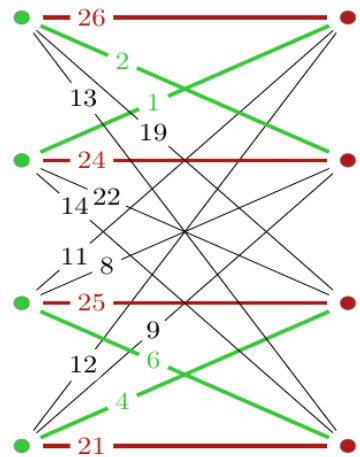

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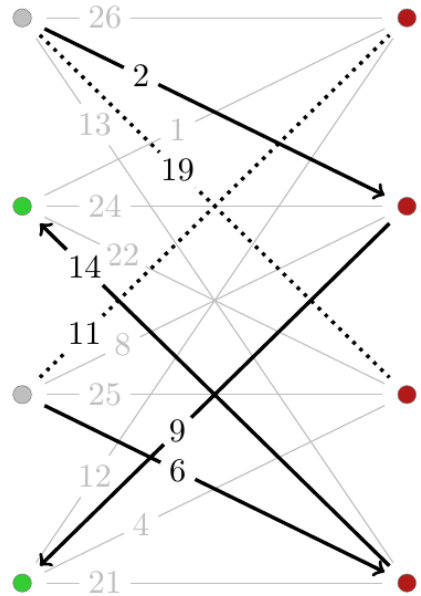

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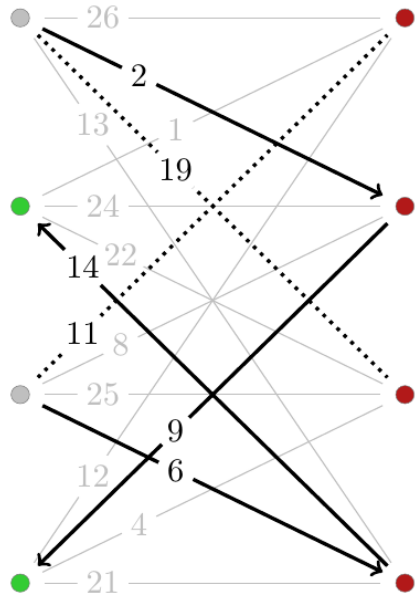

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In each step i:

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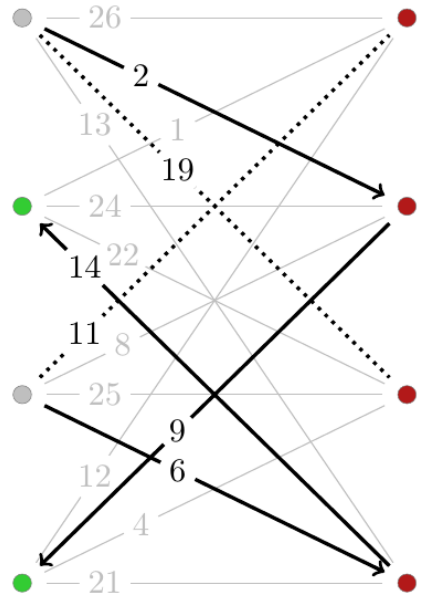

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Conclusion (entire algorithm):

$\rightarrow$ $\exists$ spanner of size $O(n) + O(n \log n) = \boldsymbol{O(n \log n)}$. $\square$

Good news 2:

Spanners of size $2n + o(n)$ almost surely exist

in random temporal graphs

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Temporal analog of Erdös-Reyni graphs, same parameters n and p

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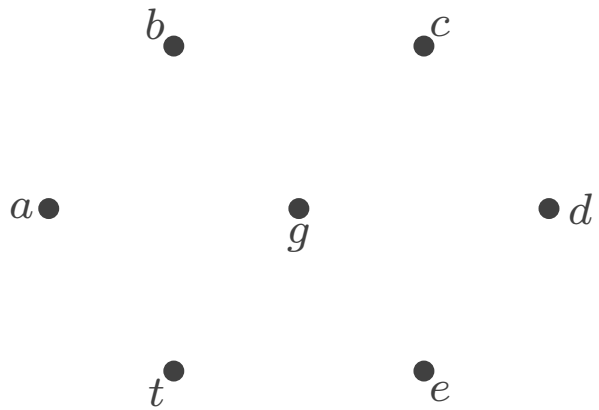

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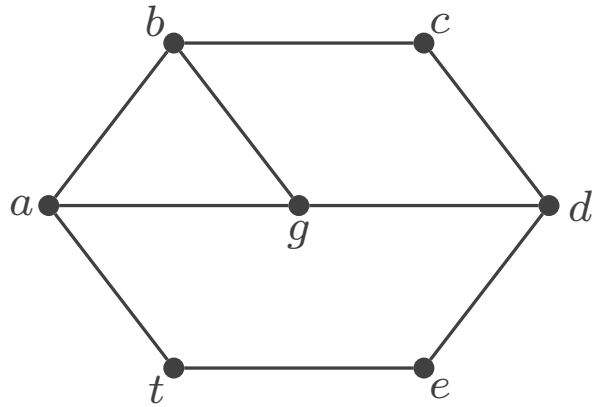

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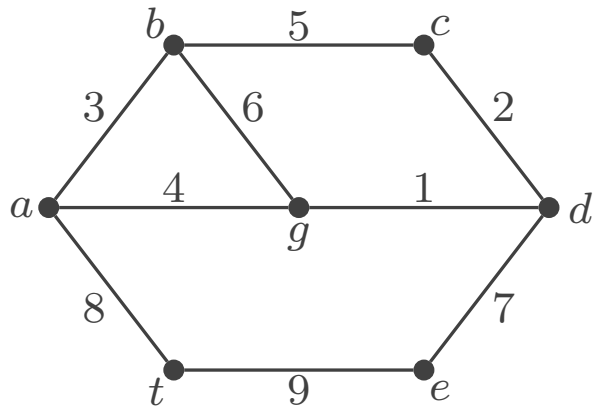

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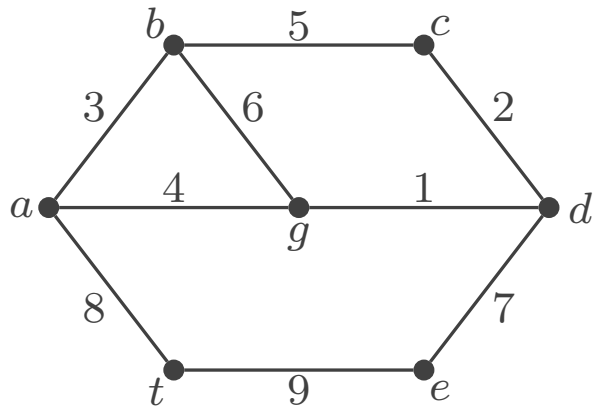

Another point of view:

1. Take a complete graph K_n
2. Assign random real times in $[0, 1]$ to every edge
3. Restrict your attention to $\mathcal{G}_{[0,p]}$

$\rightarrow$ Better for analysis.

Timeline of temporal reachability in $\mathcal{G}_{n,p}$

For sufficiently large n, what happens when p increases?

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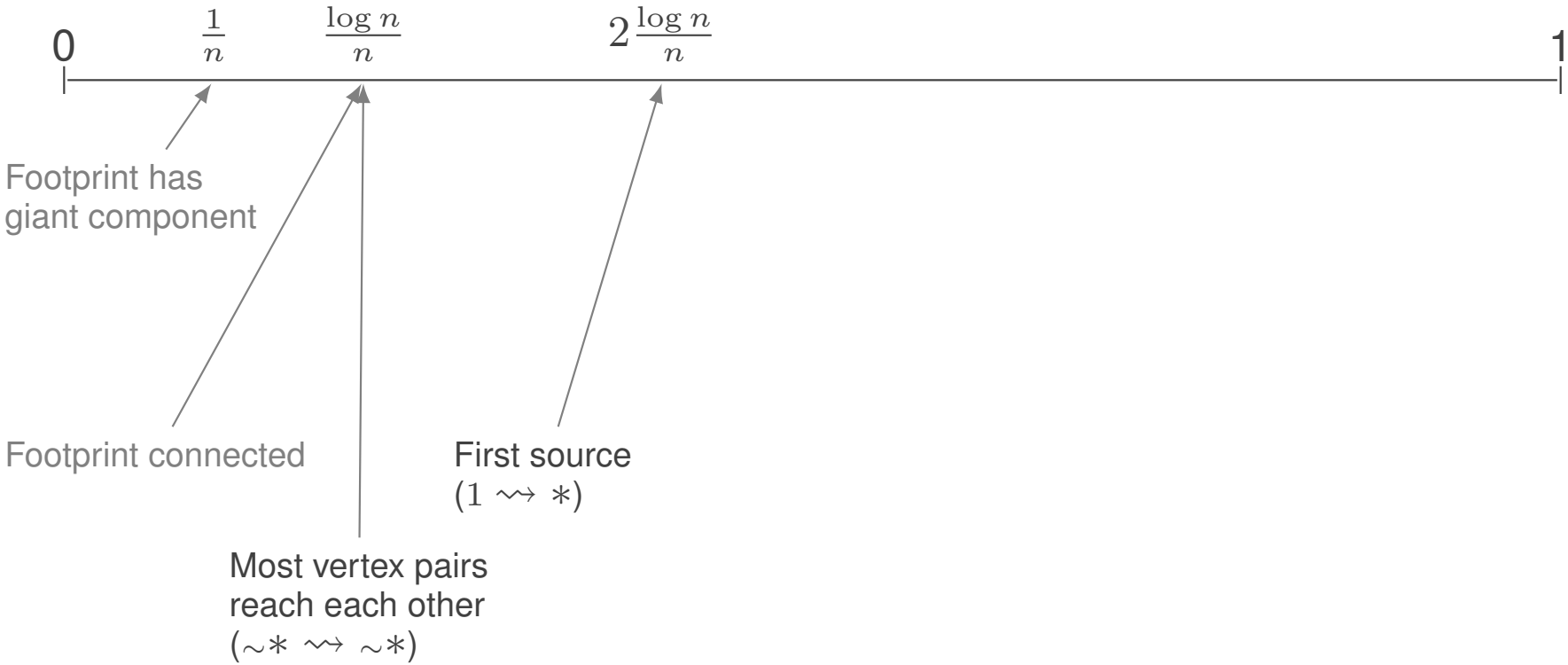

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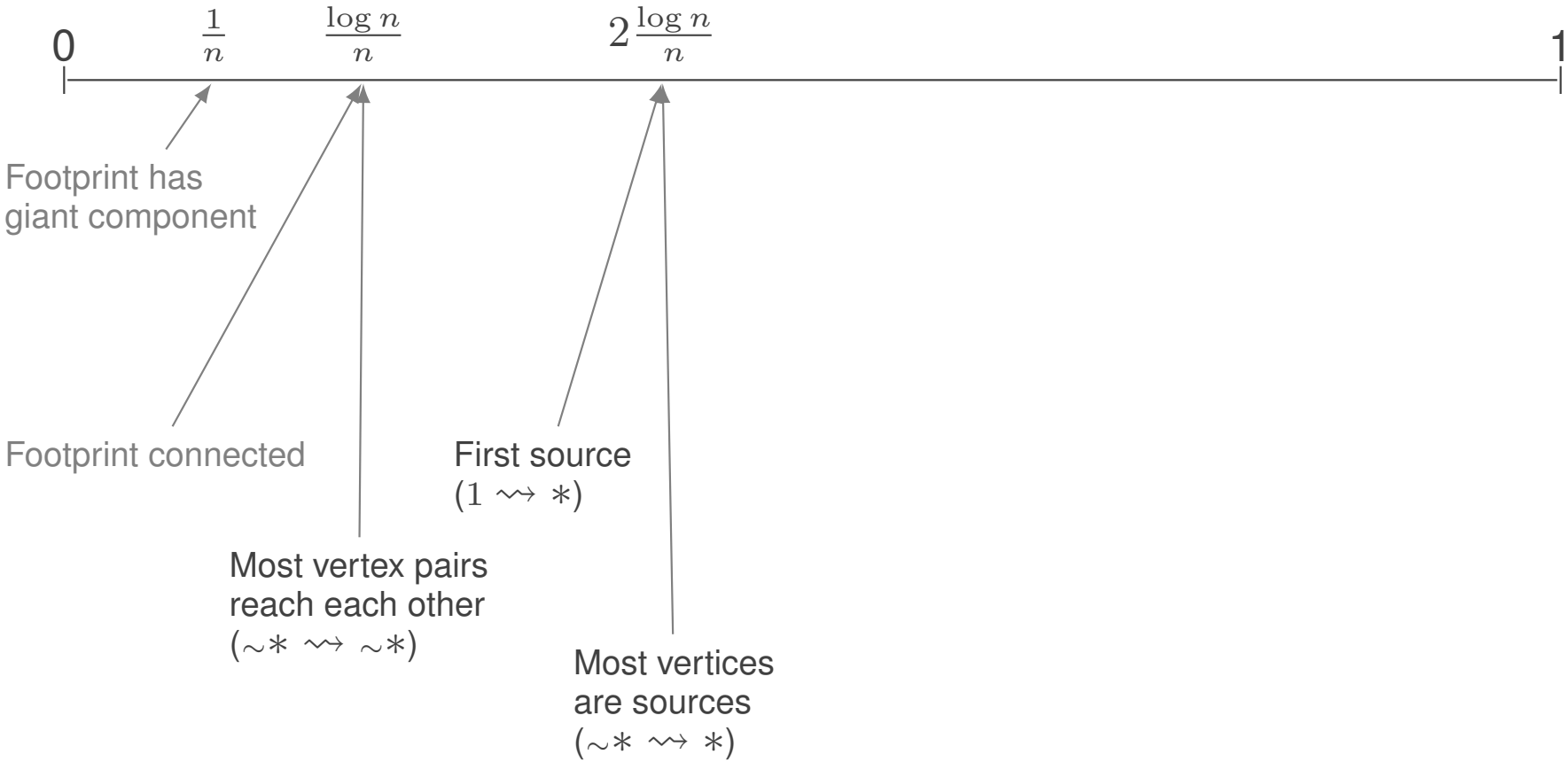

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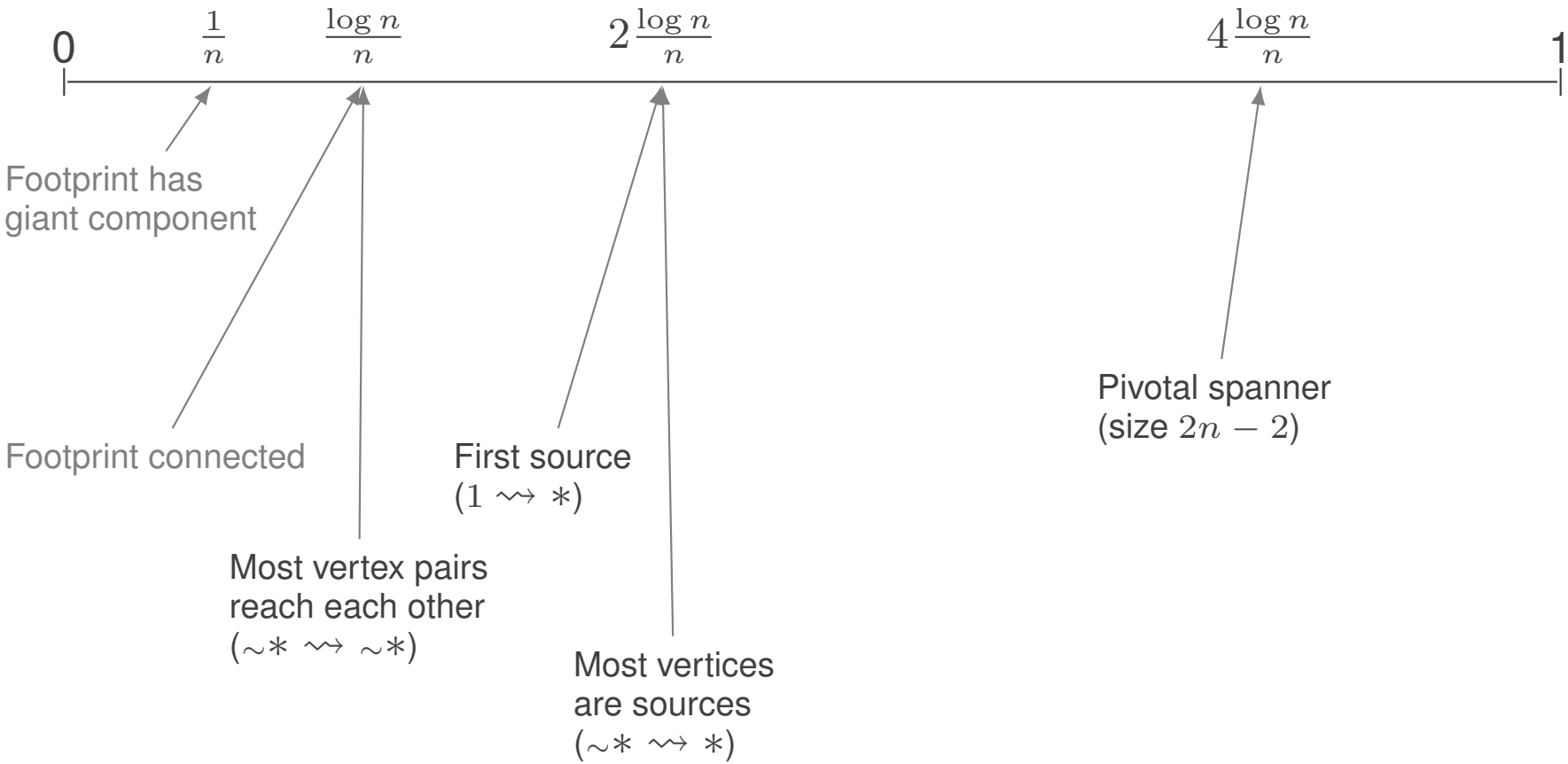

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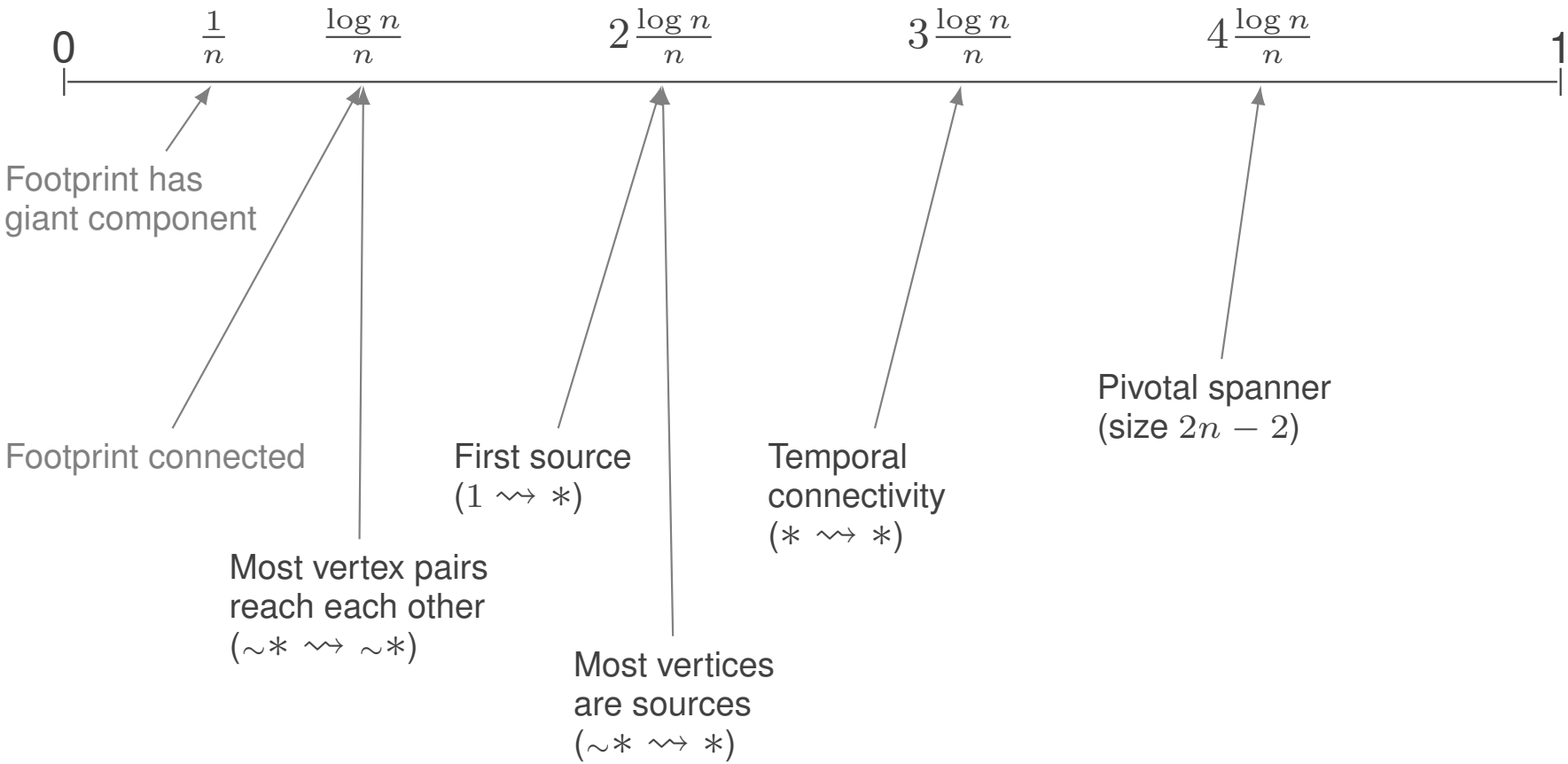

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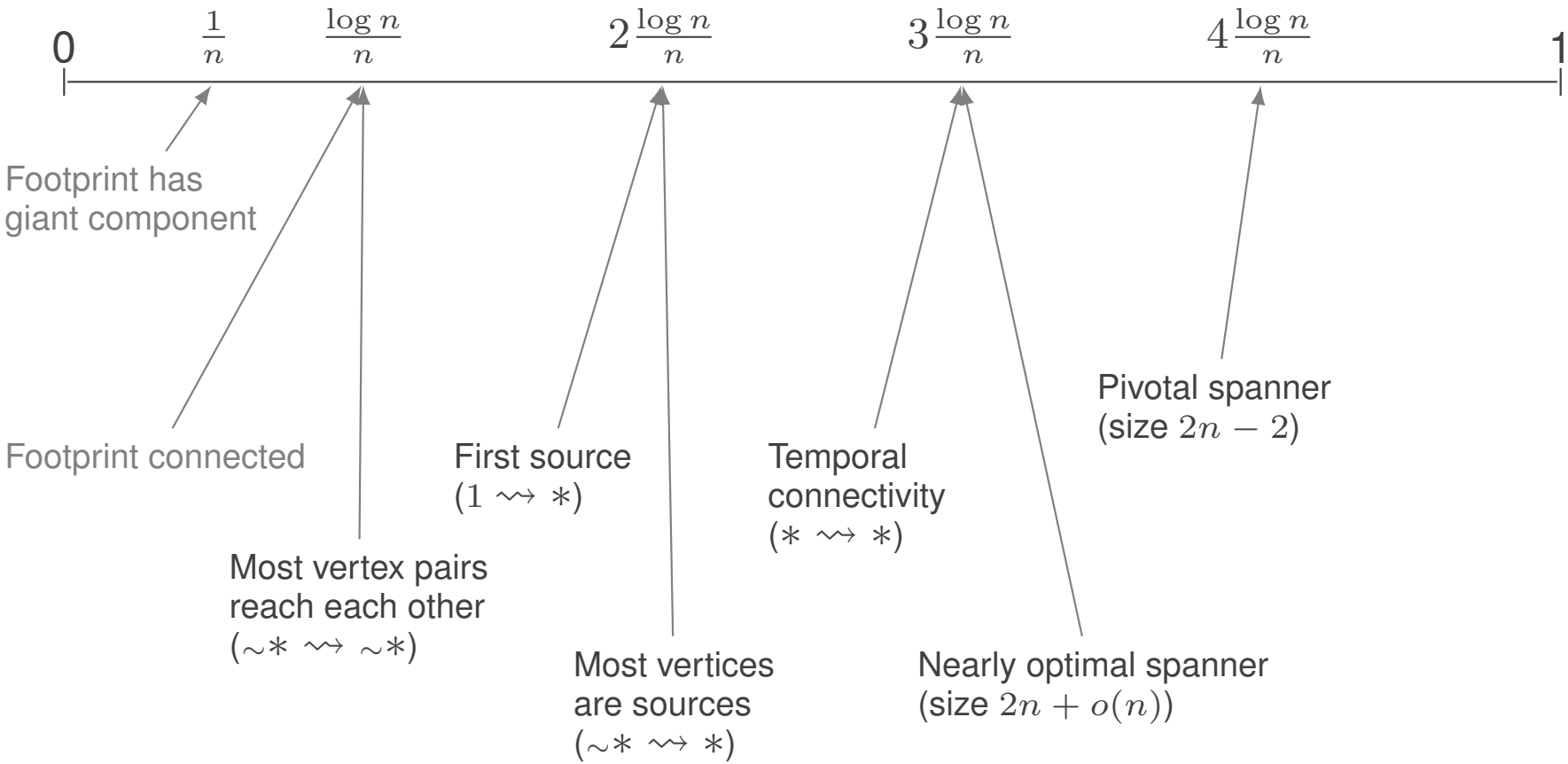

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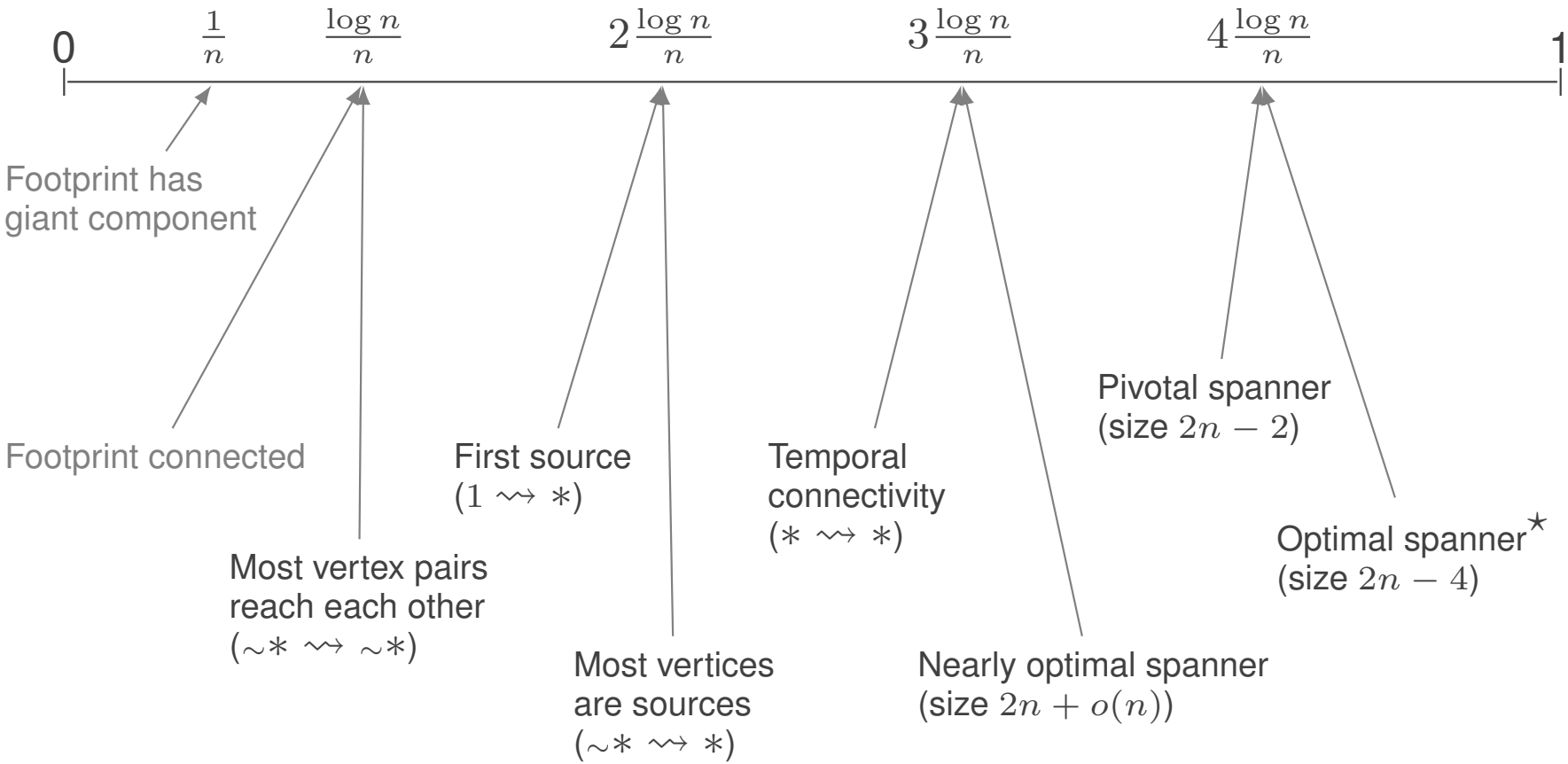

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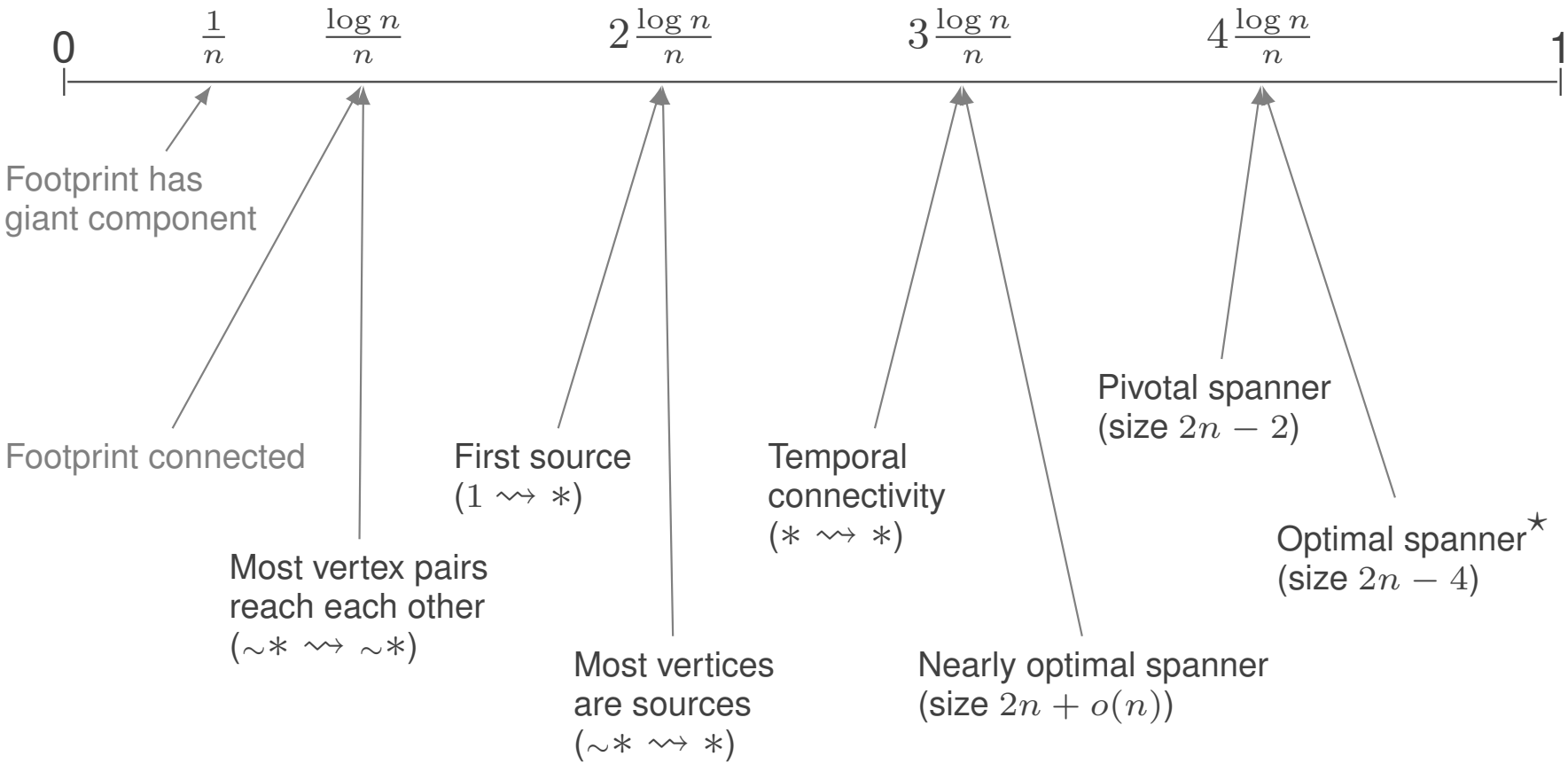


All the thresholds are sharp, except $\star$ (open problem)

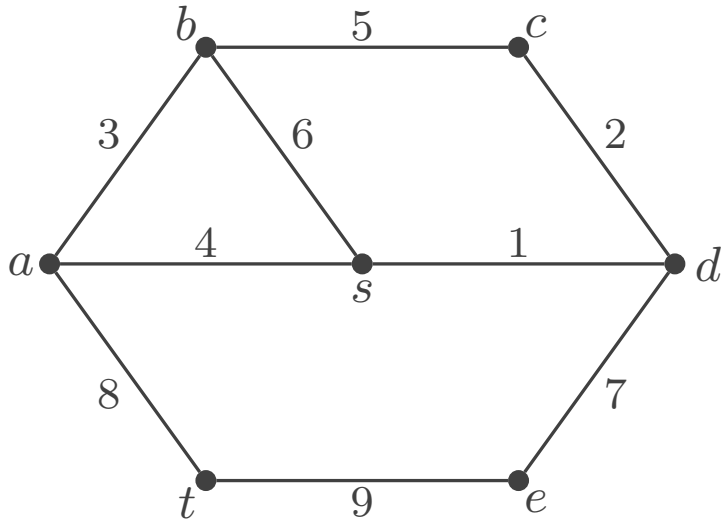
(sharp: $\exists \epsilon(n) = o(1)$, not true at $(1 - \epsilon(n))p$, true at $(1 + \epsilon(n))p$)

Main technical tool: growth of a foremost tree

Foremost tree (from s)

- Foremost temporal paths from s to all

- "Prim-like" algorithm.



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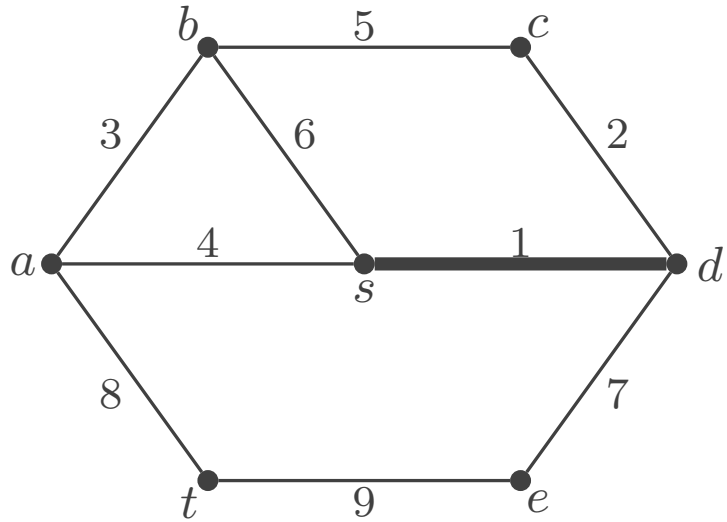

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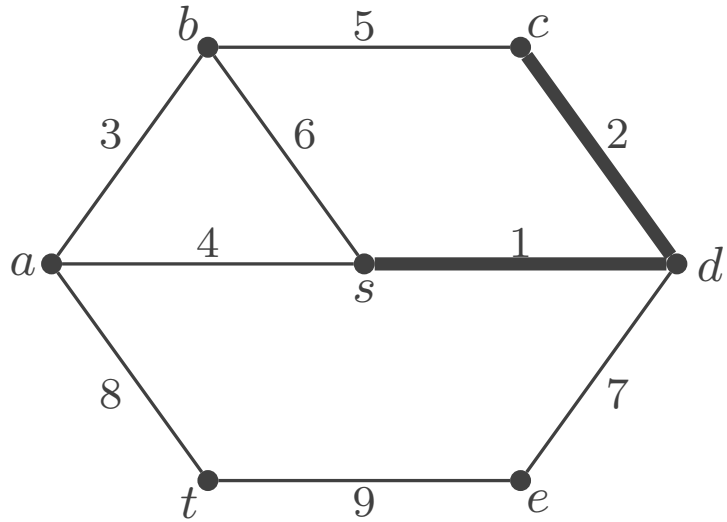
Main technical tool: growth of a foremost tree

Foremost tree (from s)

- Foremost temporal paths from s to all

- "Prim-like" algorithm.



$t = 2$

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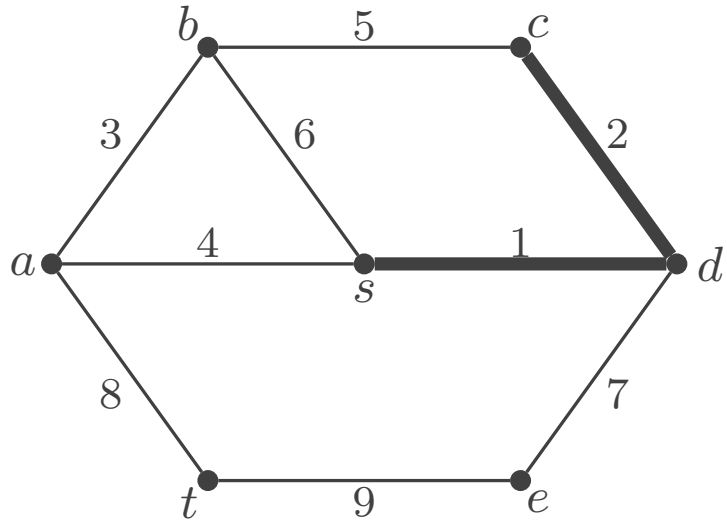

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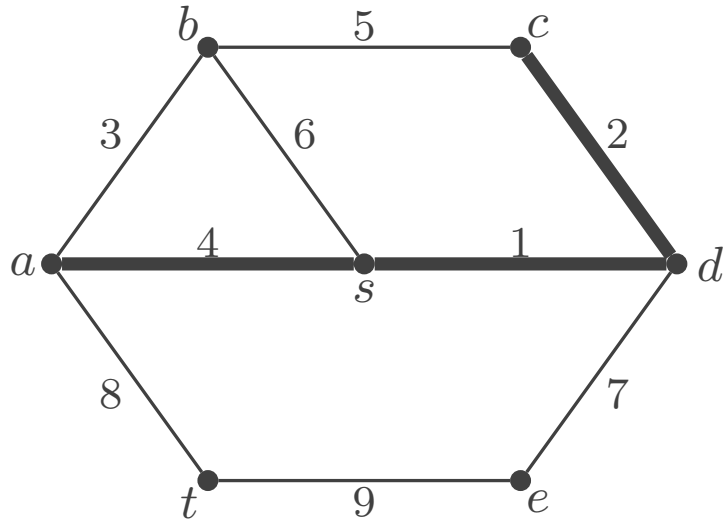

$t = 4$

Main technical tool: growth of a foremost tree

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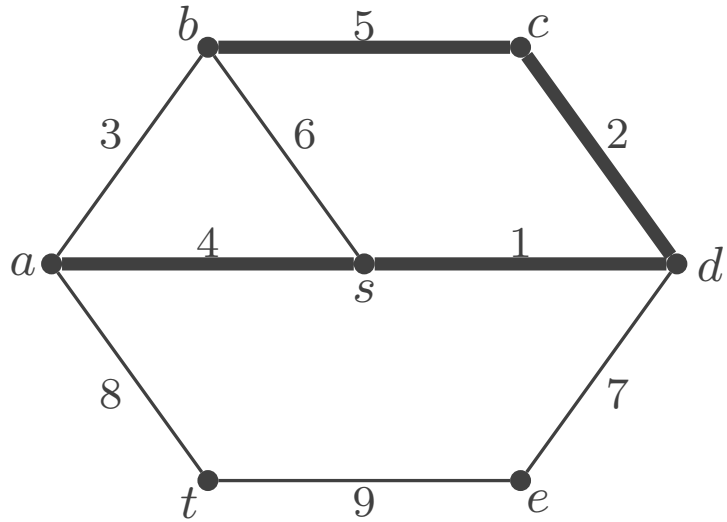

$t = 5$

Main technical tool: growth of a foremost tree

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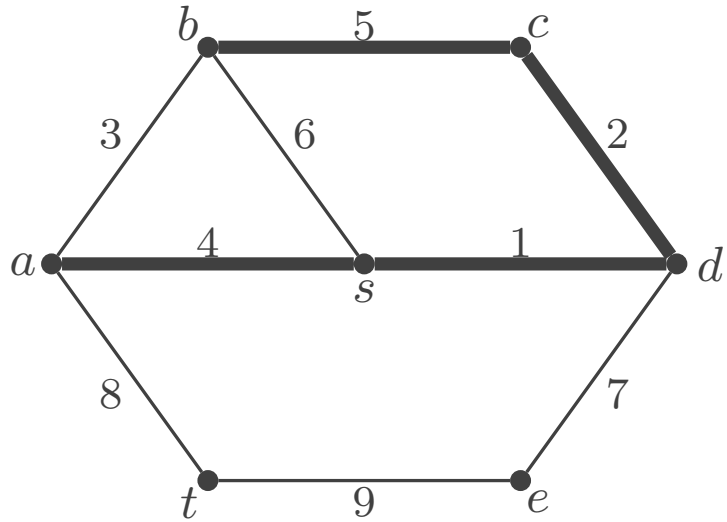

$t = 6$

Main technical tool: growth of a foremost tree

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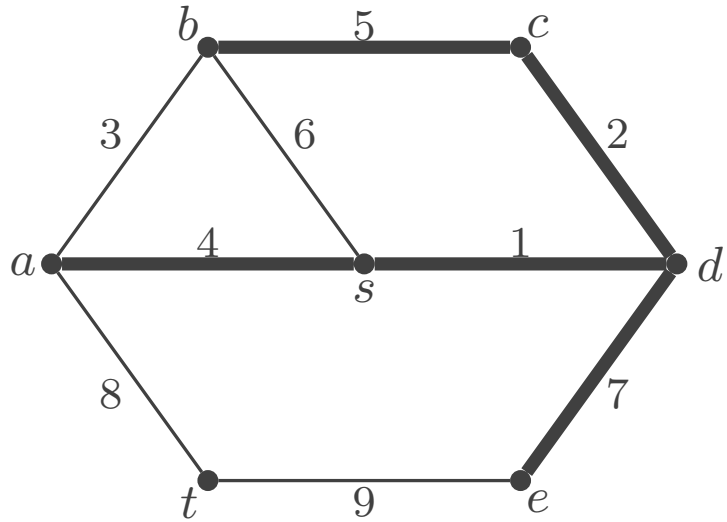

$t = 7$

Main technical tool: growth of a foremost tree

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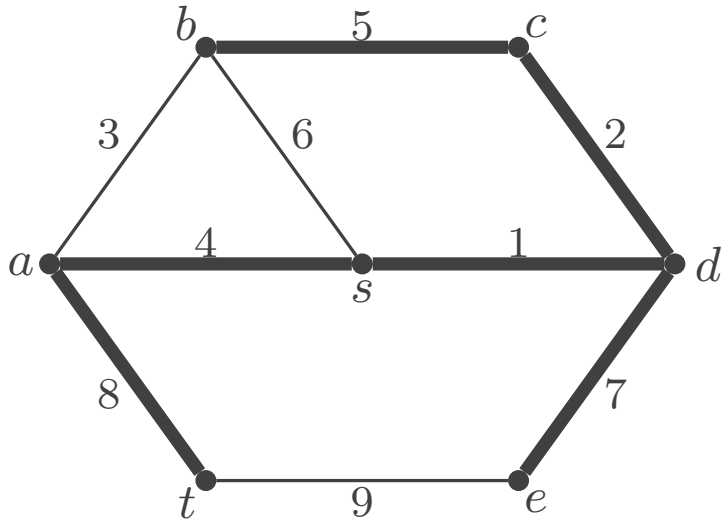

$t = 8$

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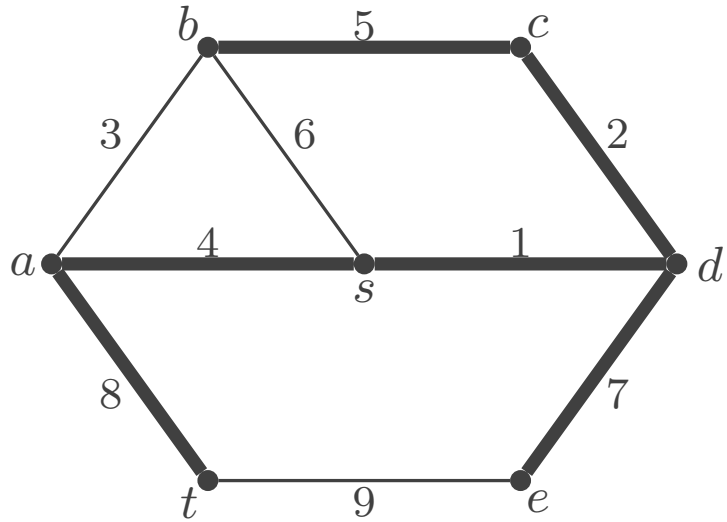

Analysis

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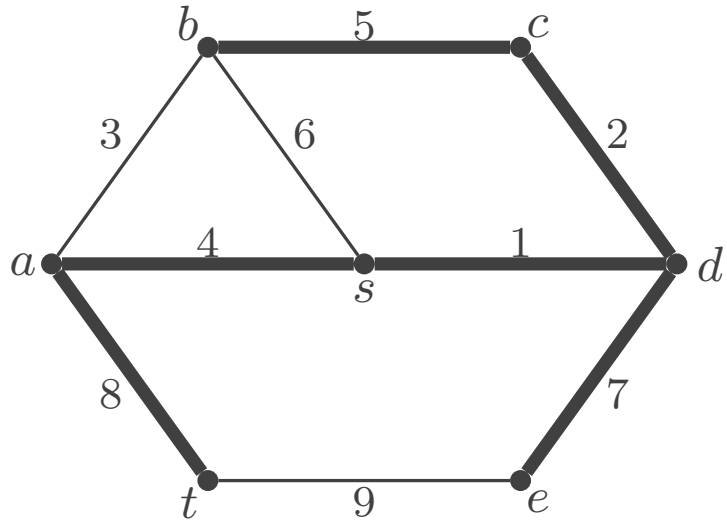

Analysis

- Consider "growing" a foremost tree in $\mathcal{G} \sim \mathcal{G}_{n,1}$ from a vertex s.

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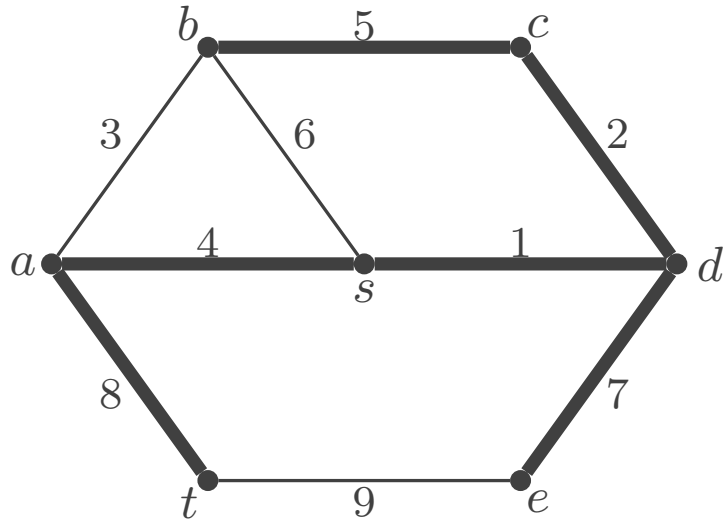

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- Consider "growing" a foremost tree in $\mathcal{G} \sim \mathcal{G}_{n,1}$ from a vertex s.
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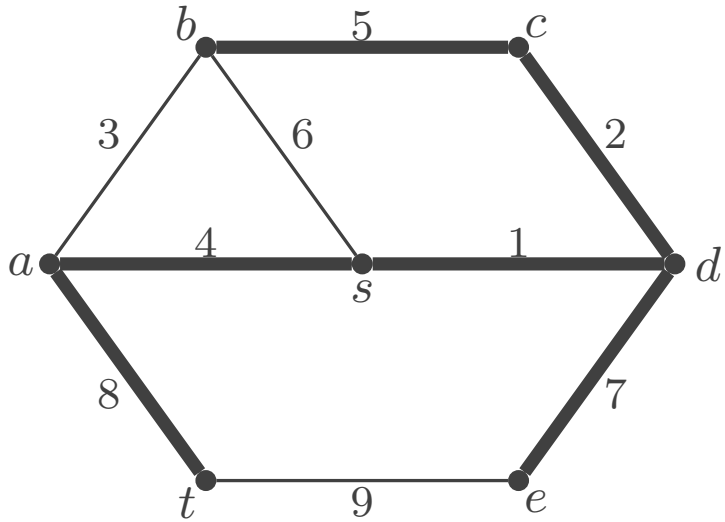

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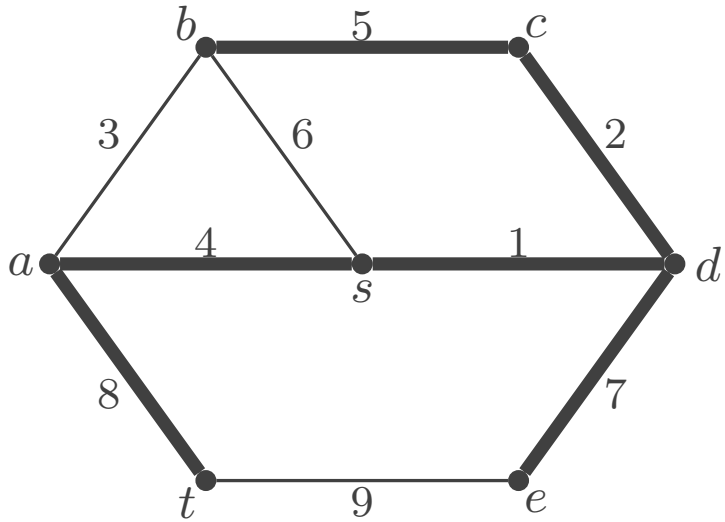

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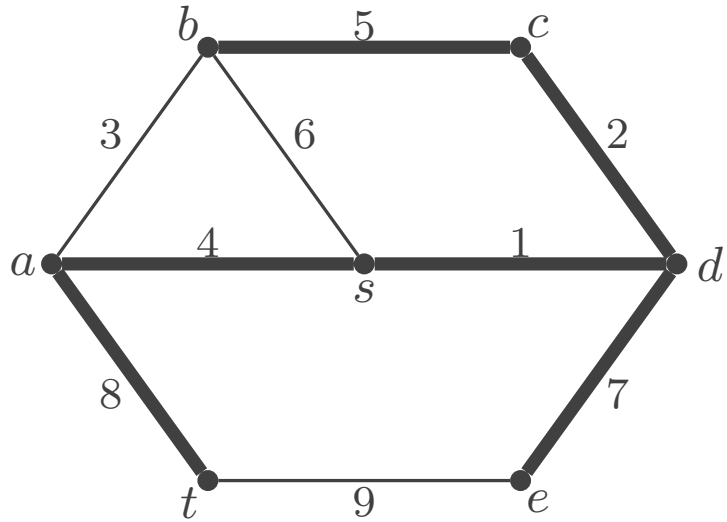

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- Azuma's inequality for concentration.

Derived arguments

We note $foremost(u)$ the set of vertices reached by a foremost tree from u.

Reachability thresholds

- $\sim* \rightsquigarrow \sim* \iff \forall u, \forall v, a.a.s.\ v \in foremost(u)$ $(\log n/n)$

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  LB: $(* \rightsquigarrow 1) + (\log n/n)$, each non sink must have at least one new edge.

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  Pivotal square. Sharp ?

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- Nearly optimal spanner (size $2n + o(n)$) $(3 \log n/n)$

  LB: Trivial (not temporally connected)

  UB: Explicit construction

  Three intervals of length $\log n/n$:

  - $\sim * \rightsquigarrow 1$ (say u) — By $\log n/n$
  - $u \rightsquigarrow \sim *$ — between $2 \log n/n$ and $3 \log n/n$.
  - $missing \rightsquigarrow u$ — between 0 and $2 \log n$
  - $u \rightsquigarrow missing$ — between $\log n/n$ and $3 \log n/n$
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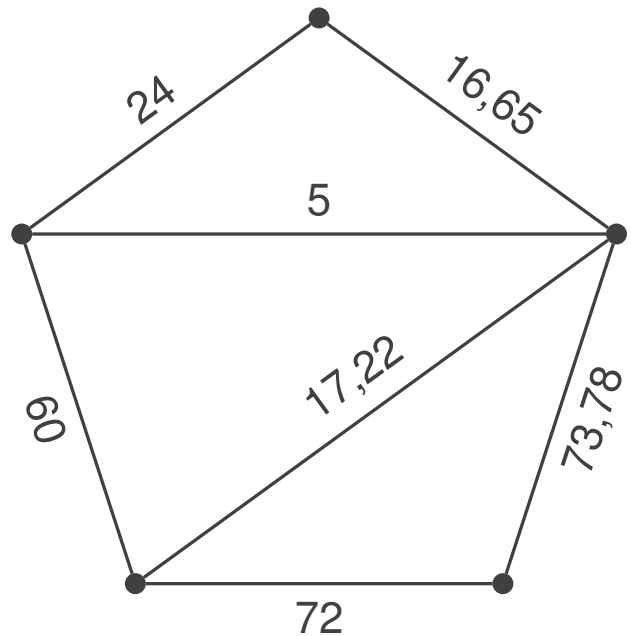
Random Non-Simple Temporal Graphs

$\mathcal{H}_{n,p}$: Each edge independently appears according to a rate 1 Poisson process stopped at time p.



Theorem

All our thresholds also hold for $\mathcal{H}_{n,p}$.

Open questions (deterministic)

Better spanners for temporal cliques

- Is $O(n \log n)$ optimal for cliques? Is $O(n)$ possible?

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Experiments:

n	sparsest spanner (# edges)	
4	4 or 5	exhaustive search
5	6 or 7	exhaustive search
6	8 or 9	exhaustive search
7	10 or 11	exhaustive search
...	...	
20	36 or 37	millions random instances

$\rightarrow$ $2n - 4 \leq OPT \leq 2n - 3$?

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  $\rightarrow$ Recall that $\exists$ unsparsifiable graphs of density $\Theta(n^2)$

  $\rightarrow$ Is there a family of graphs of density < 1 which admits sparse spanners?

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Thank you!