

Consequences for adjacency labelling schemes

Cyril Gavoille

LABORATOIRE
BORDELAIS
DE RECHERCHE
EN INFORMATIQUE

LaBRI - *University of Bordeaux*

JCRAALMA on Product Structure Theorems
@Home on BBB
May 29, 2020

Representation of a Graph



adjacency list

1	2,5
2	1,3
3	2,4,5
4	3
5	1,3

matrix

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	0	0
3	0	1	0	1	1
4	0	0	1	0	0
5	1	0	1	0	0

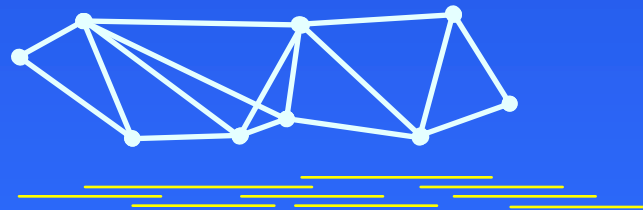
1 node = 1 pointer in the data-structure
(it does not carry any specific information)

Implicit Representation

To associate with the nodes more information, typically adjacency, and to remove the data-structure.

Interval graphs: $u \mapsto I(u) \subseteq [1, 2n]$

Edges: $u-v \Leftrightarrow I(u) \cap I(v) \neq \emptyset$



Compact representation: $O(\log n)$ bits/node

Possibly time $O(n)$ algorithms vs. $O(n+m)$

Labelling Schemes

P = a graph property defined on pairs of nodes

F = a graph family

A P -labelling scheme for F is a pair (λ, f) such that $\forall G \in F, \forall u, v \in V(G)$:

- [labelling] $\lambda(u, G)$ is a binary string
- [decoder] $f(\lambda(u, G), \lambda(v, G)) = P(u, v, G)$

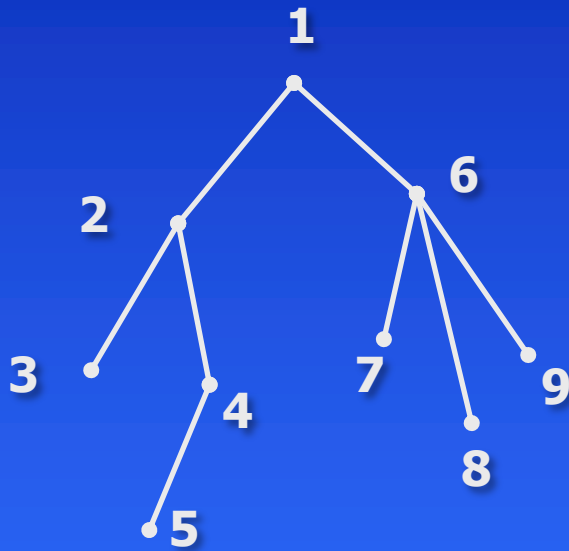
Goal: to minimize the maximum label size

In this talk: $P(u, v, G)$ is TRUE $\Leftrightarrow uv \in E(G)$

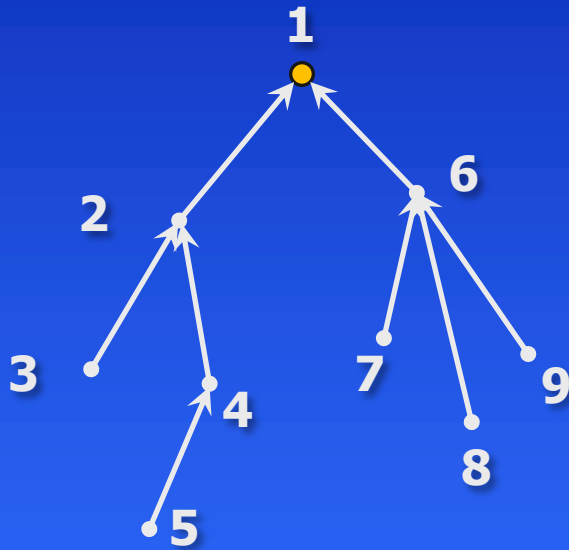
Basic Example: Trees



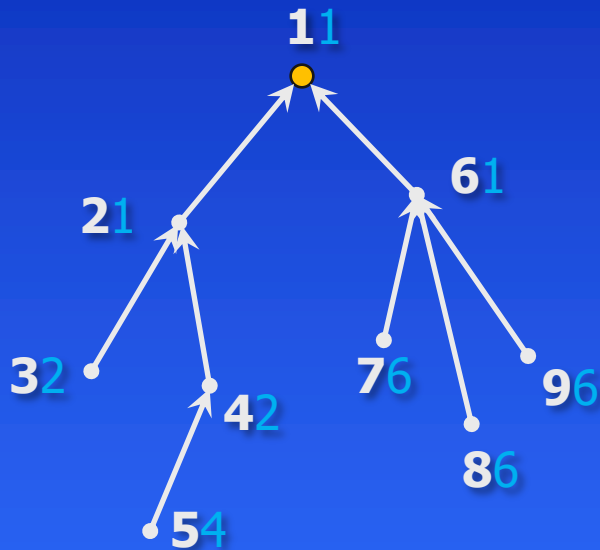
Basic Example: Trees



Basic Example: Trees



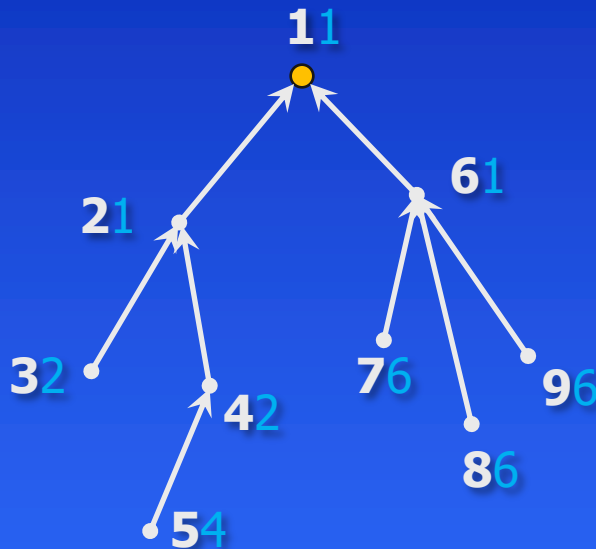
Basic Example: Trees



$\lambda(u, T) = (u, \text{parent}(u))$ or (u, u)

$f(uv, xy) = (v = x$ or $u = y)$

Basic Example: Trees



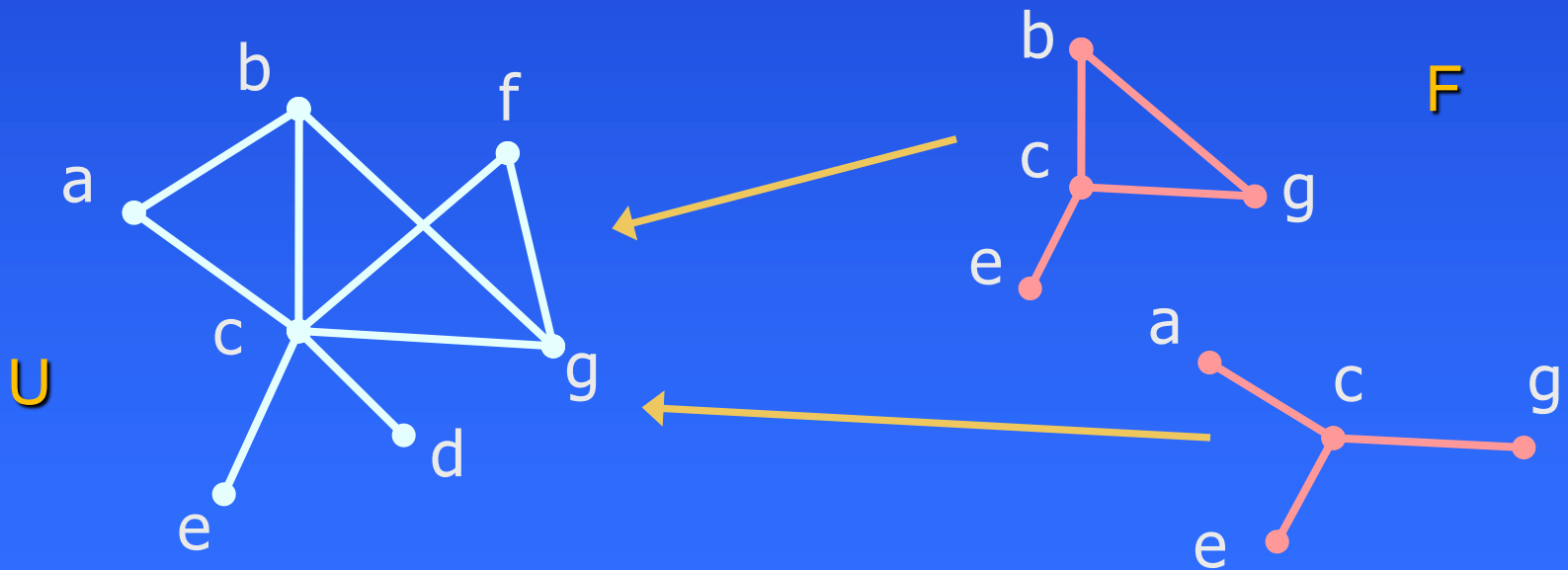
$$\lambda(u, T) = (u, \text{parent}(u)) \text{ or } (u, u)$$
$$f(uv, xy) = (v=x \text{ or } u=y)$$

For trees with n nodes: $\sim 2 \log n$ bits/node
(the constant does matter [Abiteboul et al. - SICOMP '06])

Induced-Universal Graphs

[Babai, Chung, Erdős, Graham, Spencer '82]

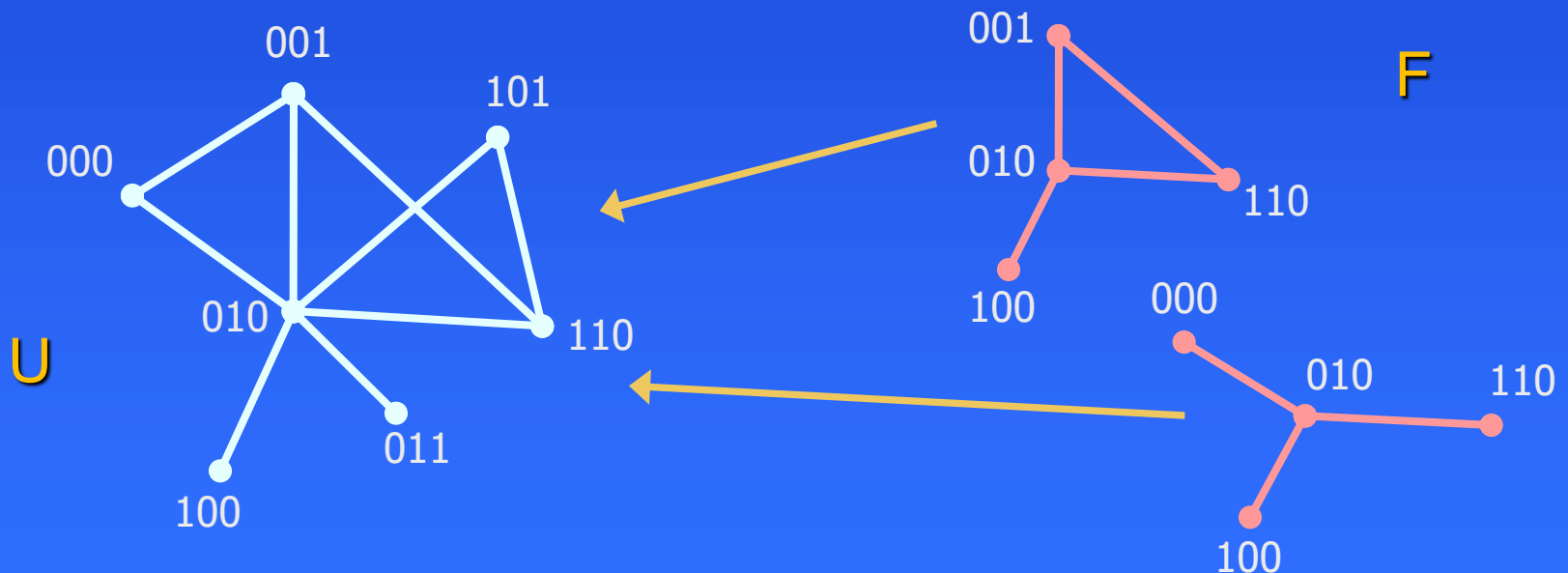
A graph U is an **induced-universal** graph for the family F if every graph of F is isomorphic to an *induced* subgraph of U .



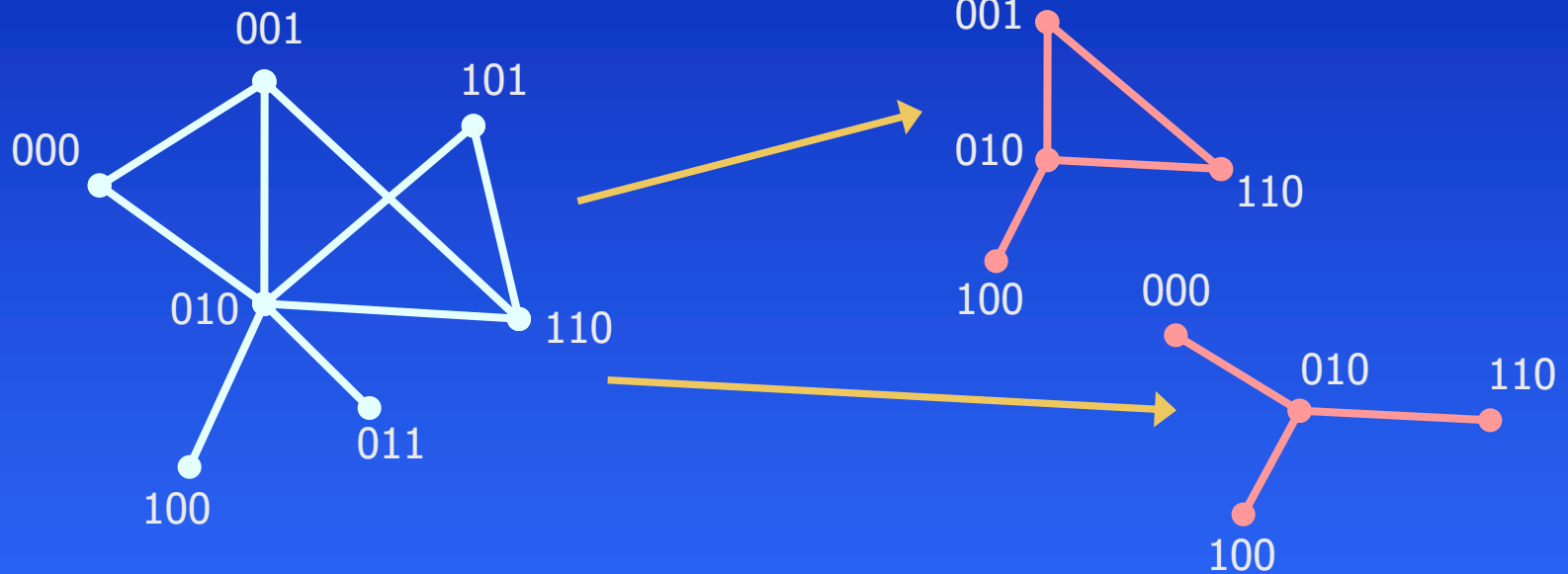
Induced-Universal Graphs

[Babai, Chung, Erdős, Graham, Spencer '82]

A graph U is an **induced-universal** graph for the family \mathcal{F} if every graph of \mathcal{F} is isomorphic to an *induced* subgraph of U .



Induced-Universal Graphs



induced-universal graph U

graphs of F

$c \log n$ -bit labelling \Leftrightarrow induced-universal graph of n^c nodes

Universal Graphs for Trees (for $n=6$ nodes)

From the $(u, \text{parent}(u))$ labelling
 \Rightarrow induced-universal graph of $n^2=36$ nodes



Using DFS for T: (u, v)
 $\Rightarrow u > v$ or $u = v = 1$

Universal Graphs for Trees (for $n=6$ nodes)

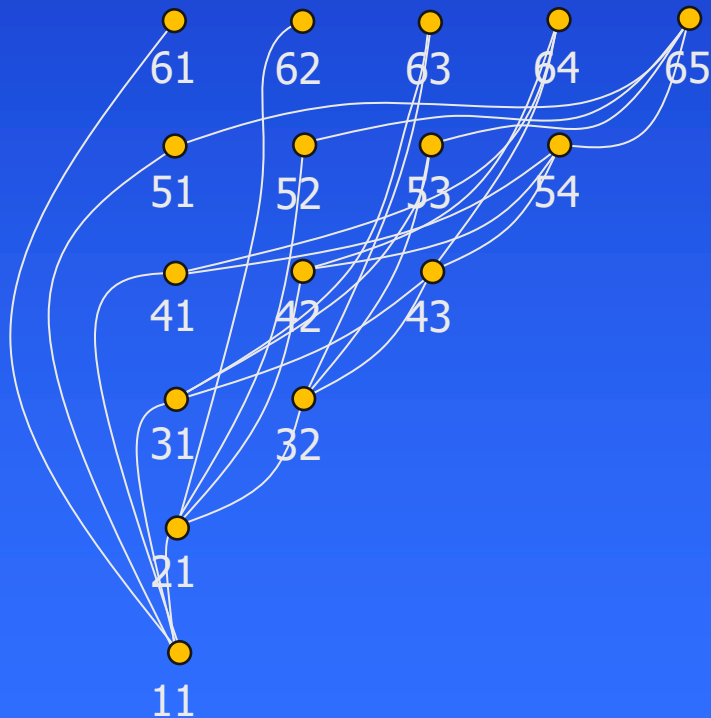
From the $(u, \text{parent}(u))$ labelling
 \Rightarrow induced-universal graph of $n^2=36$ nodes



Using DFS for T: (u, v)
 $\Rightarrow u > v$ or $u = v = 1$
 $\Rightarrow n(n-1)/2 + 1 = 16$ nodes

Universal Graphs for Trees (for $n=6$ nodes)

From the $(u, \text{parent}(u))$ labelling
 \Rightarrow induced-universal graph of $n^2=36$ nodes



Using DFS for $T: (u, v)$
 $\Rightarrow u > v$ or $u = v = 1$
 $\Rightarrow n(n-1)/2 + 1 = 16$ nodes

Universal Graphs for Trees (universal trees)

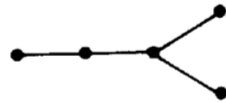
$n = 1$:



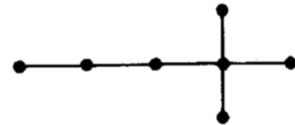
$n = 2$:



$n = 3$:



$n = 4$:



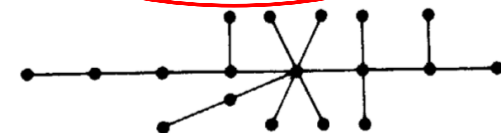
$n = 5$:



$n = 6$:



$n = 7$:

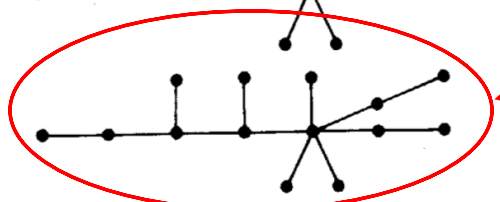


ON GRAPHS WHICH CONTAIN ALL SMALL TREES, II.

F.R.K. CHUNG — R.L. GRAHAM — N. PIPPENGER

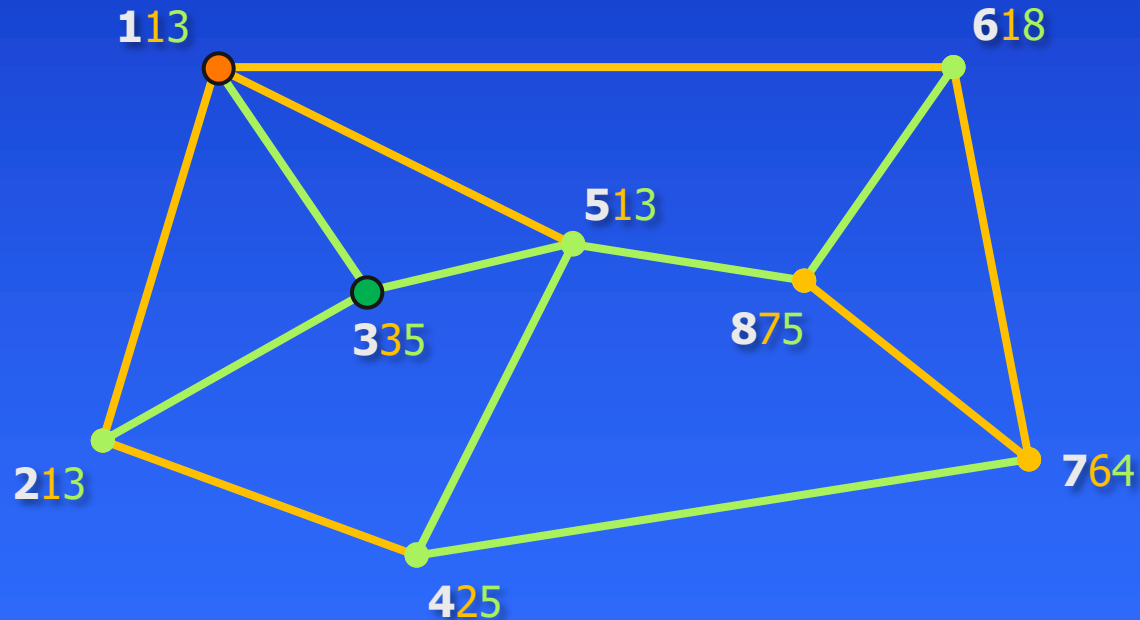
COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
18. COMBINATORICS, KESZTHELY (HUNGARY), 1976.

14 nodes



Labelling Schemes for Planar Graphs

Edge partition: combining schemes



Arboricity- k graphs: $(k+1)\log n$ bits

\Rightarrow Planar ($k=3$): **4** $\log n$ bits [KNR - STOC'88]

Better Labelling Schemes

For trees: $\log n + O(\log^* n)$, $\log n + O(1)$

[Alstrup,Rauhe – FOCS'02]

[Alstrup,Dahlgaard,B.T.Knudsen – FOCS'15 & JACM'17]

⇒ Arboricity- k : $k \log n + O(1)$

⇒ Planar: **3** $\log n + O(1)$

For treewidth- k : $\log n + O(k \log \log n)$

[G.,Labourel – ESA'07]

⇒ Planar & Minor-free: **2** $\log n + O(\log \log n)$

Bonamy, G., Pilipczuk - SODA'20

For planar & bounded genus: $\frac{4}{3} \log n + O(\log \log n)$
 \Rightarrow

**Induced-universal graph of $n^{\frac{4}{3}+o(1)}$
nodes for n -node planar graphs
(and bounded genus graphs)**

Labelling the nodes is polynomial
Decoding adjacency takes constant time

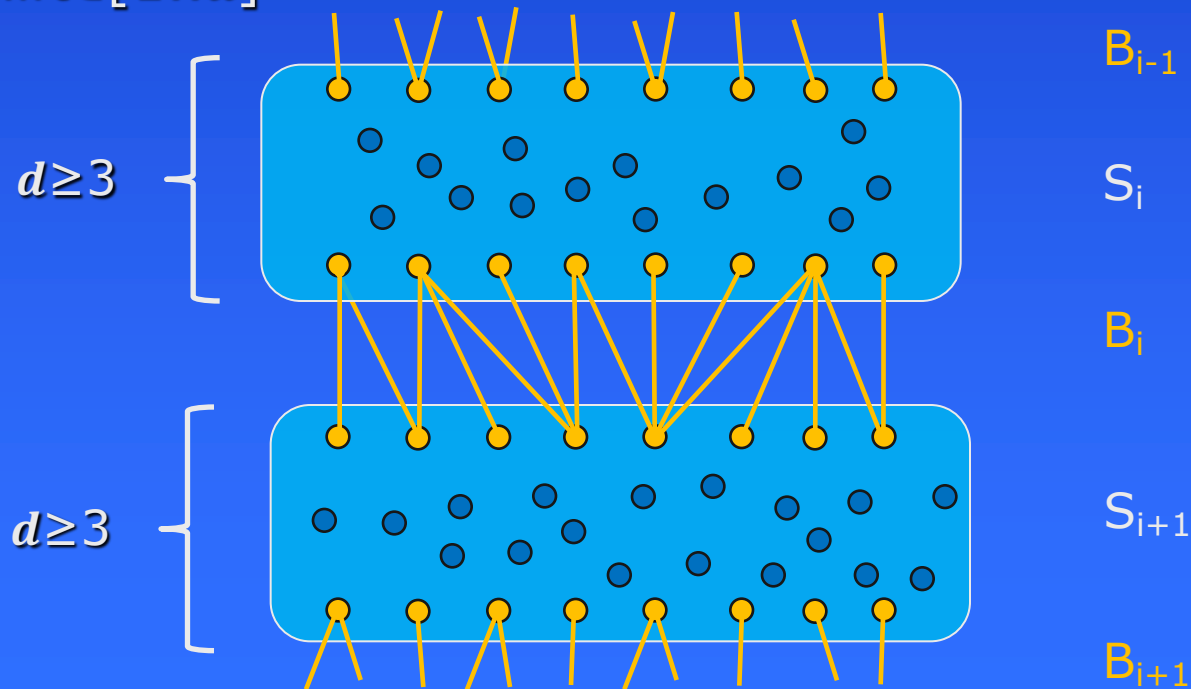
Sketch of Proof (1/2)

Edge partition: $G = S \cup B$ (Strips & Border)

S : components have d layers $\sim n^{1/3}$

B : has $\text{treewidth} \leq 5$ and $n/d \sim n^{2/3}$ nodes

BFS & $\text{shift} \in [1..d]$



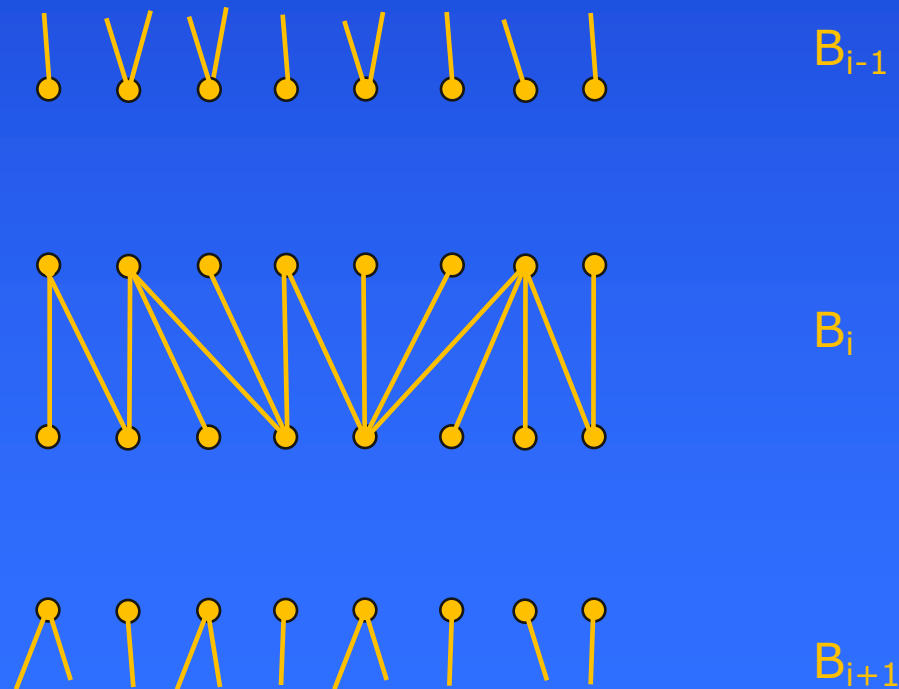
Sketch of Proof (1/2)

Edge partition: $G = S \cup B$ (Strips & Border)

S: components have d layers $\sim n^{1/3}$

B: has $\text{treewidth} \leq 5$ and $n/d \sim n^{2/3}$ nodes

BFS & $\text{shift} \in [1..d]$



Sketch of Proof (2/2)

Labelling for B: $\log(n/d)$



Labelling for S: $\log n + \log d$

new!

[up to $+O(\log \log n)$ terms]

Problem: nodes in $V(B)$ pay both labels

$\Rightarrow \log(n/d) + \log n + \log d = 2 \log n$ ☹️

Improved labelling for S: nodes in $V(B)$ pay only $\log |B| = \log(n/d)$ bits!

\Rightarrow nodes in $S \setminus V(B)$: $\log n + \log d = \frac{4}{3} \log n$ 😊

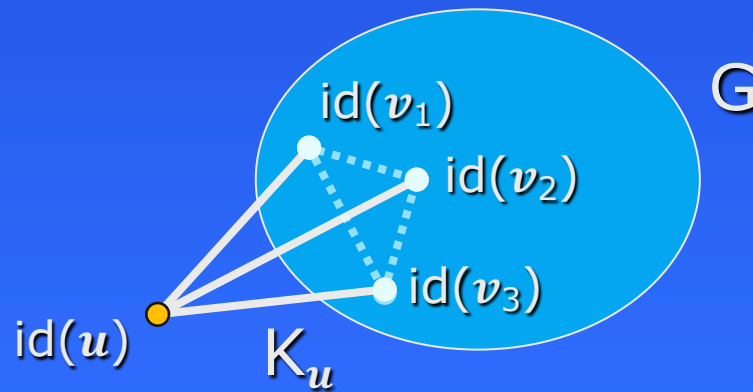
\Rightarrow nodes in $V(B)$: $\log(n/d) + \log(n/d) = \frac{4}{3} \log n$ 😊

Improved Scheme for Treewidth- k

$G = \text{treewidth-}k$, $V(G) = V_1 \cup V_2$, $K_u = N[u] =$ simplicial complex of u , $|K_u| \leq k+1$.

Lemma. G has a scheme providing, for each u , $\text{id}(u)$ and $\lambda(u)$ st. $\forall v \in K_u$ $\text{id}(v)$ can be extracted from $\lambda(u)$. Moreover, for $u \in V_i$

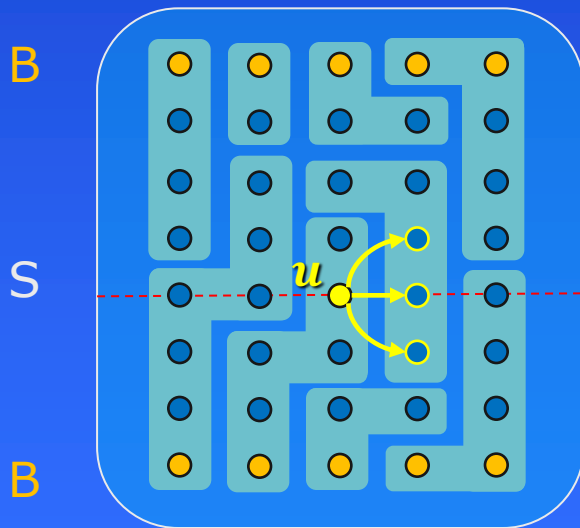
$$|\lambda(u)| = \log |V_i| + O(k \log \log |V(G)|).$$



$$u-v \Leftrightarrow \text{id}(v) \in \{\text{id}(K_u)\} \text{ or } \text{id}(u) \in \{\text{id}(K_v)\}$$

Labelling Scheme for S

Key lemma. [2018,2019] If G is planar, there is a node partition into monotone paths taken from any given BFS such that contracting each one leads to a treewidth-8 graph.



Label $\lambda(u)$ consists of:

- the treewidth-8 scheme
- the depth of u in S (unless $u \in B$)
- 3 bits/path in the treewidth-8 scheme, i.e., $3 \times 8 = 24$ extra bits

[Extend to genus- g graphs]

Improved Bound: Dujmovic, Esperet, G., Joret, Micek, Morin

Theorem. The family of n -vertex subgraphs of $H \boxtimes P$ has a labelling scheme with $\log n + o(\log n)$ bit labels, where H has bounded treewidth and P is a path.



**Induced-universal graph of $n^{1+o(1)}$ nodes
for n -node planar graphs**

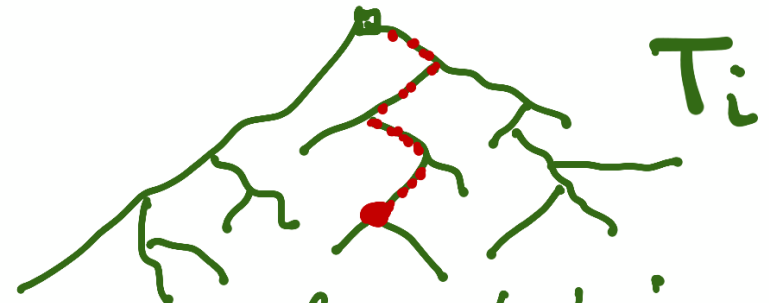
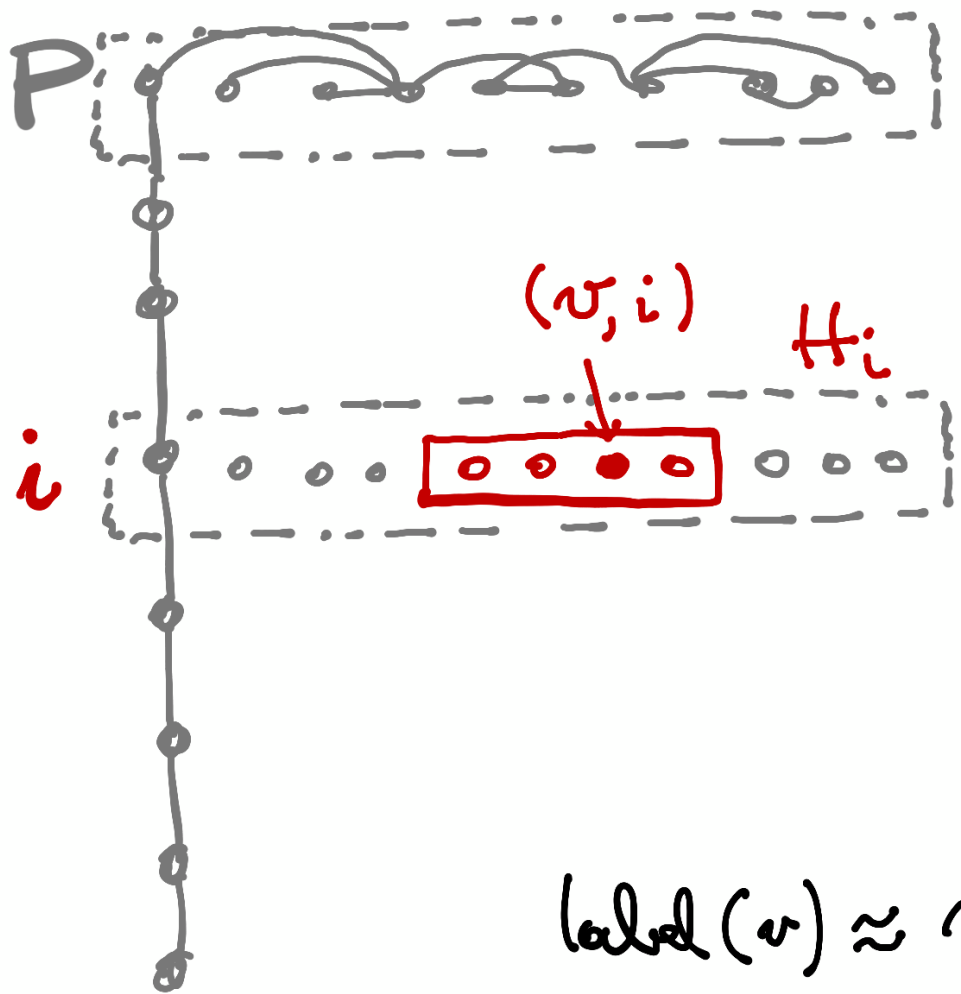
(and many other families of graphs)

Labelling the nodes takes $O(n \log n)$ time

Decoding adjacency takes $\sim \sqrt{\log n}$ time

①

H



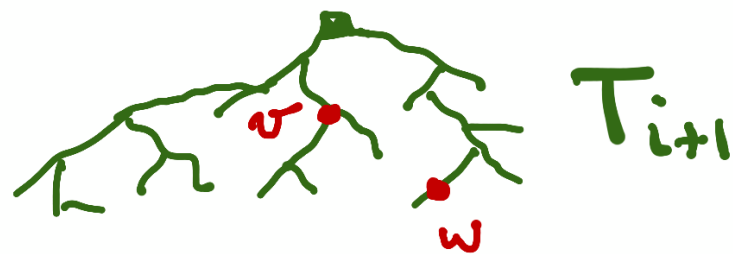
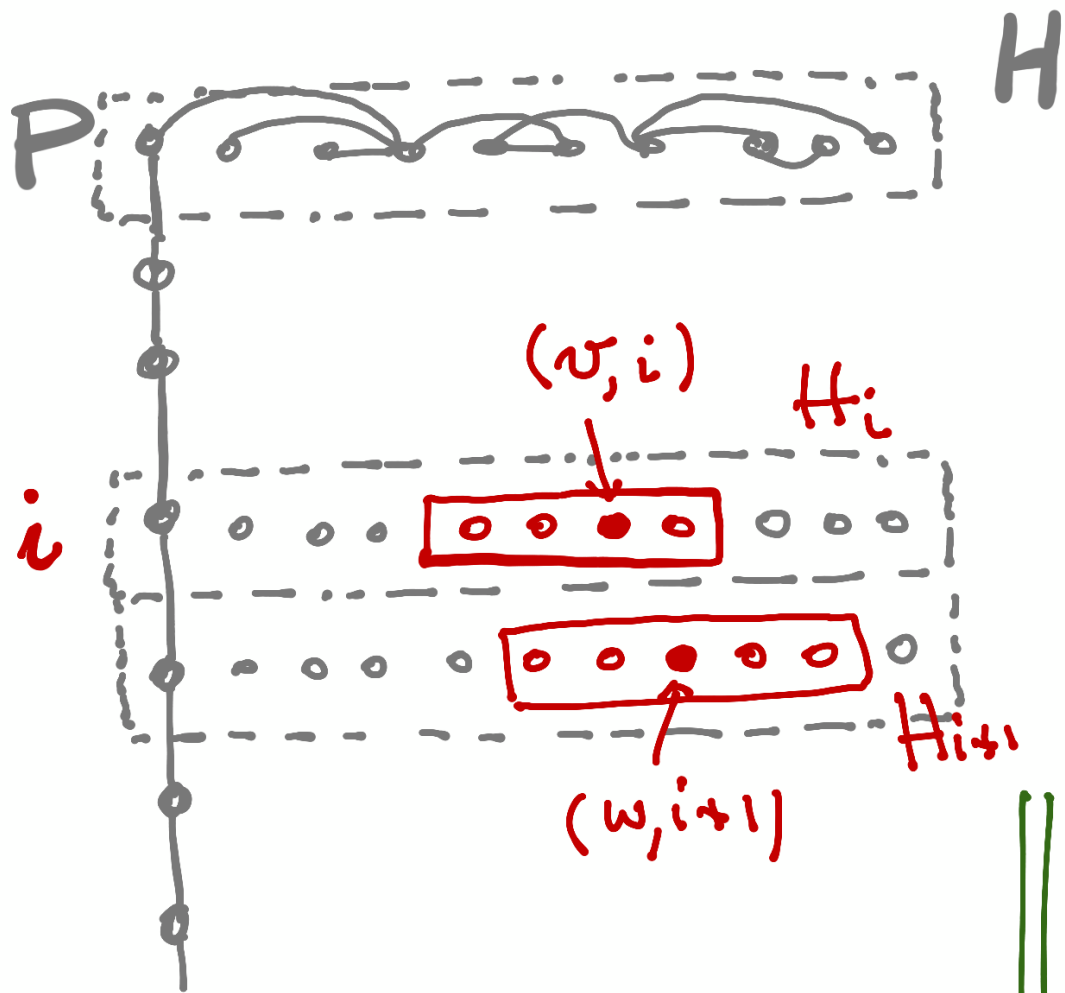
balanced binary search tree

$v \rightarrow$ position in T_i

$label(v) \approx$ root-to-position (v) path

adj: $(v, i) - (v', i)$ ✓

②



T_{i+1} spans labels of H_{i+1} and H_i

PB: Need to know position of v in T_{i+1} from position of v in $T_i \dots$

PB: Need to know position
of v in T_{i+1} from position of v in T_i ...

③

Key obs.: T_{i+1} and T_i are not indep.

\Rightarrow Given $o(\log n)$ bits we can extract
position of v in T_{i+1} given its pos. in T_i .

Tools: $T_i \rightarrow T_{i+1}$

- bulk insertions
- bulk deletions
- rebalancing

Open Problems

1. Improve the 2nd order term, $\log n + O(1)$?
2. Extend to minor-free graphs
3. Improve to $\log n + \theta(k)$ for treewidth- k
4. Prove lower bounds for planar or minor-free

Best lower bound for planar: $\log n + \Omega(1)$ ☹️

No families with $n! 2^{O(n)}$ labelled graphs like trees, planar, bounded genus, bounded treewidth, minor-free (hereditary)... is known to require labels of $\log n + \omega(1)$ bits.

THANK YOU!

