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A Note on Implicit Representation of Graphs

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Abstract

An implicit representation for a given family of graphs is a node labeling scheme of all the n -node graphs of the family by binary strings of length $O(\log n)$ such that adjacency between any two nodes can be decided from their labels only. In this note, we design a $3 \log n$ -bit labeling scheme for planar graphs constructible in $O(n)$ time and with constant time adjacency test, improving by a $\log n$ additive factor the result of Kannan, Naor and Rudich [KNR88]. More generally, we show that bounded k -arboricity graphs support an asymptotic $k \log n$ -bit labeling scheme with $O(k)$ adjacency test. For large k , we construct a $O(k \log(n/k))$ -bit labeling scheme with constant adjacency test, and show that the label length is asymptotically optimal.

Keywords: compact graph representation, implicit representation, partial k -trees, k -decomposable graphs, arboricity, degeneracy

1 Introduction

There are two basic ways of representing an n -node m -edge graph: matrix or edge list representation. The first one uses $O(n^2)$ bits and answers in constant time whether any two nodes are adjacent or not. The list representation uses $O(n + m)$ memory words¹, whereas such a data-structure does not support in the worst-case constant time adjacency queries.

So matrix representation is time-efficient but not space-efficient for sparse graphs, like planar graphs that have at most $3n - 6$ edges. However, for several graph families, there are ways to compact the representation of a graph. For instance, Chang, Lin and Lu showed in [CLL01] that planar graphs support a $2m + 2n + o(n)$ bits representation with constant time queries. Although this representation is rather compact (this is $8n + o(n)$ bits for $m = 3n - o(n)$ bits), it does not allow *label-based* graph representation. A label-based representation allows retrieving useful information about arbitrary functions or substructures

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¹In this paper we assume that in constant time one can compute standard arithmetic operations on words of length $O(\log n)$ bits.

in a graph in a *localized* manner, i.e., using only local pieces of information available to, or associated with, the nodes under inspection, and not having to search for additional global information. It is clearly a desirable property for a graph in the framework of distributing computing where individual processor element of a network want to communicate with its neighbors but has not enough local memory resources to store all the underlying topology of the network (see [GP01] for an overview).

In this spirit, Kannan, Naor, and Rudich [KNR88] have proposed to label each node x of a planar graph by a pair $(x, P(x))$ where $P(x)$ is a set of at most three neighbors of x such that x and y are adjacent if and only if either $y \in P(x)$ or $x \in P(y)$. The nodes being represented by a unique integer taken from $\{1, \dots, n\}$, this yields to a representation of every planar graph with local labels of $4 \log n$ bits and supporting constant time adjacency queries. A similar $4 \log n$ -bit labeling scheme has been obtained by Chrobak and Eppstein [EC91]. This naturally leads to the following definition.

Definition 1 *A graph family \mathcal{F} has a $f(n)$ -adjacency labeling scheme if there exists a pair (L, A) of functions, called respectively labeling and adjacency decoder, such that for every n -node graph G of \mathcal{F} , and for all nodes x, y of G ,*

1. $L(x, G)$ is a binary string of length at most $f(n)$;
2. $A(L(x, G), L(y, G)) = 1$ if and only if x and y are adjacent in G .

As defined in [KNR88], a graph family \mathcal{F} has an *implicit representation* if \mathcal{F} has an $O(\log n)$ -adjacency labeling scheme². Several graph families support an implicit representation: some intersection graph families, like interval graphs, paths graphs (intersection of paths in a tree), circle graphs (intersection of chords in a circle), permutation graphs, circular-arc graphs, graphs with bounded interval number ... (see Golumbic for definitions of these families [Gol80]), but also graphs with bounded arboricity [KNR88]. The *arboricity* of a graph G is defined³ to be

$$k = \max_H \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil$$

where H range over all possible induced subgraphs of G containing at least two nodes. Arboricity is related to *degeneracy* of graphs. A graph is d -degenerated if it has a node of degree at most d whose deletion leave the graph d -degenerated. The degeneracy is the smallest integer d that makes the graph d -degenerated. It is easy to see that every graph of arboricity k and degeneracy d satisfies $d/2 \leq k \leq d$. Moreover, $k \leq \Delta/2 + 1$ if the graph is of maximum degree Δ . Many families of graphs have bounded arboricity: bounded

²In this paper all the logarithms are in based two.

³Usually, there is no ‘‘ceiling’’ in the definition of arboricity. However, by the Nash-Williams’ Theorem quoted hereafter, there is no restrictions to define integral arboricity.

degree graphs, bounded genus graphs, bounded tree-width graphs, geometric neighborhood graphs [MTV91], bounded pagenumber graphs, and bounded degeneracy graphs.

The Nash-Williams' Theorem [CMW⁺94] states that the edges of a k -arboricity graph can be decomposed into k forests. Finding a decomposition in exactly k forests is non-trivial. There is an $O(nm \log n)$ algorithm [GW92] using matroid technique, that is an $O(kn^2 \log n)$ time algorithm since the number of edges m is at most $k(n-1)$. A simple $O(kn)$ algorithm exists for decomposition into $2k-1$ forests [AMZ97].

In [KNR88] it is shown that graphs of arboricity bounded by k have a $(k+1) \log n$ -adjacency labeling scheme. Combining several recent results we improve in this note implicit representations based on arboricity.

In Section 2 we show that graphs with large arboricity k support a $k \log(n/k)$ -bit labeling scheme with constant time queries, and we show that the bound on the label length is tight. In Section 3 we show a labeling with $k \log n$ -bit labels and supporting $O(k)$ adjacency queries, providing a $3 \log n$ -bit labeling for planar graphs and a $2 \log n$ -bit labeling for series-parallel graphs. Finally, in section 4, we show that k -decomposable graphs (or partial k -trees) have a $k \log n$ -bit labeling scheme and that for this family we prove a $\Omega(k + \log n)$ lower bound on the label length. We leave in Section 5 as an open problem the design of a $3 \log n$ -bit labeling scheme for all genus 1 graphs, i.e., all the graphs having an embedding on the torus.

2 A Scheme for Large Arboricity Graphs

In this section we consider the family of n -node graphs whose arboricity is bounded by some function $k(n)$.

Let G be such a graph, and w.l.o.g. assume that the node set is $\{1, \dots, n\}$. By the Nash-Williams' Theorem, the edges of G can be decomposed into $k = k(n)$ forests. Let assume that the forest decomposition is given, namely F_1, \dots, F_k . Clearly, two nodes x and y are adjacent in G if and only if there is a forest F_i in which x is adjacent to y . We assume that each tree of each F_i is rooted at some arbitrary node.

As done in the scheme of Kannan, Naor and Rudich [KNR88] presented previously, we associate to each node x of G the set $P(x) = \bigcup_{i=1}^k \{y \text{ parent of } x \text{ in } F_i\}$. The label of x is set to $L(x, G) = (x, P(x))$. The adjacency decoder is defined by the following two membership queries: $A((x, P(x)), (y, P(y))) = 1$ if and only if $y \in P(x)$ or $x \in P(y)$.

Observe that $P(x) \subset \{1, \dots, n\}$ and that $|P(x)| \leq k$. Let $B = \lceil \log \binom{n}{k} \rceil$ be the number of bits for coding a k -element subset of $\{1, \dots, n\}$. The label $L(x, G) = (x, P(x))$ might be coded by a binary string of $\lceil \log n \rceil + B$ bits.

In [BM99], Brodnik and Munro have showed that every k -element subset of $\{1, \dots, n\}$ can be encoded into a data-structure (polynomially-time constructible) of $B + O(B / \log^{(3)} n)$ bits supporting constant time membership queries under a standard word-RAM computa-

tional model. So, the label of x can be encoded with $\log n + B + o(B)$ bits such that the adjacency decoder A can be computed in constant time. We have $(n/k)^k \leq \binom{n}{k} \leq (ne/k)^k$, thus $B = k \log(n/k) + O(k)$. On the other hand $B \geq \log n$ for every $k \notin \{0, n\}$, thus the label length for x is at most $k \log(n/k) + O(k + \log n) + o(k \log(n/k))$.

Disregarding the $o(k \log(n/k))$ term when n is large enough, we have:

Theorem 1 *The family of bounded k -arboricity graphs have an asymptotic $k \log(n/k) + O(k + \log n)$ -adjacency labeling scheme with constant time decoder. Moreover, all the labels are polynomially-time constructible.*

Actually the label length upper bound given in Theorem 1 is tight since:

Theorem 2 *For all n and k , every $f(n, k)$ -adjacency labeling scheme on the family of bounded k -arboricity graphs requires $f(n, k) > \frac{1}{2}k \log(n/k) - \frac{1}{2}k - 1$.*

Proof. As seen in the introduction, since every graph is of maximum degree $n - 1$, one can restrict our attention to arboricity $k \leq \lfloor (n - 1)/2 \rfloor + 1 = \lceil n/2 \rceil$. Let $p = \lfloor n/2 \rfloor$ and $q = \lceil n/2 \rceil$. We consider the family \mathcal{F} of labeled graphs defined as follows. The node set is $X \cup Y$ with $X = \{1, \dots, p\}$ and $Y = \{p + 1, \dots, p + q\}$. We have $|X \cup Y| = p + q = n$, and $k \leq q \leq \lceil n/2 \rceil$. The edges link the nodes of X to the nodes of Y only (so the graphs of \mathcal{F} are bipartite) such a way that the degree of every node in X is exactly k .

Consider any $G \in \mathcal{F}$. We can partition the edges of G such that, for each $x \in X$, the k edges incident to x belong to a distinct forest. Since G is bipartite, every cycle in some forest would implies that two edges incident to a certain $x_0 \in X$ belongs to this cycle, contradicting the edge assignment of x_0 's edges. It follows that G is of arboricity bounded by k .

Every graph in \mathcal{F} has n nodes, and the number of graphs in \mathcal{F} is $|\mathcal{F}| = \binom{q}{k}^p$. Let (L, A) be any $f(n, k)$ -adjacency labeling scheme on \mathcal{F} , for some function $f(n, k)$. Consider the set $\mathcal{L} = \{(L(1, G), \dots, L(n, G)) \mid G \in \mathcal{F}\}$. We have, $|\mathcal{L}| \geq |\mathcal{F}|$. Indeed, if $|\mathcal{L}| < |\mathcal{F}|$ then there exists two distinct graphs $G, H \in \mathcal{F}$ such that $L(v, G) = L(v, H)$ for all node $v \in \{1, \dots, n\}$, that is impossible. On the other hand, if all the labels are binary strings of length at most $f(n, k)$ bits, then $|\mathcal{L}| \leq S^n$ where $S = \sum_{i=0}^{f(n, k)} 2^i = 2^{f(n, k)+1} - 1$ is the number of binary strings of length at most $f(n, k)$. Thus we have

$$|\mathcal{F}| = \binom{q}{k}^p \leq |\mathcal{L}| < 2^{(f(n, k)+1)n} . \quad (1)$$

Because $\binom{q}{k} \geq (q/k)^k$, we have

$$\log \binom{q}{k}^p = \lfloor \frac{n}{2} \rfloor \log \binom{\lceil n/2 \rceil}{k} \geq \frac{1}{2}nk \log \left(\frac{n}{2k} \right) .$$

This implies that $f(n, k) > \frac{1}{2}k \log(n/k) - \frac{1}{2}k - 1$, that completes the proof. \square

3 An Improved Labeling Scheme

In this section we consider the family of n -node graphs whose arboricity is bounded by some constant k . The next result is more compact than Theorem 1.

Theorem 3 *The family of bounded k -arboricity graphs have an asymptotic $k \log n$ -adjacency labeling scheme with a $O(k)$ time decoder. Moreover, all the labels can be constructed in $O(kn^2 \log n)$ time, and in the case of planar graphs ($k = 3$) the time can be reduced to $O(n)$.*

Proof. Let G be a graph of arboricity at most k . By the Nash-Williams' Theorem, the edges of G can be decomposed into k forests. Firstly, let assume that the forest decomposition is given, and let F_1, \dots, F_k be these forests, each one spanning the nodes of G (so eventually F_i 's may contain zero degree node). Clearly, x and y are adjacent in G if and only if there is a forest F_i in which x and y are adjacent. For each $i \in \{1, \dots, k\}$, we apply the following construction: For each connected component of F_i pick an arbitrary node x_i^j and form a tree T_i rooted at a new node linked to all the x_i^j 's node of F_i .

In [AR01, KM01], it is showed that n -node trees have an $\log n + O(\sqrt{\log n})$ -adjacency labeling scheme with constant time decoder (under a standard word-RAM computational model). Moreover the time to construct the n labels is $O(n)$. So, we associate such a binary label $\ell(x, T_i)$ to each node x of T_i as done in [AR01, KM01]. The length of the label is $\log |V(T_i)| + O(\sqrt{\log |V(T_i)|})$ that is $\log n + O(\sqrt{\log n})$ bits since T_i has $n+1$ nodes. Finally, to each node x of G we associate the k -tuple $L(x, G) = (\ell(x, T_1), \dots, \ell(x, T_k))$ of length $k \log n + O(k\sqrt{\log n})$. To test adjacency it suffices to test adjacency in each component of the labels. Observe that once the forests have been constructed it takes $O(kn)$ time to set all the labels, and $O(k)$ time to decode adjacency.

There is an $O(kn^2 \log n)$ time algorithm to find a decomposition of G into k forests [GW92]. For planar graphs ($k = 3$) there is an $O(n \log n)$ time algorithm [GL98]. We can improved this result by the use of *realizer* of plane triangulation⁴, notion introduced by [Sch90] for grid embedding of planar graphs (a similar linear time algorithm can be founded in [Epp94]). A *realizer* of a plane triangulation is a particular partition of the interior edges in three sets of directed edges: T_1, T_2, T_3 . The three sets are computable in $O(n)$ time [Epp94], and [Sch90] proved that each T_i is a directed tree including all interior nodes and exactly one exterior node distinct for each i . So, three edge-disjoint forests can be constructed in $O(n)$ time for every plane triangulation. Since in linear time one can triangulate and embed on the plane any planar graph G , it follows that the partition of the edges of G into three forests can be done in $O(n)$. This completes the proof. \square

⁴A plane triangulation is a planar graph embedded on the plane whose all its faces are triangles.

4 An Improved Scheme for k -Decomposable Graphs

A graph G is k -decomposable if either G has $k + 1$ or fewer nodes or there is a subgraph S of G with at most k nodes such that $G - S$ augmented by S with completely connected nodes is k -decomposable [AP87]. In [KNR88], it was showed that k -decomposable graphs have an asymptotic $k \log_{3/2} n \approx 1.7k \log n$ -bit labeling scheme, observing that S can be chosen such that $G - S$ can be partitioned in two sets of nodes, A and B , with no edges linking A to B and with $|A|, |B| \leq 2|V(G)|/3$. The scheme we proposed here is based on a characterization of k -decomposable graphs in terms of partial k -trees and uses $k \log n$ -bit labels.

The family of k -trees is the set of all the graphs obtained as follows: a complete graph with k nodes is a k -tree, and a k -tree with $n + 1$ nodes is obtained by adding a node connected with k nodes of a complete subgraph of an n -node k -tree [BP69]. This definition clearly implies that k -trees are k -degenerated, and thus have arboricity bounded by k . Actually, this bound is exact for every k -tree with n nodes and n large enough, since the number of edges is $m = kn - k(k + 1)/2$, and thus the arboricity must be at least $m/(n - 1) \geq k - O(k^2/n)$.

Arnborg and Proskurowski showed in [AP87] that k -decomposable graphs are exactly the families of *partial k -trees*, i.e., a subgraph of a k -tree with the same number of nodes. Thus if we are able to find in t time a k -tree H whose G is a partial k -tree then one can apply Theorem 3, and one can construct in $O(kn + t)$ time a $k \log n$ -adjacency labeling scheme for G . Unfortunately, the embedding of G into a k -tree is NP-complete in general [ACP87], even for graphs of degree bounded by 9 [BT97]. However Bodlaender has constructed a linear time algorithm to embed, if possible, G into a k -tree [Bod96]. His algorithm has a hidden constant at least exponential in k . An alternative solution is to use the algorithm of Arnborg, Corneil, and Proskurowski [ACP87] that runs in $O(n^{k+2})$ time. The smallest k such that G is a partial k -tree is also called the *tree-width* of G (see [Bod98] for a survey).

Theorem 4 *For every constant k , the family of partial k -trees (or the family of tree-width at most k graphs) has an asymptotic $k \log n$ -adjacency labeling scheme computable in linear time and supporting constant time queries. Moreover, for every $k \leq n/2$, every $f(n, k)$ -adjacency labeling scheme on the family of partial k -trees requires $f(n, k) = \Omega(k + \log n)$.*

Proof. The upper bound holds from the above discussion. Let us prove the lower bound. Consider the family \mathcal{K} of all subgraphs of a complete graphs of $k + 1$ nodes. Every graph of \mathcal{K} is a partial k -tree, and its arboricity is at most k . By Theorem 2, every $f_1(n_1, k_1)$ -adjacency labeling scheme on \mathcal{K} requires $f_1(n_1, k_1) = \Omega(k_1 \log(n_1/k_1))$ that is $\Omega(k)$ for $n_1 = k + 1$ and $k_1 = k$. Consider now the family \mathcal{T} of labeled n -node trees, a sub-family of partial k -trees. By the Cayley's formula, $|\mathcal{T}| = n^{n-2}$. Using Eq. 1 in the proof of Theorem 2, for every $f_2(n, k)$ -adjacency labeling scheme on \mathcal{T} , we must have:

$$|\mathcal{T}| = n^{n-2} < 2^{(f_2(n,k)+1)n} \Rightarrow f_2(n, k) = \Omega(\log n) .$$

Therefore, every $f(n, k)$ -adjacency labeling scheme on the family of n -node k -trees requires $f(n, k) \geq \min \{f_1, f_2\} = \Omega(k + \log n)$ completing the proof. \square

An application of Theorem 4 is that series-parallel graphs (that are partial 2-trees), have a $2 \log n$ -adjacency labeling scheme computable in linear time.

5 Open Problem

We show in this note how to construct in linear time a $3 \log n$ -adjacency labeling scheme for planar graphs, and more generally a $k \log n$ -bit labeling for bounded k -arboricity graphs. To conclude we leave as an open problem the problem to prove or disprove that the family of graphs having an embedding on a torus (genus 1 graphs) support an asymptotic $3 \log n$ -adjacency labeling scheme. Theorem 3 gives only a $6 \log n$ -bit labeling since bounded g genus graphs are $\lfloor 6 + 12(g - 1)/n \rfloor$ -degenerated (they have at most $3n + 6(g - 1)$ edges). Thus, genus 1 graphs are 6-degenerated and of arboricity at most 6.

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