

Smaller Universal Graphs for Caterpillars and Graphs of Bounded Path-Width

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Abstract

We give an explicit construction of a graph \mathcal{U}_n with at most $8n$ vertices with the property that every n -vertex caterpillar graph is isomorphic to some induced-subgraph of \mathcal{U}_n . Previous constructions of so-called induced-universal graph for caterpillars used $256n$ vertices in the best (Bonichon et al. SIROCCO'06 and Alstrup et al. FOCS'15).

We extended this result to path-width- p graphs with an induced-universal graph of $n \cdot 2^{O(p)}$ vertices. This is complemented with a $n \cdot 2^{\Omega(p)}$ lower bound. With bounded path-width graphs, our construction considerably enlarge the family of n -vertex graphs having an induced-universal graph of $O(n)$ vertices, which is optimal. Such unexpected property was known only for trees, bounded maximum degree outerplanar graphs and maximum degree-2 graphs.

The extended result is obtained from an algorithmic approach by designing an adjacency labeling scheme for these graphs using $\log n + O(p)$ bit vertex labels and with a constant time adjacency test.

1 Introduction

A graph containing as induced subgraph all the graphs of a given family is called *induced-universal graph*. In a famous paper, Alstrup et al. [ADTK15] have proved that trees² with n vertices has an induced-universal graph with only $O(n)$ vertices. This optimal result is ultimately based on a small induced-universal graph for *caterpillars*, a subfamily of trees where non-leaf vertices induce a path, and for which the design of an induced-universal graph with $O(n)$ is already very challenging. This latter challenge has been tackled by Bonichon et al. [BGL06] who proved an upper bound of $256n$ for an induced-universal graph of n -vertex caterpillars³. This involved construction has been simplified in [ADTK15, ADTK17] and generalized in order to apply to trees. For caterpillars, their construction leads to a bound of $384n$ vertices⁴.

A part for trees [ADTK15], optimal $O(n)$ -vertex induced-universal graphs are known only for bounded maximum degree outerplanar graphs [Chu90, AR14] and maximum degree-2 graphs [But06, ELO08, AAH⁺20]. Earlier constructions with this unexpected property were known for subfamily of trees: caterpillars [BGL06], bounded degree trees [Chu90], and bounded depth trees [FK10].

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²Actually this holds for forest since every forest on n vertices is an induced subgraph of some tree with $n + 1$ vertices.

³Based on a labeling scheme, the original construction used $2^6 n$ labels, but the analysis of the construction has been corrected latter to $2^8 n$, cf. [Gav20].

⁴Based on a labeling scheme as in [BGL06], their scheme constructs labels composed of 5 specific bits (for a degree-2 vertex) plus some ID, an integer taken from $[0, 12n)$. This leads to $2^5 \cdot 12n = 384n$ vertices for the induced-universal graph.

2 Caterpillars

Our first contribution is an improved construction by an order of magnitude. Our construction actually holds for path-width one graphs, a slightly more larger graph family, caterpillars being exactly maximal graphs of path-width one.

Theorem 1. *Graphs of path-width one with $n \geq 3$ vertices have an induced-universal graph \mathcal{U}_n with less than $8n$ vertices and $6n \lceil \log(2n/3) \rceil$ edges.*

Both bounds are optimal up to a multiplicative constant. Indeed, the number of vertices must be at least $\lceil 3n/2 \rceil$ since such universal graph must contain a path and a star that can share at most an independent set of size $\lceil n/2 \rceil$ (for $n > 4$). And by considering all regular caterpillars with power-two degree vertices, the number of edges is at least $n \log n - O(n)$.

The vertex set of \mathcal{U}_n consists of $1 + \lceil \log(n/3) \rceil$ layers $L_0, L_1, L_2, \dots, L_{\lceil \log(n/3) \rceil}$, $n \geq 3$. Each vertex v of \mathcal{U}_n is associated with some point (x, y) of the plane with integral coordinates, where y is the index of the layer containing v . More precisely, the layers and the edges are defined as follows (see Fig. 1(a)), where $f(y) := 3 \cdot 2^y$:

- $L_0 := \{(x, 0) : x \in [0, 6n) \cap \mathbb{N}\}$ and $L_y := \{(i \cdot f(y), y) : i \in [0, \lfloor n/2^{y-1} \rfloor] \cap \mathbb{N}\}$ for $y \geq 1$.
- Each vertex $(i \cdot f(y), y) \in L_y$, for $y \geq 1$, is connected to each vertices of $\{(x, 0) \in L_0 : i \cdot f(y) \leq x < (i+1) \cdot f(y)\} \cup \{(i \pm 1) \cdot f(y), j) \in L_j : 1 \leq j \leq y\}$.

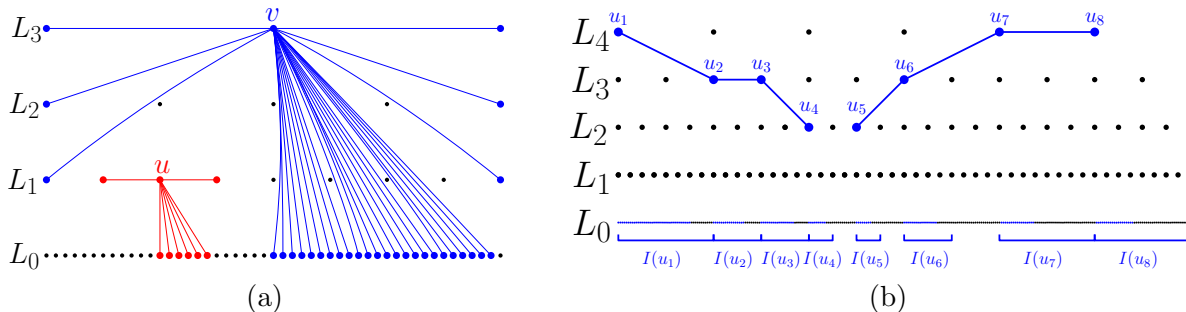


Figure 1: (a) The neighbors of u in $L_1 \cup L_0$ and of v in $L_3 \cup L_2 \cup L_1 \cup L_0$. Layer L_0 is a stable of $6n$ vertices. (b) A graph G (with blue vertices) embedded in \mathcal{U}_{48} (that has 288 points). No edges are represented, but the edges of G between its internal vertices. The internal vertices of G induced two paths: u_1, \dots, u_4 and u_5, \dots, u_8 . All the leaves attached to a vertex u_i embedded in L_y falls into an interval $I(u_i) \subset L_0$ of length $f(y)$ large enough.

The leaves of G are embedded into some points of L_0 whereas its internal vertices (of degree ≥ 2) are embedded to some points of L_y for some level $y \geq 1$. The larger the degree of an internal vertex u_i , the higher the level y where u_i embeds in is. Whereas the embedding of u_i is quite simple whenever u_i has a lower degree than u_{i-1} (in a greedy manner from left to right), the embedding requires some backtracking in the embeddings of the u_j 's, $j < i$, if the degree of u_i increases w.r.t. u_{i-1} .

3 Bounded Path-Width

Our second contribution is much more general, since it proposes a tight bound for an induced-universal graphs for path-width p graphs. Actually, our result is expressed as an *adjacency labeling scheme*.

Recall that an *adjacency labeling scheme* for a graph family \mathcal{F} is a pair (L, f) of algorithms where L is in charge of assigning a label $L(u, G)$ with each vertex u of every graph G in \mathcal{F} , and where f is in charge of decoding adjacency from labels. More precisely given any two labels $\ell_1 = L(u, G)$ and $\ell_2 = L(v, G)$, $f(\ell_1, \ell_2)$ is true if and only if u, v are adjacent.

It is well-known (see [KNR92]) that a graph family \mathcal{F} has an adjacency labeling scheme with k -bit labels⁵ if and only if \mathcal{F} has an induced-universal \mathcal{U} of 2^k vertices. The labeling can be seen as the embedding of each vertex of each graph in the induced-universal graph whose vertex set is the label set used for the family \mathcal{F} . So, small induced-universal graphs can be derived from labeling schemes with short labels.

As we will now see, it is not very hard to design a labeling scheme for path-width- p graphs with $2 \log n + O(p)$ bits, so leading to an induced-universal graph of $n^2 \cdot 2^{O(p)}$ vertices. For this, one can triangulate the input graph by replacing each bag of any path-decomposition of width p by a clique. This latter graph is an interval graph with maximal clique size $p + 1$. Consider an interval representation for this graph where intervals associated with vertices have boundaries that are distinct integers taken from $[1, 2n]$. (This can be done by sorting all together the $2n$ boundaries and by considering their ranks.) Then, one can orient each edge $u \rightarrow v$ if v is a right neighbor of u , i.e., if v is a neighbor of u whose right boundary is greater than u 's right boundary. It follows that u has at most p right neighbors because the intervals of its right neighbors share a common point (the u 's right boundary) and the clique size is at most $p + 1$ (including u). Lastly, one can compute a $(p+1)$ -coloring of the interval graph, say in a greedy manner by scanning all the intervals from right to left. Eventually, one store in the label of each vertex u the following three pieces of information:

- the two interval boundaries (on $2 \lceil \log(2n) \rceil$ bits);
- its color (on $\lceil \log p \rceil$ bits); and
- a binary string indicating, for each vertex color i , whether the colored- i right neighbor of u is a neighbor in the original graph (this can be implemented with p bits given the color of u).

Overall, this leads to a simple $2 \lceil \log(2n) \rceil + \lceil \log p \rceil + p$ bit labeling scheme whose adjacency decoder is straightforward. We observe that the implementation of this construction is essentially best possible because, from theoretic-information lower bounds, general interval graphs with n vertices require asymptotically $2 \log n$ bits per label [GP08] and general graphs on $p + 1$ vertices, so of path-width at most p , require $p/2$ bits per label, see [Alo17] for instance. But these lower bounds do not imply anything about the tightness of the construction it-self. For example, there is no reason why the optimal construction should be based on interval representation.

Unfortunately, this above construction is far to be optimal, since it leads to an induced-universal of $O(n^2)$ vertices for caterpillars ($p = 1$) whereas from Theorem 1, we know that induced-universal graphs with $8n$ vertices do exist for this family.

Our main result is:

⁵Assuming that in each graph no two vertices can receive the same label.

Theorem 2. *Graphs with n vertices and path-width at most p have an adjacency labeling scheme with $\log n + O(p)$ bit labels supporting constant time adjacency test.*

It follows that n -vertex path-width- p graphs have an induced-universal graph with $n \cdot 2^{O(p)}$ vertices. Using an Information-Theoretical lower bound derived counting the number of labeled n -vertex of path-width- p graphs, we can show that:

Theorem 3. *Every adjacency labeling scheme for n -vertex graphs of path-width p requires labels of $\log n + p - o(p)$ bits for every $p = o(n)$.*

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