

Distance Labeling in Hyperbolic Graphs

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ISAAC 2005



Plan

1 Distance Labeling Problem

- Preliminaries
- Exact Distance Labeling Scheme
- State of the Art
- Approximate Distance Labeling Scheme
- Our result

2 Hyperbolic Graphs

- A Bit of History
- Definition

3 Distance Labeling in Hyperbolic Graphs

- A $\delta \log n$ -Additive Error Labeling Scheme
- Lower Bound

4 Open Problems



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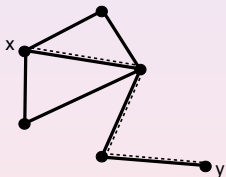


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We consider *undirected, unweighted, simple, connected graphs* defined by pairs of form (V, E) where V denotes the set of *vertices* and E the set of *undirected edges*.



$$d_G(x, y) = 3$$

Definition

The *distance* between two vertices x and y of a graph G :

$d_G(x, y)$ = the length of a *shortest path* between x and y in G .





Let G be a graph (a network)

Distributed data structure

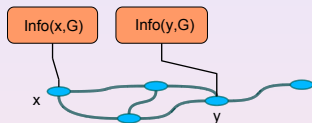
Put on each vertex some information depending on itself and G such that one can compute $d_G(x, y)$ just from information put on x and y .

No centralized data structure to compute distances.

Goal

Minimize the quantity of information





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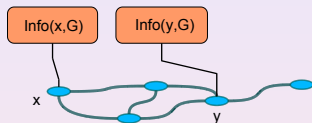
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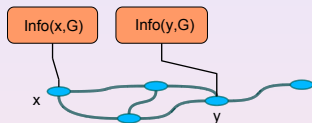
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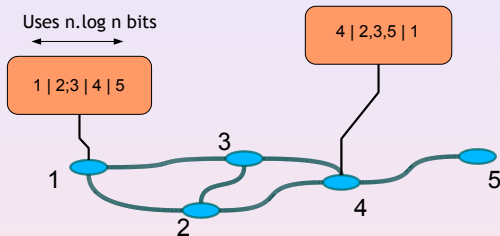
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Minimize the quantity of information





Definition (Distance Labeling Scheme – Peleg'00)

Let \mathfrak{F} be a family of graphs.

A *distance labeling scheme* for \mathfrak{F} is a pair $\langle L, f \rangle$ of functions such that

- $\forall G \in \mathfrak{F}$ and $\forall x \in V_G$, $L(x, G)$ is a binary string (the data attached to x).
- $\forall G \in \mathfrak{F}$ and $\forall x, y \in V_G$,

$$f(L(x, G), L(y, G)) = d_G(x, y)$$

f computes the distance between x and y in G *only* from the data put on x and y .

Goal

Minimize $\max_x |L(x, G)|$ relatively to $|G|$



\mathfrak{F}	Maximal label length	Lower bound	
Arbitrary graphs	$O(n)$	optimal	Gavoille-Peleg-Pérennès-Raz'04
Trees	$O(\log^2 n)$	optimal	Peleg'00, Alstrup-Bille-Rauhe'03
Bounded tree-width graphs	$O(\log^2 n)$	optimal	Gavoille-Peleg-Pérennès-Raz'04
Distance-hereditary graphs	$O(\log^2 n)$	optimal	Gavoille-Paul'03
Interval graphs	$O(\log n)$	optimal	Gavoille-Paul'03

- $O(g(n))$ means that the DLS assigns to vertices of any graph G labels of length at most $O(g(|V_G|))$
- "optimal" means that any other DLS suitable for the family of all the graphs has maximal-length labels in $\Omega(g(n))$.



Definition (Approximate Distance Labeling Scheme – Gavoille-Katz-Katz-Paul-Peleg'01)

Let \mathfrak{F} be a family of graphs.

An (s, r) -approximate distance labeling scheme for \mathfrak{F} is a pair $\langle L, f \rangle$ of functions such that

- $\forall G \in \mathfrak{F}$ and $\forall x \in V_G$, $L(x, G)$ is the information of $x \rightsquigarrow$ a binary string.
- $\forall G \in \mathfrak{F}$ and $\forall x, y \in V_G$,

$$d_G(x, y) \leq f(L(x, G), L(y, G)) \leq s \cdot d_G(x, y) + r$$



\mathfrak{F}	Approximation	Maximal label length	Lower bound	
Arbitrary graphs	$(2k - 1)$ mult.	$O(n^{1/k} \log^2 n)$	$\Omega(n^{1/k})$	T-Z'05
Trees	$(1 + 1/\log n)$ mult.	$O(\log n \cdot \log \log n)$	optimal	G-K-K-P-P'01
Bounded Tree-Length	$O(1)$ add.	$O(\log^2 n)$	optimal	G-K-K-P-P'01
α Doubling Dimension	$(1 + \varepsilon)$ mult.	$O(\varepsilon^{-O(\alpha)} \log n \cdot \log \log n)$	optimal	S'05



Theorem

The family of δ -hyperbolic graphs support a $\delta \log n$ -additive approximate DLS which uses $O(\log^2 n)$ bits per vertex.

Theorem (Lower Bound)

There exists a family \mathfrak{F} of graphs of bounded hyperbolicity such that for any (s, r) -approximated DLS $\langle L, f \rangle$ on it:

$$|L| = O(\log^a n) \Rightarrow s + r = \Omega(\log \log n)$$



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Lobachevski – beginning of XIXth century: Hyperbolic Geometry

↪ independency of the 5th Euclid's Axiom.

Mostov 1968: Rigidity of Hyperbolic Manifolds

Hyperbolic manifold are characterized by their fundamental groups.

↪ Particular case of Poincarre's conjecture

Gromov 1987: Hyperbolic Groups

Generalization of fundamental groups of hyperbolic manifolds

Strong algorithmic properties

- Hyperbolic groups are automatic [Cannon 1984]
- Isomorphism Problem is decidable [Sela 1995]

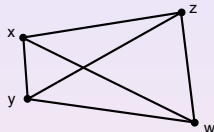


Definition (Gromov 1987)

A metric space X is said to be δ -hyperbolic if and only if:
 $\forall x, y, w, z \in X$, the two larger of the three sums

$$\begin{aligned}d(x, y) + d(w, z) \\d(x, w) + d(z, y) \\d(x, z) + d(w, y)\end{aligned}$$

differs of at most δ

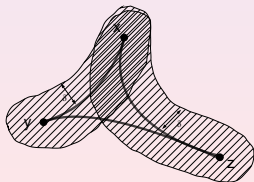


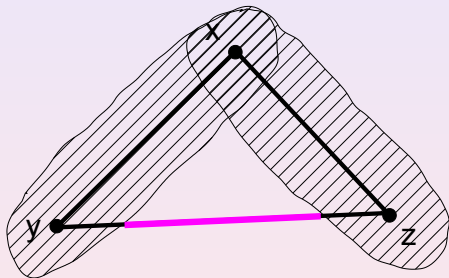
Rips characterization

A metric space X is said to be δ -hyperbolic if and only if for any geodesic triangle xyz ,

$$[xz] \subset \delta\text{-neigh}([xy] \cup [yz])$$

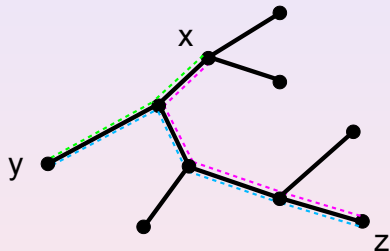
Rips's $\delta \sim$ Gromov's δ .





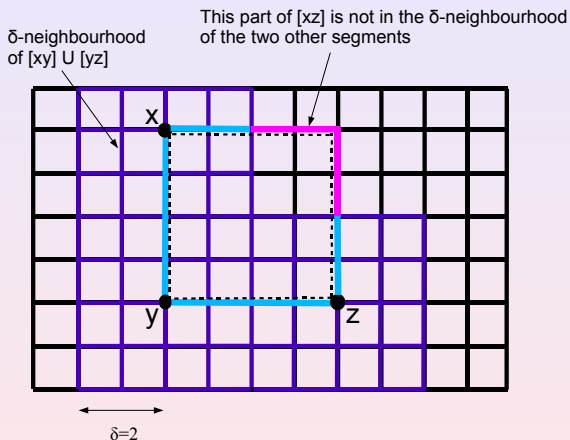
The euclidian space is not hyperbolic





Trees are 0-hyperbolic





For any δ there exists a grid which is not δ -hyperbolic



Definition (Hyperbolic graphs)

A graph is said to be δ -hyperbolic if it is δ -hyperbolic as a metric space for the metric of shortest paths.

The *hyperbolicity* of a graph G is the minimum δ such that G is δ -hyperbolic.

Hyperbolicity: A Treelikeness Measure

- 0-hyperbolic graphs are exactly the *block graphs* [Brinkmann-Koolen-Moulton 2001].
In particular *trees are 0-hyperbolic*.
- Chordal graphs are 2-hyperbolic [BKM'2001]
- $\text{Hyp}(G) = O(\text{TL}(G))$ and $\text{TL}(G) = O(\text{Hyp}(G) \log |G|)$ [G.-L.'2004]
- For any δ , there exists a square grid which is not δ -hyperbolic.



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Method

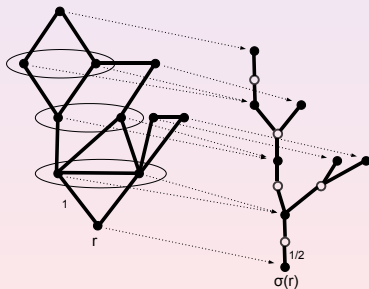
- 1 Approximation of a δ -hyperbolic graph by a tree with $O(\delta \log n)$ additive error
- 2 Exact distance labeling scheme on the tree ($\log^2 n$ bits)



Lemma (Approximation by trees – Gromov 1987)

Let G be a δ -hyperbolic graph. Then there exists a tree T and a mapping $\sigma : V_G \rightarrow V_T$ such that

$$\forall x, y \in V_G : d_T(\sigma(x), \sigma(y)) \leq d_G(x, y) \leq d_T(\sigma(x), \sigma(y)) + 2\delta \log |G|$$



Definition (Gromov Product)

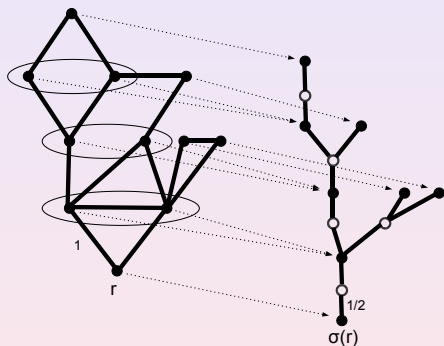
Let G be a graph and $x, y, r \in V_G$. We call *Gromov product* of x and y regarding r the following number:

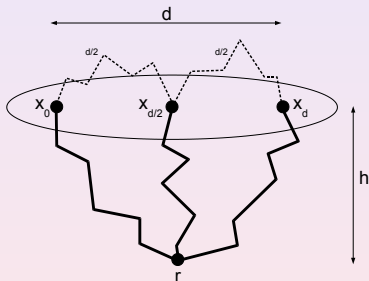
$$(x|y)_r = \frac{1}{2}(d_G(x, r) + d_G(y, r) - d_G(x, y))$$

Lemma

G is δ -hyperbolic if and only if $\forall x, y, z \in V_G: (x|y)_r \geq \min\{(x|z)_r, (z|y)_r\} - \delta$.







$$(x_0|x_d)_r \geq \min\{(x_0|x_{d/2})_r, (x_{d/2}|x_d)_r\} - \delta$$

$$(x_0|x_{d/2})_r \geq \min_{i < d/2} \{(x_i|x_{i+1})_r\} - \delta \log d/2$$

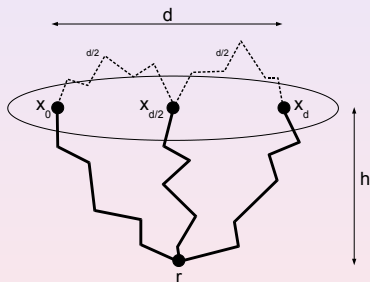
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$$\Rightarrow (x_0|x_d)_r \geq \min_i \{(x_i|x_{i+1})_r\} - \delta(\log d - 1) - \delta$$

$$\Rightarrow (x_0|x_d)_r \geq \min_i \{(x_i|x_{i+1})_r\} - \delta \log d$$

$$\Rightarrow d \leq 1/2 + 2\delta \log d \leq 1/2 + 2\delta \log n$$





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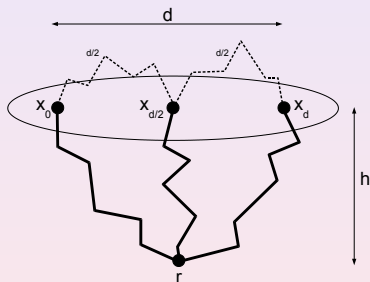
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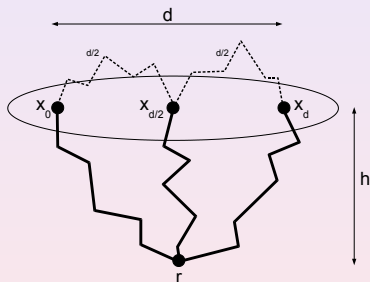
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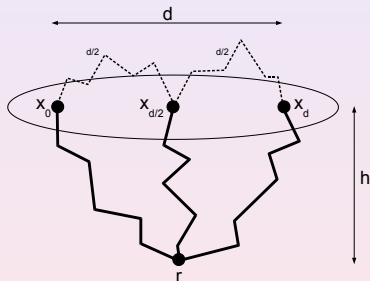
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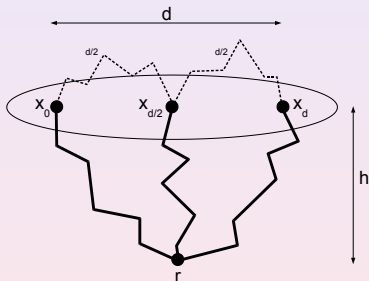
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Goal

Quantify the following principle:

If one does not store enough information, then one makes errors.

Theorem (Lower Bound)

There exists a family \mathfrak{G} of graphs of bounded hyperbolicity such that for any (s, r) -approximated DLS $\langle L, f \rangle$ on it:

$$|L| = O(\log^a n) \Rightarrow s + r = \Omega(\log \log n)$$



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Lemma

There exists a family $\mathfrak{F}_{n,k}$ of graphs of bounded hyperbolicity such that for any (s, r) -approximated DLS $\langle L, f \rangle$ on it:

$$s + r < 2 \log(4k/3 + 2) - 3 \Rightarrow |L| = \Omega((n/\log k)^{1/k})$$

Lemma (Lazebnik-Ustimenko-Voldar 1995)

For any n there is a graph of girth $4k/3 + 3$ with n vertices and a maximal number of edges $\Omega(n^{1+1/k})$.

(Related to Erdős's conjecture)

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- $G_{n,k} \rightsquigarrow$ a graph of girth $4k/3 + 3$ with n vertices and $\Omega(n^{1+1/k})$ edges.
- $\mathfrak{F}_{n,k} \rightsquigarrow$ the family of subgraphs of $G_{n,k}$ with the same vertices and a subset of edges.

$$\rightarrow |\mathfrak{F}_{n,k}| = \Omega(2^{n^{1+1/k}}).$$

Let (L, F) be an DLS on $\mathfrak{F}_{n,k}$ such that $|L| = o(n^{1/k})$.

- For each $H \in \mathfrak{F}_{n,k}$ let us consider the word
 $\bar{L}(H) = L(1, H) \# L(2, H) \# \dots \# L(n, H)$

$$\bullet |\{\bar{L}(H) | H \in \mathfrak{F}_{n,k}\}| = o(2^{n^{1/k}})$$

$\rightarrow \exists H_1 \neq H_2$ such that $\bar{L}(H_1) = \bar{L}(H_2)$.

- let $(x, y) \in (E_{H_1} \setminus E_{H_2})$

- $\bar{L}(H_1) = \bar{L}(H_2) \Rightarrow x$ and y have the same labels in H_1 and $H_2 \Rightarrow$ the DLS gives the same distance between them for H_1 and for H_2 .

- But the girth of $G_{n,k}$ is $4k/3 + 3$ thus $d_{H_2}(x, y) \geq 4k/3 + 2$.

- And $(x, y) \in E_{H_1} \Rightarrow d_{H_1}(x, y) = 1$

Finally Our DLS makes errors of at least $4k/3 + 2$.



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All the distances > 5 are shortened
by \log

Hyperbolicity is bounded





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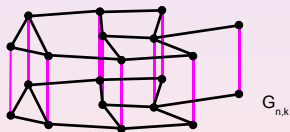
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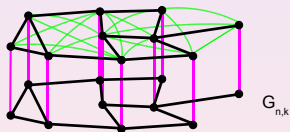
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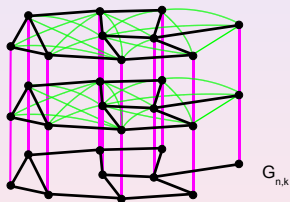
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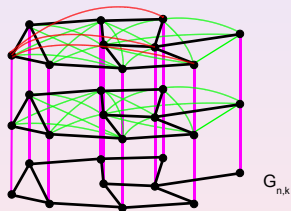
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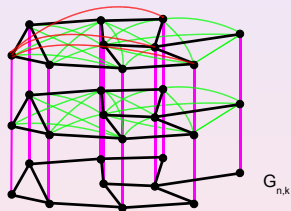




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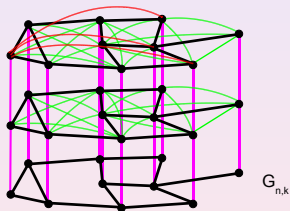




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Plan

- 1 Distance Labeling Problem
 - Preliminaries
 - Exact Distance Labeling Scheme
 - State of the Art
 - Approximate Distance Labeling Scheme
 - Our result
- 2 Hyperbolic Graphs
 - A Bit of History
 - Definition
- 3 Distance Labeling in Hyperbolic Graphs
 - A $\delta \log n$ -Additive Error Labeling Scheme
 - Lower Bound
- 4 Open Problems



Fill the gap between upper and lower bound
(somewhere in between $\log n$ and $\log \log n$...)

Collective tree spanners for hyperbolic graphs ?

Thank you for your attention