On the Complexity of Distributed Coloring

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A k-coloring is a color assignment $V(G) \rightarrow \{1, ..., k\}$ such that adjacent vertices of a graph G get different colors.

A **Maximal Independent Set** is set of pairwise non-adjacent vertices and which is maximal (for inclusion).

Both problems arise in many partitioning & clustering problems in many aeras.

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- **O** Greedy coloring: choose the smallest available color
- **Largest-first coloring**: color first high degree vertices with the smallest possible available color

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Secret goal: Prove that these problems are inherently non-local

(in finding fast distributed algorithms)

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 $\begin{array}{l} \Delta = \mbox{maximum degree of the graph} \\ N_v = \mbox{set of neighbors of } v \\ N_v^\geqslant = \mbox{set of neighbors of } v \mbox{ with degree at least } \deg(v) \end{array}$

Model of Computation

Linial's free model, a.k.a. \mathcal{LOCAL} model

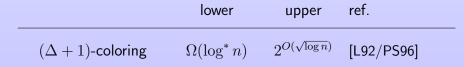
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 \Rightarrow model to understand the LOCALITY of a problem. In time t a vertex knows at most its distance-t neighborhood



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$(\Delta+1) ext{-coloring}$	$\Omega(\log^* n)$	$2^{O(\sqrt{\log n})}$	[L92/PS96]
Max. Indep. Set	$\Omega(\sqrt{\frac{\log n}{\log \log n}})$	$2^{O(\sqrt{\log n})}$	[KMW04/PS93]

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largest-first colorin	_	$O(\sqrt{n})$	[KK06/ <mark>here</mark>]

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- Step 2 If girth(G) $\ge \ell$ and $\chi(G) > \ell/2$, then **A** requires at least $\ell/2 1$ rounds for G.
- Step 3 There are *n*-vertex graphs with girth $\ge \ell$ and $\chi > \ell/2$ for $\ell = \frac{1}{4} \log n / \log \log n$.

Thank You for your attention