

# On the Complexity of Distributed Greedy Coloring\*

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**Abstract.** Distributed Greedy Coloring is an interesting and intuitive variation of the standard Coloring problem. It still consists in coloring in a distributed setting each node of a given graph in such a way that two adjacent nodes do not get the same color, but it adds a further constraint. Given an order among the colors, a coloring is said to be *greedy* if there does not exist a node for which its associated color can be replaced by a color of lower position in this order without violating the coloring property. We provide lower and upper bounds for this problem in Linial's model and we relate them to other well-known problems, namely *Coloring*, *Maximal Independent Set (MIS)*, and *Largest First Coloring*. Whereas the best known upper bound for Coloring, MIS, and Greedy Coloring are the same, we prove a lower bound which is strong in the sense that it now makes a difference between Greedy Coloring and MIS.

We discuss the vertex coloring problem in a *distributed network*. Such a network consists of a set  $V$  of processors and a set  $E$  of bidirectional communication links between pairs of processors. It can be modeled by an undirected graph  $G = (V, E)$ . We denote  $n = |V|$ ,  $m = |E|$  and for each vertex  $v$  define its *neighborhood*  $N_v = \{u : \{u, v\} \in E\}$  and *vertex degree*  $\deg_G v = |N_v|$ . The set of neighbours of high degree is denoted by  $N_v^\geq = \{u \in N_v : \deg u \geq \deg v\}$ .

To *color* the vertices of  $G$  means to give each vertex a positive integer color value in such a way that no two adjacent vertices get the same color. If at most  $k$  colors are used, the result is called a *k-coloring*. In many practical considerations, such as code assignment in wireless networks [1], it is desirable to minimise the number of used colors. The smallest possible positive integer  $k$  for which there exists a  $k$ -coloring of  $G$  is called the *chromatic number*  $\chi(G)$ . This value is bounded from above by  $\Delta + 1$ , where  $\Delta$  denotes the maximal vertex degree of the graph; consequently, a graph always admits a  $(\Delta + 1)$ -coloring.

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The problems discussed in this paper can be formulated using a local definition: the goal is to achieve a state of the system in which the local state variables associated with each node fulfill certain constraints with respect to the local state variables of its neighbours.

**Definition 1.** *The problems are defined by the following constraints on the local variable<sup>1</sup>  $c$  at vertex  $v$ :*

$$\begin{aligned}
 (\Delta + 1)\text{-Coloring (COL):} & \quad c(v) \in \{1, \dots, \Delta + 1\} \setminus c(N_v) \\
 \text{Greedy Coloring (G-COL):} & \quad c(v) = \min \{1, \dots, \Delta + 1\} \setminus c(N_v) \\
 \text{Largest-First Coloring (LF-COL):} & \quad c(v) = \min \{1, \dots, \Delta + 1\} \setminus c(N_v^\geq) \\
 \text{Maximal Independent Set (MIS):} & \quad c(v) \neq 0 \Leftrightarrow c(N_v) = \{0\}
 \end{aligned}$$

We provide lower and upper bounds on the deterministic distributed (Linial’s model) time complexity of *Greedy Coloring* (G-COL) with respect to *Coloring* (COL), *Maximal Independent Set* (MIS) and *Largest First Coloring* (LF-COL). A summary of the results is contained in Table 1, where (\*) indicates the new results obtained in this paper. In particular, we derive new upper bounds for G-COL and LF-COL and a new lower bound for G-COL. Whereas the upper bounds for the COL, MIS, and G-COL are the same, we prove a strong lower bound in the sense that our lower bound now makes a difference between G-COL and MIS.

**Table 1.** The time complexity of Greedy Coloring with respect to other well-known problems in the distributed setting. The table can be read also vertically as  $(\Delta + 1)$ -Coloring  $\leq$  Maximal Independent Set  $\leq$  Greedy Coloring  $\leq$  LF-Coloring.

<i>Problem</i>	<i>Lower Bound</i>	<i>Upper Bound</i>
$(\Delta + 1)$ -Coloring (COL)	$\Omega(\log^* n)$ [2]	$2^{O(\sqrt{\log n})}$ [3,4]
Maximal Independent Set (MIS)	$\Omega\left(\sqrt{\frac{\log n}{\log \log n}}\right)$ [5]	$2^{O(\sqrt{\log n})}$ [3,4] $O(\Delta + \text{COL})$
Greedy Coloring (G-COL)	$\Omega\left(\frac{\log n}{\log \log n}\right)$ (*)	$2^{O(\sqrt{\log n})}$ (*) $O(\Delta + \text{COL})$ (*) $O(\Delta^2 + \log^* n)$ [2]
Largest-First Coloring (LF-COL)	$\Omega(\sqrt{n})$ [6]	$O(\sqrt{n})$ (*) $O(\Delta \cdot \text{MIS})$ [6]

## References

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<sup>1</sup> This variable represents the color or the characteristic function of the MIS.

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