Lower Bounds for Induced-Universal Graphs

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Abstract

We give a series of new lower bounds on the minimum number of vertices required by a graph to contain every graph of a given family as induced subgraph. In particular, we show that this induced-universal graph for n-vertex planar graphs must have at least $10.52n$ vertices. We also show that the number of conflicting graphs to consider in order to beat this lower bound is at least 137. In other words, any family of less than 137 planar graphs of n vertices has an induced-universal graph with less than $10.52n$ vertices, stressing the difficulty in beating such lower bounds. Similar results are developed for other graph families, including but not limited to, trees, outerplanar graphs, series-parallel graphs, $K_{3,3}$-minor free graphs. As a byproduct, we show that any family of t graphs of n vertices having small chromatic number and sublinear pathwidth, like any proper minor-closed family, has an induced-universal graph with less than $15/7\sqrt{t} \cdot n$ vertices. This is achieved by making a bridge between equitable colorings, combinatorial designs, and path-decompositions.

Keywords: planar graphs, universal graphs, bounded pathwidth graphs, equitable coloring, and combinatorial designs.

1 Introduction

Universality plays an important role in Graph Theory and Computer Science. For the famous Traveling Salesperson Problem, [JLN+05] showed that every set S of n points in a metric space has some universal tour, computable in polynomial time, that approximates, up to poly-logarithmic factor, the best tour of every subset of S by taking the induced subtour of this universal ordering. In graph theory, the celebrated Excluded Grid Theorem of Robertson and Seymour [RST94] (which states that graphs excluding as minor a fixed planar graph have bounded treewidth), relies on the fact that every planar graph is a minor of some relatively small grid. So, excluding such a grid as minor implies excluding as minor every sufficiently small planar graph. Therefore, rather than considering individually each graph of a given graph family, it is much simpler to manipulate one single graph (here a

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grid), whose properties is close enough to those of the graphs of the family (here the property is planarity). This grid-minor universality has been extended by [GH23], by showing that for every n and g, there is some fixed graph of Euler genus g and with only $O(g^2(n+g)^2)$ vertices that is minor-universal for all n-vertex graphs of Euler genus g. Here the size of the minor-universal graph is an important parameter, and its polynomial dependency in n and g is the crux for the very recent polynomial bound for the Graph Minor Structure Theorem [GSW25].

Induced-universal graphs. In this paper, we focus on universality under induced-subgraph containment. More precisely, a graph U is an *induced-universal graph* for a graph family $\mathcal{F}$ if every graph of $\mathcal{F}$ is isomorphic to an induced-subgraph of U. Here, the term "graph family" must be considered in a broad sense: it can contain a finite set of graphs (like two bounded size graphs, or two trees with n vertices) or a countably finite set of graphs (like the class of all planar graphs). We denote by $\mathcal{U}(\mathcal{F})$ the smallest number of vertices of an induced-universal graph for $\mathcal{F}$, and by $\mathcal{F}_n$ the sub-family composed only of n-vertex graphs of $\mathcal{F}$ (if any).

For the family $\mathcal{G}$ of all graphs, Alon [Alo17] showed that $\mathcal{U}(\mathcal{G}_n) = (1+o(1)) \cdot 2^{(n-1)/2}$. For the family $\mathcal{P}$ of all planar graphs, a result of [DEG+21, GJ22] implies that $\mathcal{U}(\mathcal{G}_n) \leqslant n \cdot 2^{O(\sqrt{\log n})}$. And for the family $\mathcal{T}$ of all trees, [ADBTK17] showed that $\mathcal{U}(\mathcal{T}_n) \leqslant cn$, where $c \gg 768$ is a rather large constant[1]. The results for trees and planar graphs come from the design of *adjacency labeling schemes*, which can be viewed as an alternative definition of induced-universal graphs with algorithmic applications [KNR92].

Whereas for all graphs, the first order of $\mathcal{U}(\mathcal{G}_n)$ is well-established, the situation is very different for planar graphs and trees. Up to the best of our knowledge, the only non-trivial better-than-n lower bound is $\mathcal{U}(\mathcal{P}_n) \geqslant 11\lfloor n/6 \rfloor \approx 1.83n$, coming from the family of paths and cycles [ELO08, Claim 1]. For trees, the lower bound is even weaker. A folklore bound $\mathcal{U}(\mathcal{T}_n) \geqslant 3\lfloor n/2 \rfloor$ can be obtained by considering a family of two trees that can only share $\lceil n/2 \rceil$ vertices: a star (a depth-1 tree) and a path, each with n vertices. These both lower bounds are ad-hoc techniques based on the impossibility of coexisting a certain number of "conflicting graphs" in a single graph that is too small.

On the difficulty of proving lower bounds. It is perhaps worth mentioning that it can be very difficult to prove strict lower bounds on the size of universal objects for specific combinatorial properties. For instance, consider *universal planar point sets*, i.e., a fixed set of points such that each n-vertex planar graph has a straight-line drawing on it. The best upper bound is $2n^2$ (an $2n \times n$-grid), and the current best lower bound has been improved only recently to $1.293n$ for n large enough [SSS20], a large gap. Interestingly, [SSS20] showed that for $n = 11$, no universal planar point sets of exactly n points exist. In other words, the lower bound is at least $n+1$ for $n = 11$. Proving an $(n+1)$-lower bound, even for small n, is not so easy. Technically, they have considered (computer-assisted) a family of 49 "conflicting" graphs, each with $n = 11$ vertices, which cannot be simultaneously drawn on any set of n points. They show that more than 36 conflicting graphs are required. In other words, for $n = 11$, in order

[1] It is not given explicitly in [ADBTK17]. Their involved construction is based on a solution for caterpillars that contains at least $12 \cdot 2^5 n = 384n$ vertices. Extending the solution for trees requires to at least double the caterpillar solution. An estimate of 10^5 or 10^6 for c seems to be closer to reality.

to beat the trivial n-lower bound any conflicting family has size at least 37. It is only conjectured that the n-lower bound can be beaten for every $n \geqslant 11$. There are even conjectures made about the size of any conflicting family w.r.t the trivial n-lower bound [BCD+07, (3), p. 129].

For induced-universal graphs for trees, it is not clear if a lower bound as low as say $2.1n$ can be achieved. As previously said, the folklore $3\lfloor n/2 \rfloor$-lower bound for trees is achieved by considering a conflicting family of two caterpillars[2] on n vertices: a star and a path. Naively, one may believe that by considering a conflicting family of three trees (instead of two caterpillars) may allow us to easily beat this folklore $3n/2$-lower bound. Unfortunately, one of our results implies that $\mathcal{U}(\mathcal{F}) \leqslant 3n/2 + O(\log n)$ for any family $\mathcal{F}$ composed of three trees with n vertices. So, at least 4 trees need to be considered in any conflicting family w.r.t. the $3n/2$-lower bound. Even worse, we show that in order to beat $2.1n$, we will need to consider at least 7 trees. Clearly, a lower bound proof involving 7 trees is far more complicated than one involving 2 trees, as pairwise intersections of the trees in an induced-universal graph solution may be hard to control.

Our contributions. First, we use some classical Information-Theoretic lower bound to improve the current lower bounds based on conflicting families. This concerns trees, planar and some other families (see Table 1 and Table 3 columns c). We also develop another lower bound for specific conflicting families containing small cliques (see Table 2). The bound is actually optimal for unions of cliques of bounded size. This captures the folklore $3n/2$-lower bound for trees, and the $11n/6$-lower bound for paths and cycles. As intermediate step, it gives a simple $25n/12$-lower bound for planar graphs by considering cliques of size at most four. Using our Information-Theoretic lower bound, the bound for planar graphs is improved to $10.52n$.

Second, we use block designs and special vertex-coloring to construct small induced-universal graphs for any family $\mathcal{F}$ with t graphs of n vertices. The result depends on two more integral parameters on $\mathcal{F}$: p and k. More precisely, we show in Theorem 3 that, if every graph of $\mathcal{F}$ has a set of p vertices whose removal leaves a graph that is *equitably k-colorable* (a k-coloring with same color-class sizes), then

$$\mathcal{U}(\mathcal{F}) \leqslant s \cdot \frac{n}{k} + tp \,. \tag{1}$$

The number $s = s(t,k)$ is related to some block designs and the Coding-Theoretic Function as defined in Section 3.1. Actually, we show in Theorem 4 that parameter p is at most the pathwidth of any graph in $\mathcal{F}$ times the chromatic number k of $\mathcal{F}$.

For instance, for any family $\mathcal{F}$ composed of $t = 3$ trees with n vertices, Eq. (1) implies $\mathcal{U}(\mathcal{F}) \leqslant 3n/2 + O(\log n)$, since $k = 2$, $p = O(\log n)$ (trees are bipartite of logarithmic pathwidth), and because $s = 3$ in this case. For $\mathcal{F}$ composed of $t = 6$ trees, we have $s = 4$, and thus $\mathcal{U}(\mathcal{F}) \leqslant 2n + O(\log n)$. If $\mathcal{F}$ is composed of $t = 20$ planar graphs, $\mathcal{U}(\mathcal{F}) \leqslant 4n + O(\sqrt{n})$, since in that case $k = 4$, $p = O(\sqrt{n})$, and $s = 16$.

From above, at least 7 conflicting trees would be required in order to beat a hypothetical $2.1n$-lower bound for trees, as $\mathcal{U}(\mathcal{F}) \leqslant 2n + O(\log n)$ for $t = 6$ trees. Actually, we have showed

[2] Recall that a caterpillar is a tree in which the nodes of degree at least two induced a path. They are also exactly the maximal graphs of pathwidth one.

a lower bound of $1.626n$ for tree families, and $\mathcal{U}(\mathcal{F}) \leqslant 3n/2 + O(\log n)$ for $t = 3$ trees. Thus, at least 4 conflicting trees are required in order to beat the $1.626n$-lower bound (cf. the first line of Table 1). For planar graph families, $k = 4$, $p = O(\sqrt{n})$, and $s = 42$ if $t = 136$. So, at least 137 conflicting graphs are required in order to beat the $10.52n$-lower bound.

This framework applies to several families of graphs, and the series of new lower bounds we obtained, on $\mathcal{U}$ and conflicting family size, are summarized in columns c and t of Table 1.

$\mathcal{F}$	c	t	k	$A(\lceil ck \rceil - 1, 2k-2, k)$
forests	1.626	4	2	$A(3,2,2) = 3$
outer-planar	3.275	13	3	$A(9,4,3) = 12$
series-parallel	3.850	17	3	$A(11,4,3) = 16$
K_5^--minor-free[3]	6.264	51	4	$A(25,6,4) = 50$
planar	10.520	137	4	$A(42,6,4) = 136$
$K_{3,3}$-minor-free	10.521	124	5	$A(52,8,5) \in \{123, 124\}$[4]

Table 1: New lower bounds given by Theorem 2 for various family $\mathcal{F}$ of the form $\mathcal{U}(\mathcal{F}_n) \geqslant cn - o(n)$. Here t is a lower bound on the size of a conflicting family needed to beat the corresponding cn-lower bound. From Theorem 5, we have $t = 1 + A(\lceil ck \rceil - 1, 2k-2, k)$, where k is the chromatic number of $\mathcal{F}$, and A, related to block designs, is the Coding-Theoretic Function defined in Section 3.1.

Motivated by the quest for an $\Omega(n \log n)$-lower bound for planar graph, we show that a conflicting family w.r.t. to this lower bound (if it exists) must have a size of at least $\Omega(\log^2 n)$. More generally, we show in Theorem 6 that any family $\mathcal{F}$ of t graphs having a small chromatic number and sublinear pathwidth, like any proper minor-closed family as planar or bounded genus graphs, verifies $\mathcal{U}(\mathcal{F}_n) < 15/7\sqrt{t} \cdot n$.

Organization of the paper. Section 2 presents the union-of-clique lower bound, and the Information-Theoretic lower bound. Section 3 gives our construction, as in Eq. (1), using block designs, and its applications to conflicting families. Section 4 presents our upper bound in $O(\sqrt{t} \cdot n)$.

2 Lower Bounds

2.1 Union of cliques

The first lower bound is based on the union of small cliques. The bounds presented in Theorem 1 are optimal and generalize previously known lower bounds: the folklore lower bound of $\lfloor 3n/2 \rfloor$ for acyclic graphs of maximum degree two, and the lower bound of $11\lfloor n/6 \rfloor$ for graphs of maximum degree at most two [ELO08, Claim 1]. We remark for this latter that our lower bounds are slightly better. It actually matches ours only for n multiple of six. E.g., if $n = 6i + 5$ for integer i, $11\lfloor n/6 \rfloor = 11i$ whereas our bound gives $11i + 8$.

[3] K_5^- is the graph K_5 minus one edge.

[4] According to Andries E. Brouwer's web pages, it is not known whether $A(52,8,5) = 123$ or 124.

Theorem 1 *Let $\mathcal{K}_{n,k}$ be the family of graphs with at most n vertices composed of a disjoint union of cliques each with at most k vertices. Then, $\mathcal{U}(\mathcal{K}_{n,k}) = \sum_{i=1}^{k} \lfloor n/i \rfloor \geqslant (n+1)\ln(k+1) - k$.*

Proof.

Lower bound. For every $i \in \{1,\dots,k\}$, let G_i be the n-vertex graph composed of $\lfloor n/i \rfloor$ disjoint cliques of i vertices plus n mod i isolated vertices, and let $F_i = \{G_1,\dots,G_i\}$. Clearly, $F_i \subseteq \mathcal{K}_{n,i}$, and thus in particular $\mathcal{U}(\mathcal{K}_{n,k}) \geqslant \mathcal{U}(F_k)$. Our goal is to show that $\mathcal{U}(F_k) = \sum_{i=1}^{k} \lfloor n/i \rfloor$. For convenience, let $N_i = \mathcal{U}(F_i)$. Obviously, $N_1 = n$ since $F_1 = \{G_1\}$ and G_1 has n vertices. Assume $k \geqslant 2$. We will show that $N_k \geqslant \lfloor n/k \rfloor + N_{k-1}$.

Let U be a minimum induced-universal graph for F_k, thus with N_k vertices. We construct as follows a subgraph U' of U that is an induced-universal graph for F_{k-1} with no more than $N_k - \lfloor n/k \rfloor$ vertices. First, consider any induced-subgraph embedding of G_k in U, and denote by $C_1,\dots,C_{\lfloor n/k \rfloor}$ be all its cliques embedded in U with $k \geqslant 2$ vertices. Thus, the subgraph $\bigcup_{i=1}^{\lfloor n/k \rfloor} C_i$ is an induced union of cliques of U, each with $k \geqslant 2$ vertices. Second, for each C_i, select an arbitrary vertex x_i. Finally, we set $U' = U \setminus X$, where $X = \{x_1,\dots,x_{\lfloor n/k \rfloor}\}$. Note that X is a stable set in U, because there are no edges in U between any two distinct cliques of $\bigcup_{i=1}^{\lfloor n/k \rfloor} C_i$.

Clearly, $|V(U')| = |V(U)| - |X| = N_k - \lfloor n/k \rfloor$. It remains to show that U' is an induced-universal graph for F_{k-1}. Consider $G_t \in F_{k-1}$, for some $t < k$, and denote by $D_1, D_2, \dots$ be all its clique components that are embedded in U. The subgraph $\bigcup_i D_i$ is an induced union of cliques of U, each with 1 or t vertices. Each clique D_i can intersect X in at most one vertex, because X is a stable set. Suppose that D_i intersects X in vertex x_j for some i, j. The key point is that the clique C_j containing x_j intersects no other D_m with $m \neq i$. Otherwise, $D_i \cup D_m$ would be not an induced union of disjoint cliques in U. Also, we have $|V(C_j)| = k > t = |V(D_i)|$. It follows that, for each such clique D_i, one can move its embedding into $C_j \setminus \{x_j\}$ instead of D_i while preserving that their union is disjoint. One can check that their union is also induced in U if U' is *minimal*, i.e., with no proper subgraphs that is induced-universal for the same family. So, this new embedding is preserved as well in $U \setminus \{x_j\}$, since x_j is not useful anymore. By applying this embedding transformation for each clique D_i intersecting X, we have shown that G_t is induced in $U' = U \setminus X$. This applies for each $G_t \in F_{k-1}$.

Therefore, U' is an induced-universal for F_{k-1}, and thus $N_{k-1} \leqslant |V(U')| \leqslant N_k - \lfloor n/k \rfloor$. In other words, we have established that:

$$\forall k \geqslant 2, \quad N_k \;\geqslant\; \left\lfloor \frac{n}{k} \right\rfloor + N_{k-1} \,.$$

By iterating the previous inequality down to $k = 2$, and using the fact that $N_1 = n$, we get:

$$\mathcal{U}(\mathcal{K}_{n,k}) \;\geqslant\; N_k \;\geqslant\; \sum_{i=1}^{k} \left\lfloor \frac{n}{i} \right\rfloor .$$

We remark[5] that $\sum_{i=1}^{k} \lfloor n/i \rfloor \geqslant \sum_{i=1}^{k} ((n+1)/i - 1) = (n+1)\cdot(\sum_{i=1}^{k} 1/i) - k \geqslant (n+1)\cdot\int_1^{k+1} \frac{1}{x}dx - k$, and thus $\mathcal{U}(\mathcal{K}_{n,k}) \geqslant (n+1)\ln(k+1) - k$.

[5] Using the fact that $\lfloor a/b \rfloor + 1 \geqslant (a+1)/b$ for integers a and $b \neq 0$.

Upper bound. For every $i \in \{1,\dots,k\}$, we inductively construct a graph U_i as follows. For $i = 1$, we let U_1 be a stable set with n vertices. Then, for $i \geqslant 2$, assuming that U_{i-1} contains $\lfloor n/(i-1) \rfloor$ disjoint cliques of $i-1$ vertices, the graph U_i is obtained from a copy of U_{i-1} in which we replace $\lfloor n/i \rfloor$ cliques of $i-1$ vertices each by a clique of i vertices. In other words, we increase by one vertex $\lfloor n/i \rfloor$ cliques taken among the $\lfloor n/(i-1) \rfloor$ cliques of $i-1$ vertices of U_{i-1}. So, U_i has exactly $\lfloor n/i \rfloor$ vertices more than U_{i-1}, and it contains $\lfloor n/i \rfloor$ disjoint cliques of i vertices. Moreover, U_i contains U_j as induced subgraph for each $j \leqslant i$. Thus, U_i contains $\lfloor n/j \rfloor$ disjoint cliques of at least j vertices.

In particular, $|V(U_k)| = \lfloor n/k \rfloor + |V(U_{k-1})| = \sum_{i=1}^{k} \lfloor n/k \rfloor$, and U_k contains $\lfloor n/j \rfloor$ disjoint cliques of at least j vertices, for every $j \in \{1,\dots,k\}$. Also observe that U_k has a total of n disjoint cliques. Denoted by C_i be the ith largest clique of U_k.

Let us show that U_k is induced-universal for $\mathcal{H}_{n,k}$. Consider a graph $G \in \mathcal{H}_{n,k}$, and let $D_1,\dots,D_t$ be its $t \leqslant n$ disjoint cliques ordered by non-increasing size. The cliques of G are embedded in U_k in a greedy way: D_i is embedded into C_i, for each $i = 1, 2, \dots, t$.

To show that this embedding does not fail, it is sufficient to show that $|V(C_i)| \geqslant |V(D_i)|$. Let $d_i = |V(D_i)|$. Since $G \in \mathcal{H}_{n,k}$, $d_i \in \{1,\dots,k\}$. The main observation is that $d_i \leqslant |V(G)|/i \leqslant n/i$. This is because cliques of G are considered by non-increasing order. We have seen that U_k contains $\lfloor n/j \rfloor$ disjoint cliques with at least j vertices for every $j \in \{1,\dots,k\}$. So, since $d_i \leqslant n/i$, U_k contains $\lfloor n/d_i \rfloor \geqslant \lfloor n/(n/i) \rfloor = i$ disjoint cliques with at least d_i vertices. Therefore, the ith largest cliques in U_k has at least d_i vertices, that is $|V(C_i)| \geqslant d_i = |V(D_i)|$ as claimed. □

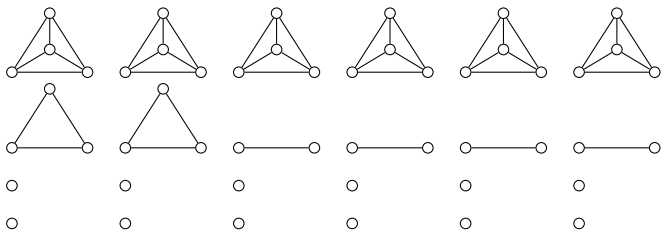

Figure 1: The minimum induced-universal graph for $\mathcal{H}_{24,4}$, the family of graphs with at most 24 vertices that are union of disjoint cliques with at most 4 vertices. Its has $\mathcal{U}(\mathcal{H}_{24,4}) = 24 \cdot (1 + 1/2 + 1/3 + 1/4) = 50$ vertices.

Note that graphs in $\mathcal{H}_{n,2}$ are forests (it is actually a matching), graphs in $\mathcal{H}_{n,3}$ have maximum degree at most two and thus are outer-planar, and $\mathcal{H}_{n,4}$ are planar. Thus, for these families, for infinitely many n (for n divisible by $k!$), we have respectively $3n/2$, $11n/6$, and $25n/12$ as lower bounds on the number of vertices for their minimum induced-universal graphs (see Table 2). However, for forests, outer-planar and planar graphs, the next lower bounds, as presented in Theorem 2, will be better (see also Table 3).

2.2 Information-Theoretic lower bound

Given an infinite graph family $\mathcal{F}$, the *unlabeled constant-growth* of $\mathcal{F}$ is the following number (if it exists),

$$g(\mathcal{F}) \;=\; \limsup_{n\to\infty} (u_n)^{1/n}$$

$\mathscr{F}$	lower bounds on $\mathcal{U}(\mathscr{F}_n)$
caterpillar forests $\supset \mathscr{K}_{n,2}$	$3n/2$
union of cycles and paths $\supset \mathscr{K}_{n,3}$	$11n/6$
planar $\supset \mathscr{K}_{n,4}$	$25n/12$
union of cliques with $\leqslant k$ vertices $= \mathscr{K}_{n,k}$	$(n+1)\ln(k+1)-k$
union of cliques $= \mathscr{K}_{n,n}$	$(n+1)\ln(n+1)-n$

Table 2: Summary of lower bounds derived from Theorem 1.

where u_n counts the number of unlabeled graphs of $\mathscr{F}$ having n vertices.

Theorem 2 *Let $\mathscr{F}$ be a infinite graph family with unlabeled constant-growth at least*[6] *$g > e/2$. For every $0 < \varepsilon < g/e - 1/2$, and for infinitely many n, $\mathcal{U}(\mathscr{F}_n) > (c-\varepsilon)n$, where c is solution of $c^c/(c-1)^{c-1} = g$. We have $c \geqslant g/e + 1/2$. Moreover, if $\lim_{n\to\infty}(u_n)^{1/n}$ does exist, then $\mathcal{U}(\mathscr{F}_n) \geqslant cn$ for every large enough n.*

Proof. Consider any minimum induced-universal graph U for $\mathscr{F}_n$, let $N = \mathcal{U}(\mathscr{F}_n)$ be its number of vertices, and let u_n be the number of unlabeled graphs in $\mathscr{F}_n$.

From the universality of U, every graph of $\mathscr{F}_n$ is isomorphic to an n-vertex induced subgraph of U. So, the number of pairwise non-isomorphic n-vertex induced subgraphs in U must be an upper bound on u_n. Such induced subgraphs can be obtained by selecting a subset of n vertices taken among the N of U. Thus, there are at most $\binom{N}{n}$ induced subgraphs in U having n vertices. It follows that $u_n \leqslant \binom{N}{n}$.

We have the well-known upper bound for the binomial numbers (see [Bol78, p. 255] for instance):

$$\forall\, 0 < b < a, \quad \binom{a}{b} < \left(\frac{a}{b}\right)^b \left(\frac{a}{a-b}\right)^{a-b} = \left(\alpha\left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1}\right)^b = f(\alpha)^b$$

where $\alpha = a/b > 1$ and $f : x \mapsto x^x/(x-1)^{x-1}$ is related to the entropic-function.

Assume $N \leqslant cn$ for all n and some constant $c > 1$. We have, for every $n \geqslant 1$:

$$\begin{aligned} u_n &\leqslant \binom{N}{n} \leqslant \binom{cn}{n} < f(c)^n \\ \Rightarrow \quad (u_n)^{1/n} &< f(c)\,. \end{aligned}$$

The latter inequality implies that $\limsup_{n\to\infty}(u_n)^{1/n} \leqslant f(c)$, since[7] $f(c)$ is a constant independent of n. In other words, if $N \leqslant cn$ for every n and $c > 1$, then $g \leqslant g(\mathscr{F}) \leqslant f(c)$, where g is a lower bound on the unlabeled constant-growth of $\mathscr{F}$. A basic property of lim sup is that, for every sequence x_n, if $\limsup_{n\to\infty} x_n > \ell$, for some constant ℓ, then, $x_n > \ell$ for infinitely many n. It turns out that if $g > f(c)$, then $(u_n)^{1/n} > f(c)$, and thus $N > cn$ for infinitely many n.

[6] Here $e = \exp(1) = 2.7182818284\ldots$.

[7] This is because, for every sequence x_n, $\limsup_{n\to\infty} x_n = \inf_{n\geqslant 0}\left\{\sup_{m\geqslant n} x_m\right\}$.

One can check that the derivative of f is $f'(x) = f(x)\ln(x/(x-1)) > 0$ for $x > 1$, and that $f(x) \to ex - e/2$ when $x \to \infty$ (using the facts that $\lim f'(x) = e$ and $\lim f(x) - ex = -e/2$). In fact, analyzing the second derivative of f, we have $f(x) \leqslant ex - e/2$ for all $x > 1$. In particular, $f(x)$ is non-decreasing for $x > 1$, and the real solution of the equation $f(c) = g$ is some c_0 satisfying $g \leqslant ec_0 - e/2$, i.e., $c_0 \geqslant g/2 + 1/2$.

We have $0 < \varepsilon < g/e - 1/2$ by assumption. Let $c = c_0 - \varepsilon$. Since $c_0 \geqslant g/2 + 1/2$, we have $c \geqslant g/e + 1/2 - \varepsilon > g/e + 1/2 - (g/e - 1/2) > 1$. So, $f(c) = f(c_0 - \varepsilon) < g$ as $f(x)$ is non-decreasing for $x > 1$, which in turns implies that is $\mathcal{U}(\mathcal{F}_n) = N > (c_0 - \varepsilon)n$ for infinitely many n.

Now assume that $(u_n)^{1/n}$ has a limit, which must be $g(\mathcal{F})$. Following the same argument as above, if $N \leqslant cn$ for every large enough n, then $(u_n)^{1/n} < f(c)$. In particular, $\lim_{n\to\infty}(u_n)^{1/n} \leqslant f(c)$, and thus $g \leqslant g(\mathcal{F}) \leqslant f(c)$. In other words, if $g \geqslant f(c)$, then $N \geqslant cn$ for every large enough n. Therefore, for c solution of $f(c) = g$, we indeed have $N \geqslant cn$ for every large enough n.

This completes the proof. □

At first glance, the condition "for infinitely many n" in the lower bound of Theorem 2 is not satisfactory, as we may expected stronger condition like "for every n, $\mathcal{U}(\mathcal{F}_n) \geqslant (c-\varepsilon)n$" or "$\mathcal{U}(\mathcal{F}_n) \geqslant cn - o(n)$". However, we observe that there are graph families where such a stronger condition cannot be true. This is the case of cubic planar graphs since there are no cubic graphs with an odd number of vertices. Nevertheless, the lower bound $\mathcal{U}(\mathcal{F}_n) \geqslant cn - o(n)$ holds for all the families considered in Table 3, as $\lim_{n\to\infty}(u_n)^{1/n}$ exists for all of them.

Now, using known results on counting graphs for various families, and combined with Theorem 2, we get the following set of new lower bounds (cf. Table 3):

$\mathcal{F}$	g	refs	c
caterpillar forests[8]	2	[HS73]	1.293
forests	2.9557	[Ott48]	1.626
outer-planar	7.5036	[BFKV07]	3.275
series-parallel	9.0733*	[BGKN07]	3.850
K_5^--minor-free	15.65*	[GN09]	6.264
planar	27.2268*	[GN09]	10.520
bounded genus	27.2268*	[McD08]	10.520
$K_{3,3}$-minor-free	27.2293*	[GGNW08]	10.521

Table 3: Lower bounds for various family $\mathcal{F}$ of the form $\mathcal{U}(\mathcal{F}_n) \geqslant cn - o(n)$, where c is the solution of $g = c^c/(c-1)^{c-1}$ and where g is a lower bound on the unlabeled constant-growth of $\mathcal{F}$. All figures in the numbers are correct (not rounded). Starred numbers correspond to the *labeled* constant-growth of $\mathcal{F}$ defined as $\limsup_{n\to\infty}(\ell_n/n!)^{1/n}$, where ℓ_n counts the number of labeled graphs of $\mathcal{F}$ having n vertices, a lower bound on the unlabeled constant-growth of $\mathcal{F}$.

The labeled constant-growth of other families can be founded in [GN09, MK12, NRR20]. An intriguing question, we left open, is to know whether there are hereditary families $\mathcal{F}$

[8] Improved by Theorem 1.

with unlabeled constant-growth g for which the lower bound given by Theorem 2 is tight.

3 Almost Equitable Coloring

The main idea in our construction of small induced-universal graph in Theorem 3 is to reuse large stable sets found in each graph of the family. For this purpose, one need these stable sets have similar size.

Recall that a k-coloring is *equitable* if the number of vertices in two color classes differ by at most one. In other words, if G has an equitable k-coloring, then $V(G)$ has a partition into stable sets $C_1, \ldots, C_k$ such that $|C_i| \in \{\lceil n/k \rceil, \lfloor n/k \rfloor\}$ for each i, where $n = |V(G)|$. Note that $|C_i| = 0$ for some i implies that $|C_j| \in \{0, 1\}$ for all j.

It is well-known that any n-vertex graph with maximum degree Δ has an equitable $(\Delta+1)$-coloring, as proved by Hajnal and Szemerédi (see [KK08] for a shorter proof), that can be computed in time $O(\Delta n^2)$ [KKMS10]. It is conjectured that in fact every connected graph is equitably $(\Delta + 1)$-colorable [Mey73], if it is neither a clique nor an odd cycle. This has been confirmed for many graph classes. In particular, every 4-colorable graph of maximum degree 4 has an equitable 4-coloring [KK12].

We point out that there are graphs with equitable k-coloring but no equitable $(k + 1)$-coloring, e.g. a 2-coloring of $K_{3,3}$. There are also bipartite graphs with no equitable 2-coloring, for instance $K_{p,q}$ with $q > p + 1$. Even worst for $p = 1$, $K_{1,q}$ has an equitable k-coloring if and only if $k \geqslant 1 + q/2$. However, by removing p vertices, we can obtain a graph that has an equitable 2-coloring, and even so an equitable 1-coloring.

This leads to the following extension of equitable coloring that will be the crux for our main result. See Fig. 2 for an illustration.

Definition 1 *A graph G is* p-almost equitably k-colorable *if there exists a set X of at most p vertices such that $G \setminus X$ has an equitable k-coloring.*

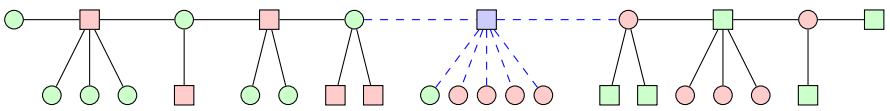

Figure 2: A caterpillar of 29 vertices with a 2-coloring that is not equitable (18 circle vs. 11 squared vertices). In fact, it has no equitable 2-coloring, but it has a 1-almost equitable 2-coloring (14 green and 14 red vertices) obtained by removing the squared vertex in the meddle of the path.

So, equitably k-colorable graphs, i.e., the graphs having an equitably k-coloring, are exactly those that are 0-almost equitably k-colorable. Clearly, every graph G with n vertices is n-almost equitably k-colorable, $G \setminus X$ has only k vertices which can be (equitably) k-colored anyway. It is also straightforward to see that:

Claim 1 *If G has n vertices and is p-almost equitably k-colorable with $k \leqslant n$, then there exists a set X with exactly $\min\{p, n-k\}$ vertices such that $G \setminus X$ has an equitable k-coloring.*

Indeed, if $|X| < \min\{p, n-k\}$, one can successively increase X set by taking one vertex at each step from the stable set with the current most frequent color. We will use Claim 1 twice in this section.

3.1 Block design based construction

Theorem 3 below relies on the well-known Coding-Theoretic Function $A(n,d,w)$ that is the maximum number of binary words of n bits that are pairwise at distance[9] at least d and of weight[10] w. For instance, with the bijection between binary words of length n to subsets of $\{1,\dots,n\}$, $A(n,2,2)$ can be seen as the number of distinct pairs of integers taken from $\{1,\dots,n\}$, since to be at distance at least two, binary words of weight two can share at most one 1. Therefore, $A(n,2,2) = \binom{n}{2}$. More generally, for each integer $k \geqslant 2$, $A(n,2k-2,k)$ is the maximum number of edge-disjoint copies of K_k taken in K_n. Such clique partitions are also called *block designs* or *incomplete block designs*, and are parts of Combinatorial Designs field [CD07].

Not all the values of $A(n,d,w)$ are known, and we refer to [BSSS90, CD07] for best known bounds on $A(n,d,w)$. However, by counting edges of the cliques, it is easy to see that $A(n,2k-2,k) \leqslant \binom{n}{2}\Big/\binom{k}{2}$. The equality holds if and only if a Steiner System $S(2,k,n)$ exists (for instance, see [BSSS90, Theorem 7]). E.g., the equality holds for any $a \in \mathbb{N}$, for any prime power k and $n = k^a$, or, for any k power of two and $n = (k-1)2^a + k$ (cf. [BSSS90, Theorem 13, Eq. (26&29)].

By combining block designs and almost equitable coloring, we can show:

Theorem 3 *Let $s,t,k \in \mathbb{N}$ such that $t \leqslant A(s,2k-2,k)$. For every family $\mathscr{F}$ with t graphs, each being p-almost equitably k-colorable with n vertices,*

$$\mathscr{U}(\mathscr{F}) \;\leqslant\; s\left\lceil \frac{n-p}{k} \right\rceil + tp\,.$$

Given the numbers $A(s,2k-2,k)$, the induced-universal graph $U_{\mathscr{F}}$ for $\mathscr{F}$ has an explicit construction. Furthermore, all embedded graphs of $\mathscr{F}$ in $\mathscr{U}_{\mathscr{F}}$ are pairwise edge disjoint, as only vertices are shared into some common stable sets. It follows that $U_{\mathscr{F}}$ is also an induced-universal graph for any edge-colored graph of $\mathscr{F}$. For the family of all edge-colored cliques with r colors, we refer to [Kou21].

Proof of Theorem 3. Consider a family $\mathscr{F}$ of t graphs with n vertices that are p-almost equitably k-colorable, where $t \leqslant A(s,2k-2,k)$ for some integer s.

We define the graph family $\mathcal{H}$ obtained from $\mathscr{F}$, by removing, for each $G \in \mathscr{F}$, exactly p vertices of G such that the resulting graph has an equitable k-coloring. The fact we can remove exactly p vertices is due to Claim 1. By construction, $|\mathcal{H}| \leqslant |\mathscr{F}| = t$.

$U_{\mathcal{H}}$ is composed of s stable sets $S_1 \dots, S_s$, each with $\lceil (n-p)/k \rceil$ vertices.

[9] I.e., the Hamming distance.
[10] I.e., the number of 1's in the binary word.

We define $\mathscr{K}_s$ be the complete graph on s vertices that are the stables $S_1, \ldots, S_s$. As we already said, it is possible to pack $A(s, 2k-2, k)$ cliques of k vertices in $\mathscr{K}_s$ in a way that no two cliques of the packing share more than one vertex. Let $Q_1, \ldots, Q_t$ be such clique packing in $\mathscr{K}_s$. Note that this is possible because $t \leqslant A(s, 2k-2, k)$. E.g., if $k = 2$, Q_i's are just K_2.

The edges of $U_{\mathcal{H}}$ are determined by the graphs of $\mathcal{H}$ as follows. We associated with each graph $H_i \in \mathcal{H}$ a unique clique Q_i of the packing. In $U_{\mathcal{H}}$, Q_i corresponds to a collection of k stables $C_1, \ldots, C_k$, taken among the S_i's. In the meanwhile, the equitable k-coloring of H_i induced a partition of $V(H_i)$ into k stables, each with at most $\lceil (n-p)/k \rceil$ vertices. We map all vertices of a given part of H_i, say those of color j, to some stable C_j in $U_{\mathcal{H}}$. Then, we add to $U_{\mathcal{H}}$ all edges of H_i, so that are between some C_j and $C_{j'}$. Mapping the vertices of color j in H_i on C_j is $U_{\mathcal{H}}$ is possible because each C_j has $\lceil (n-p)/k \rceil$ vertices.

From the above construction, one can check that each $H_i \in \mathcal{H}$ is an induced subgraph of $U_{\mathcal{H}}$, since any two graphs of $\mathcal{H}$ share at most one stable in $U_{\mathcal{H}}$. It shows that $U_{\mathcal{H}}$ is an induced-universal graph for $\mathcal{H}$ with $s\lceil (n-p)/k \rceil$ vertices.

The graph $U_{\mathscr{F}}$ is obtained from $U_{\mathcal{H}}$ by adding the following set of vertices and edges. For each $G_i \in \mathscr{F}$, we select some $H_i = G_i \setminus X_i$ in $\mathcal{H}$ with $|X_i| = p$. This is possible because G is p-almost equitably k-colorable. The graph $H_i \in \mathcal{H}$ appears in $U_{\mathcal{H}}$ as induced subgraph. Then, we add p vertices that are connected to H_i in the same way X_i is connected to its neighbors in G_i as shown in Fig. 3. Repeating the same process for each graph $G_i \in \mathscr{F}$ adds tp vertices.

Eventually, the graph $U_{\mathscr{F}}$ has $s\lceil (n-p)/k \rceil + tp$ vertices and is induced-universal for the family $\mathscr{F}$. See Fig. 3 for an illustration with $k = 2$.

□

The above construction relies on the function $A(s, 2k-2, k)$, whose values are not all known, but can be computed in practice (see for instance [Owe95]).

3.2 Bounded pathwidth graphs

We make a link between pathwidth and almost equitable coloring. Recall that a graph G has pathwidth at most p if it is a subgraph of some interval graphs with maximum clique size $p+1$.

Theorem 4 *Every k-colorable graph of pathwidth at most p is $p(k-1)$-almost equitably k-colorable.*

Because graphs of pathwidth p are $(p+1)$-colorable (as they are p-degenerated), we immediately get from Theorem 4:

Corollary 1 *Every graph of pathwidth at most p is p^2-almost equitably $(p+1)$-colorable.*

The strategy to prove Theorem 4 is as follows.

First, we select two colors that are non-equitable for the k-coloring, the least and the most frequent colors. Second, we show how to re-balance these two colors, by removing at most p vertices, so that one of the color has $n/k - p(k-1)/k$ vertices. Then, we set this color aside

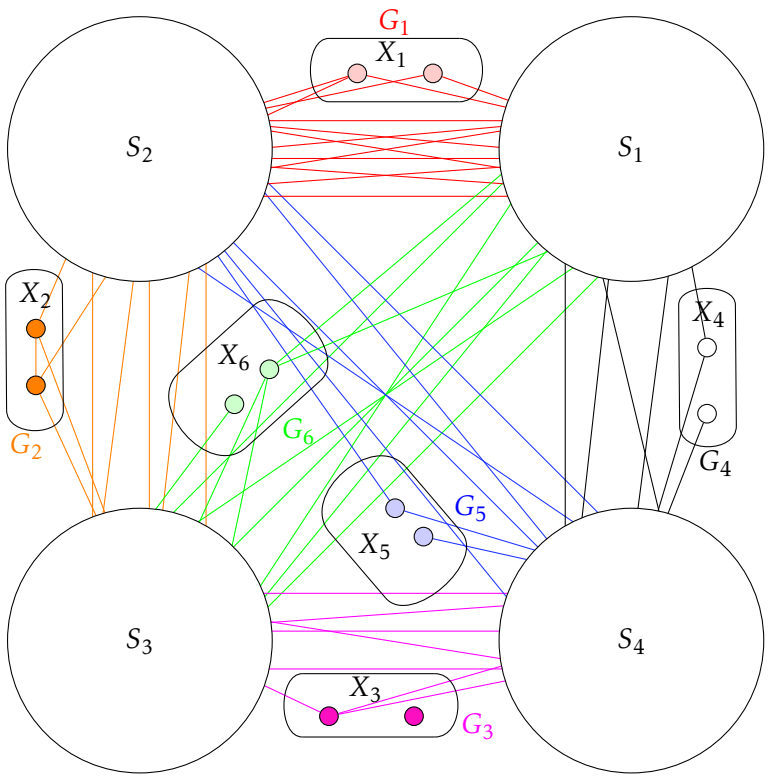

Figure 3: Construction of $U_{\mathcal{F}}$ for a family $\mathcal{F} = \{G_1, \ldots, G_6\}$ of 2-almost 2-colorable graphs. Here $p = k = 2$, $s = 4$, and $t = A(s, 2k-2, k) = A(s, 2, 2) = \binom{s}{2} = 6$. The stables $S_1, \ldots, S_4$, each of size $\lceil (n-p)/k \rceil$, are shared between the 6 graphs of $\mathcal{F}$.

and repeat this process $k-1$ times in total, until all the colors are balanced, so by removing a total of $p(k-1)$ vertices.

For the second part, we need of the following technical lemma.

Lemma 1 *Let B be a graph having a path-decomposition of width p and a 2-coloring into stables C_1, C_2 with $|C_1| \geqslant |C_2|$. Then, for every non-negative integer $a \in [|C_2| - p, |C_1|]$ there exists a set X of at most p vertices and a 2-coloring C_1', C_2' of $B \setminus X$ such that $|C_1'| = a$.*

Proof. Let B be a bipartite graph, and $P = (X_1, \ldots, X_r)$ be a path-decomposition of width p. According to [CFK+15, Lemma 7.2], we can assume that P is *nice*, meaning that X_1, X_r are empty, and that X_{i+1} is obtained from X_i either by adding a vertex not in X_i, or by removing a vertex from X_i. Some X_i's other than X_1 and X_r may be empty, in particular if B is not connected.

For every X_i of P, we define a 2-coloring C_1^i, C_2^i of $B \setminus X_i$ as follows. Informally, all the vertices of $B \setminus X_i$ *after* X_i in P (i.e., every $u \in X_j \setminus X_i$ with $j > i$) keep their color. Moreover, the ones *before* X_i (i.e., every $u \in X_j \setminus X_i$ with $j < i$) exchange their colors. More precisely, we define C_1^i as the set of vertices of $B \setminus X_i$ that are in C_1, if after X_i, and in C_2, if before X_i. Moreover, define $C_2^i = V(B) \setminus X_i \setminus C_1^i$, that is the set of vertices of $B \setminus X_i$ that are in C_2, if after X_i, and in C_1, if before X_i. Note that, for each i, C_1^i, C_2^i is a 2-coloring of $B \setminus X_i$ since, from the path-decomposition P, vertices before and after X_i are in different connected components of $B \setminus X_i$. Since there no vertices before X_1 and $B \setminus X_1 = B$, we have $C_1^1 = C_1$ and $C_2^1 = C_2$. Similarly, there no vertices after X_r and $B \setminus X_r = B$, so $C_1^r = C_2$ and $C_2^r = C_1$.

For simplicity, denote by $a_i = |C_1^i|$ and $b_i = |C_2^i|$ the color's counters for C_1^i, C_2^i. The 2-colorings defined for $i = 1$ and $i = r$ have their colors inverted. So, $a_1 = |C_1^1| = |C_1| = b_r$ and $b_1 = |C_2^1| = |C_2| = a_r$. Note that $a_1 \geqslant a_r$, since $|C_1| \geqslant |C_2|$.

Consider any non-negative integer $a \in [|C_2| - p, |C_1|]$. In other words, $a \in [\max(0, a_r - p), a_1]$. If $a \leqslant a_r$, then we can select for X any subset of C_2 with $|X| = a_r - a$ vertices. Since $a \geqslant \max(0, a_r - p)$, then $a_r - a \leqslant a_r - \max(0, a_r - p) = \min(a_r - 0, a_r - (a_r - p)) \leqslant p$, and thus we have $|X| \leqslant p$, and $|C_2 \setminus X| = |C_2| - |X| = a_r - (a_r - a) = a$. Thus, if $a \leqslant a_r$, we are done with the initial 2-coloring of B by setting $C_1' = C_2 \setminus X$ and $C_2' = C_1 \setminus X = C_1$. So, from now on, let us assume $a \in (a_r, a_1]$.

Two consecutive bags of P change by exactly one vertex (recall that P is nice). Therefore, whenever moving from X_i to X_{i+1} along P, the color's counters for the 2-coloring C_1^i, C_2^i goes from (a_i, b_i) to (a_{i+1}, b_{i+1}) for C_1^{i+1}, C_2^{i+1} such that

$$(a_{i+1}, b_{i+1}) \in \{(a_i - 1, b_i), (a_i + 1, b_i), (a_i, b_i - 1), (a_i, b_i + 1)\} .$$

Since $a \in (a_r, a_1]$, there must exist at least one bag X_j of P such that $a_j = a$ since the color's counter moves decreasingly one-by-one on each component from (a_1, b_1) to (a_r, b_r) with $a_1 \geqslant a_r$. W.l.o.g. assume j is the largest index such that $a_j = a$.

If $|X_j| \leqslant p$, then we are done by selecting $X = X_j$ and the 2-coloring $C_1' = C_1^j$ and $C_2' = C_2^j$. We will conclude now by showing that the case $|X_j| = p + 1$ is not possible.

Assume $|X_j| = p + 1$. Consider X_{j+1}, which exists in P, since $|X_j| > 0$ and thus $X_j \neq X_r = \varnothing$. Note that $|X_{j+1}| < |X_j|$, since P is nice. We also remark that $a_i + b_i + |X_i| = |V(B)|$ for every i. If $a_{j+1} = a$, then j is not maximal: a contradiction. If $a_{j+1} < a$, then $b_{j+1} = b_j$. It follows that

$a_{j+1} + b_{j+1} + |X_{j+1}| < a_j + b_j + |X_j|$: a contradiction, both sums must be $|V(B)|$. So, we are left with the case $a_{j+1} > a$. Thus, we have $a_r < a < a_{j+1}$. By considering the bags of P from X_{j+1} to X_r, and the color counters from (a_{j+1}, b_{j+1}) to (a_r, b_r), there must be an index $j' \in [j+1, r]$ such that $a_{j'} = a$: a contradiction with the maximality of j. □

Proof of Theorem 4. Let G be a graph with n vertices having a path-decomposition of width p and a k-coloration into stable sets $C_1, \ldots, C_k$. The willing k-coloring for G is obtained by applying the following process (initially the current graph is G it-self):

1. Select in the current graph the two stable sets corresponding to the least and most frequent colors, say $C_{\min}, C_{\max}$.
2. Apply Lemma 1 on the bipartite subgraph induced by $C_{\min}, C_{\max}$, and with a suitable non-negative integer $a \in [|C_{\min}| - p, |C_{\max}|]$. This provides a set X and a new 2-coloring C'_1, C'_2 with $|C'_1| = a$ and $|X| \leqslant p$. Adjust X and C'_2 such that $|X| = p$.
3. Update the current graph by moving aside vertices of X and C'_1, and repeat.

It is clear that after repeating this process $k-1$ times, we obtain a new k-coloring for G, the last color being composed of the remaining graph. The first $k-1$ colors are all the C'_1's stable sets constructed during the process. Moreover, the deleting set is the union of all the X's sets constructed. It contains exactly $p(k-1)$ vertices.

What we need to check is that such a k-coloring for G, without the deleting set, is equitable.

To be more precise, we introduce the following notations. Consider the situation obtained after applying i loops of the above process, and denote by:

- G_i the current graph, initially $G_0 = G$;
- $C^i_1, \ldots, C^i_{k-i}$ the current $(k-i)$-coloring of G_i, initially $(C^0_1, \ldots, C^0_k) = (C_1, \ldots, C_k)$.
- $C'_1, \ldots, C'_i$ the current sequence of i colors that are supposed to be equitable, initially, for $i = 0$, the sequence is empty, no such colors have yet been defined; and
- X_i the current deleting set for G, obtained by taking the union of all X's sets constructed so far, initially $X_0 = \varnothing$.

After i loops, the current state can be summarized as a partition of the vertex set of G, and can be represented by

$$\langle\, C^i_1, \ldots, C^i_{k-i} \mid X_i \mid C'_1, \ldots, C'_i \,\rangle$$

where G_i is the graph induced by $C^i_1 \cup \cdots \cup C^i_{k-i}$. Initially, for $i = 0$, we have the state

$$\langle\, C_1, \ldots, C_k \mid \varnothing \mid \varepsilon \,\rangle$$

where ε denotes the empty sequence.

So, the $(i+1)$-th step of the process can be rephrased as follows:

1. Select in G_i two stables set $C^i_{\min}, C^i_{\max}$, among its $k-i$ colors $C^i_1, \ldots, C^i_{k-i}$, with least and most cardinality.

2. Apply Lemma 1 on the induced subgraph $G[C^i_{\min} \cup C^i_{\max}]$ with a suitable parameter a_i. This provides a set X and a stable set C'_i with $|C'_i| = a_i$. Adjusting X to p vertices, we form X_i with $i \cdot p$ vertices.

3. Update G_i and its coloring $C^i_1, \ldots, C^i_{k-i}$ into G_{i+1} and $C^{i+1}_1, \ldots, C^{i+1}_{k-i-1}$.

It can also be illustrated by the state transition:

$$\langle\, C^i_1, \ldots, C^i_{k-i} \mid X_i \mid C'_1, \ldots, C'_i \,\rangle \quad \xrightarrow{i+1} \quad \langle\, C^{i+1}_1, \ldots, C^{i+1}_{k-i-1} \mid X_{i+1} \mid C'_1, \ldots, C'_i, C'_{i+1} \,\rangle$$

After $k-1$ steps, we have a deleting set X_{k-1} with exactly $p(k-1)$ vertices, and a k-coloring of $G \setminus X_{k-1}$ into stable sets $C'_1, \ldots, C'_{k-1}$ and C^{k-1}_1 (the remaining stable set, actually G_{k-1}). In other words, the last transition is:

$$\langle\, C^{k-2}_1, C^{k-2}_2 \mid X_{k-2} \mid C'_1, \ldots, C'_{k-2} \,\rangle \quad \xrightarrow{k-1} \quad \langle\, C^{k-1}_1 \mid X_{k-1} \mid C'_1, \ldots, C'_{k-1} \,\rangle$$

To show that these colors are equitable for $G \setminus X_{k-1}$, we have to show that a_i's can be chosen such that $a_i \in \{\lceil (n-|X_{k-1}|)/k \rceil, \lfloor (n-|X_{k-1}|)/k \rfloor\}$, and, similarly the for the last color C^{k-1}_1, that $C^{k-1}_1 \in \{\lceil (n-|X_{k-1}|)/k \rceil, \lfloor (n-|X_{k-1}|)/k \rfloor\}$. Furthermore, in order to apply Lemma 1, we have to check that $a_i \in [|C^i_{\min}| - p, |C^i_{\max}|]$ and $a_i \geqslant 0$.

We remark that for non-null integers m, k, we have $m = k\lfloor m/k \rfloor + r$, where $r = m \bmod k$. Observing that $\lceil m/k \rceil - \lfloor m/k \rfloor \in \{0,1\}$ depending whether k divides m or not, this can be rewritten as $m = (k-r)\lfloor m/k \rfloor + r\lceil m/k \rceil$.

So, to be equitable, we must have in $G \setminus X_{k-1}$ precisely $k-r$ stable sets of size $\lfloor (n-|X_{k-1}|)/k \rfloor$ and r stable sets of size $\lceil (n-|X_{k-1}|)/k \rceil$, where $r = (n-|X_{k-1}|) \bmod k$. Since $n-|X_{k-1}| = n-p(k-1)$, we have $r = (n+p) \bmod k$, and $(n-|X_{k-1}|)/k = (n+p)/k - p$. So, we must have r stable sets of size $\lceil (n+p)/k \rceil - p$ and $k-r$ stable sets of size $\lfloor (n+p)/k \rfloor - p$, including the last color C^{k-1}_1.

For the r first steps $i = 1, \ldots, r$, we choose $a_i = \lceil (n+p)/k \rceil - p$, and for $i > r$ we choose $a_i = \lfloor (n+p)/k \rfloor - p$. As a_i's differ by at most one, it will be more convenient to rewrite a_i as

$$\forall i \in \{1, \ldots, k-1\}, \quad a_i \;=\; \left\lfloor \frac{n+p}{k} \right\rfloor - p + d_i \quad \text{, where } d_i = \begin{cases} 1 & \text{if } r > 1 \text{ and } i \leqslant r \\ 0 & \text{if } r = 0 \text{ or } i > r \end{cases}$$

It is easy to check that $\sum_{j=1}^{i} d_i = \min(i, r)$, since up to index r, $d_i = 1$, and then $d_i = 0$ (we have always $d_i = 0$ if $r = 0$). W.l.o.g. we assume that $n \geqslant p(k-1)$, since Theorem 4 is clearly true otherwise. It follows that $(n+p)/k - p \geqslant 0$, and thus all the a_i's are $\geqslant 0$.

We assume that Lemma 1 applies correctly for the first i steps of the process, that is $a_i \in [|C^i_{\min}| - p, |C^i_{\max}|]$. Let $n_i = |V(G_i)|$ be the number of vertices in the current graph G_i. By

construction, $n_i = n - |X_i| - \sum_{j=1}^{i} |C'_i|$. Since Lemma 1 applies, $|C'_i| = a_i$, and we have

$$\begin{aligned} n_i &= n - |X_i| - \sum_{j=1}^{i} a_i \\ &= n - ip - \sum_{j=1}^{i} \left(\left\lfloor \frac{n+p}{k} \right\rfloor - p + d_j \right) \\ &= n - i \left\lfloor \frac{n+p}{k} \right\rfloor - \sum_{j=1}^{i} d_j \\ &= n - i \left\lfloor \frac{n+p}{k} \right\rfloor - \min(i, r) \,. \end{aligned}$$

We consider now Step $i+1$, for some $i \in \{0, \ldots, k-2\}$. There are $k-i$ colors in G_i, which has n_i vertices, so the least and most frequent colors in G_i satisfy $|C^i_{\min}| \leqslant n_i/(k-i) \leqslant |C^i_{\max}|$. To show that $a_{i+1} \in [|C^i_{\min}| - p, |C^i_{\max}|]$ (we have seen that a_{i+1} – in fact all the a_i's – is non-negative), it suffices to show that

$$\begin{aligned} & \frac{n_i}{k-i} - p \;\leqslant\; a_{i+1} \;\leqslant\; \frac{n_i}{k-i} \\ \Leftrightarrow\quad & \frac{n_i}{k-i} - a_{i+1} \in [0, p] \\ \Leftrightarrow\quad & n_i - (k-i) \cdot a_{i+1} \in I, \text{ where } I = [0, (k-i)p] \\ \Leftrightarrow\quad & \left(n - i \left\lfloor \frac{n+p}{k} \right\rfloor - \min(i, r) \right) - (k-i) \left(\left\lfloor \frac{n+p}{k} \right\rfloor - p + d_{i+1} \right) \in I \\ \Leftrightarrow\quad & n - k \left\lfloor \frac{n+p}{k} \right\rfloor - \min(i, r) + (k-i)(p - d_{i+1}) \in I \\ \Leftrightarrow\quad & r - p - \min(i, r) + (k-i)(p - d_{i+1}) \in I \end{aligned} \tag{2}$$

the last simplification in Eq. (2) comes from the fact that we have chosen r such that $n + p = k \lfloor (n+p)/k \rfloor + r$.

To prove Eq. (2), we consider two cases.

Case $d_{i+1} = 1$. In that case $r > 0$ and $i + 1 \leqslant r$. So $\min(i, r) = i$. Eq. (2) can be simplified as:

$$\begin{aligned} & r - p - i + (k-i)(p-1) \in I \\ \Leftrightarrow\quad & (k-i-1)p - (k-r) \in I = [0, (k-i)p] \\ \Leftrightarrow\quad & (k-i-1)p \geqslant (k-r) \;\text{ and }\; (k-i-1)p - (k-r) \leqslant (k-i)p \\ \Leftarrow\quad & (k-i-1)p \geqslant (k-i-1) \;\text{ and }\; -(k-r) \leqslant p \end{aligned}$$

which is always true as $p \geqslant 0$ and $r < k$. Thus, Eq. (2) is true in this case.

Case $d_{i+1} = 0$. In this case, $r = 0$ or $i + 1 > r$. We observe that in both cases, $r - \min(i, r) = 0$. So, Eq. (2) can be simplified in

$$\begin{aligned} & r - \min(i, r) - p + (k-i)(p-0) \in I \\ \Leftrightarrow\quad & (k-i-1)p \in [0, (k-i)p] \end{aligned}$$

which is true since $p \geqslant 0$ and $i+1 \leqslant k-1$. Thus, Eq. (2) is true in this case as well.

All together, we have shown that the first r colors satisfy $|C'_1|,\ldots,|C'_r| = \lceil (n+p)/k \rceil - p$, and that the $k-r-1$ next colors satisfy $|C'_{r+1}|,\ldots,|C'_{k-1}| = \lfloor (n+p)/k \rfloor - p$. It is easy to check that the last color C_1^{k-1} has $\lfloor (n+p)/k \rfloor - p$ vertices. Indeed, by the partition of the vertices of $G \setminus X_{k-1}$, we must have

$$\begin{aligned}
& n - |X_{k-1}| \;=\; \sum_{i=1}^{r} |C'_i| + \sum_{i=r+1}^{k-1} |C'_i| \;+\; |C_1^{k-1}| \\
\Leftrightarrow \quad & n - p(k-1) \;=\; r\left(\lceil (n+p)/k \rceil - p\right) + (k-r-1)\left(\lfloor (n+p)/k \rfloor - p\right) + |C_1^{k-1}| \\
\Leftrightarrow \quad & n \;=\; r\lceil (n+p)/k \rceil + (k-r-1)\lfloor (n+p)/k \rfloor + |C_1^{k-1}| \\
\Leftrightarrow \quad & n+p \;=\; r\lceil (n+p)/k \rceil + (k-r-1)\lfloor (n+p)/k \rfloor + (|C_1^{k-1}| + p)
\end{aligned}$$

This implies that $|C_1^{k-1}| + p = \lfloor (n+p)/k \rfloor$ by the choice of r. So, $|C_1^{k-1}| = \lfloor (n+p)/k \rfloor - p$ as required.

This completes the proof of Theorem 4. □

We left open the question about the optimality of Theorem 4. However, we can show that, for each k, the pathwidth in Theorem 4 is required. More precisely:

Proposition 1 *For each integers k and p, there is a k-colorable graph of pathwidth at most p for which every q-almost equitably k-coloration requires $q \geqslant p$.*

Proof. Let G be the complete k-partite graph with $k-1$ "small" parts of roughly $p/(k-1)$ vertices and the with one "large" part with $L = 2p + 2k - 1$ vertices. To be more precise, each small part has $\lfloor p/(k-1) \rfloor$ or $\lceil p/(k-1) \rceil$ vertices such that their size sum up to exactly p vertices. The graph G is k-colorable (each part is monochromatic, with a distinct color), and of pathwidth at most p (consider a path of L bags $B_1 - B_2 - \cdots - B_L$ where each B_i is composed of the p vertices contained in the small parts plus the ith vertex of the large part).

Consider any q-almost equitably k-coloration of G, let X be the set of vertices that makes this coloring equitable, and let $G' = G \setminus X$. Suppose $|X| < p$.

There must be one of the small parts with $t > 0$ vertices remaining in G', because their sizes sum up to p. Note that $t \leqslant \lceil p/(k-1) \rceil < p/(k-1) + 1$. It follows that $t + 1 < p/(k-1) + 2$, and thus $(t+1)(k-1) \leqslant p + 2k - 1$.

In the large part, it remains $\ell > (t+1)(k-1)$ vertices in G', because $L - |X| = 2p + 2k - 1 - |X| > p + 2k - 1 \geqslant (t+1)(k-1)$. The colors of any small part cannot appear in the large part. So, to color the vertices of the large part, at most $k-1$ colors are available. To be equitable, each color classes must have t or $t+1$ vertices: a contradiction with $\ell > (t+1)(k-1)$. □

We note that the proofs of that Theorem 4 and Lemma 1 are constructive. Assuming a with-p path-decomposition and a k-coloring of the graph are given, the proofs lead to polynomial time algorithm for constructing the deleting set and the k-coloring making the graph $p(k-1)$-almost equitably k-colored.

3.3 Conflicting family

An application of the construction in Theorem 3 is that it can give a lower bound on the size of a conflicting family w.r.t. some cn-lower bound.

Theorem 5 *Let $\mathcal{F}$ be a family of k-colorable graphs with n vertices and pathwidth $o(n/k)$. If $\mathcal{U}(\mathcal{F}) \geqslant cn - o(n)$ for some constant c, then $\mathcal{F}$ contains more than $A(\lceil ck \rceil - 1, 2k-2, k)$ graphs, for every large enough n.*

Proof. Let s be the largest integer such that $s < ck$ and let $a = A(s, 2k-2, k)$. We have $s = \lceil ck \rceil - 1$. We want to show that $\mathcal{F}$ has to contain $t > a$ graphs. By way of contradiction, assume that $t \leqslant a$.

Note that $a \leqslant \binom{s}{2} \Big/ \binom{k}{2} \leqslant (s/k)^2 < c^2$ by the choice of s. By Theorem 4, every graph of $\mathcal{F}$ is $p(k-1)$-almost k-colorable, where p is an upper bound on the pathwidth of the graph. Therefore, by Theorem 3, $\mathcal{U}(\mathcal{F}) \leqslant sn/k + tp(k-1)$. As $t \leqslant a < c^2$, the additive term $tp(k-1) < c^2 pk = o(n)$ as c is constant and $p = o(n/k)$ by assumption. Thus, $\mathcal{U}(\mathcal{F}) \leqslant sn/k + o(n)$.

This is incompatible with the assumption that $\mathcal{U}(\mathcal{F}) \geqslant cn - o(n)$, since, for every large enough n, this implies that $\mathcal{U}(\mathcal{F})/n \geqslant c - o(1)$, whereas we have $\mathcal{U}(\mathcal{F}) \leqslant sn/k + o(n)$ implies $\mathcal{U}(\mathcal{F})/n < c + o(1)$ by the choice of s.

It follows that $t > a = A(s, 2k-2, k)$ where $s = \lceil ck \rceil - 1$ as claimed. □

To illustrate Theorem 5, consider any family $\mathcal{F}_n$ of bipartite graphs with sublinear pathwidth. From Theorem 5, if $\mathcal{U}(\mathcal{F}_n) \geqslant 2.1n - o(n)$, then $\mathcal{F}_n$ must contains more than $A(\lceil 2.1 \cdot 2 \rceil - 1, 2 \cdot 2 - 2, 2) = A(4, 2, 2) = 6$ graphs. In particular, any conflicting family w.r.t. a $2.1n$-lower bound for trees must contains at least 7 trees.

Now, we can combine the lower bounds on c collected from Table 3, with Theorem 5. By this way, we obtain in Table 4 the lower bounds on the size of conflicting families. Note that all families considered therein have bounded chromatic number and sublinear pathwidth.

$\mathcal{F}$	c	t	k	$A(\lceil ck \rceil - 1, 2k-2, k)$
forests	1.626	4	2	$A(3,2,2) = 3$
outer-planar	3.275	13	3	$A(9,4,3) = 12$
series-parallel	3.850	17	3	$A(11,4,3) = 16$
K_5^--minor-free[11]	6.264	51	4	$A(25,6,4) = 50$
planar	10.520	137	4	$A(42,6,4) = 136$
$K_{3,3}$-minor-free	10.521	124	5	$A(52,8,5) \in \{123, 124\}$[12]

Table 4: Lower bounds on the size t of a conflicting families w.r.t. a cn-lower bound.

4 Towards Super-Linear Lower Bounds

The motivation of this part is mostly linked to the research of lower bounds for families of n-vertex graphs for which one suspect a super-linear lower bound, say $\lambda(n) \cdot n$ where $\lambda(n)$ is

non-constant. We show that, for many of those families, namely those with small chromatic number and sublinear pathwidth, the size of any conflicting family w.r.t. a $\lambda(n) \cdot n$-lower bound cannot be linear in $\lambda(n)$. As shown in the next theorem, it must be at least quadratic.

Theorem 6 *Let $\mathcal{F}$ be a family of n-vertex k-colorable graphs of pathwidth at most p such that $p(k^2-1) \leqslant n$. If the number of graphs in $\mathcal{F}$ is $t \geqslant \max\{4k^2, 811\}$, then*

$$\mathcal{U}(\mathcal{F}) \;<\; \frac{15}{7} \cdot \sqrt{t} \cdot n \;.$$

A consequence of Theorem 6 is that any $\Omega(n \log n)$-lower bound proof for planar graphs (if it exists), and more generally for fixed minor-closed family of graphs, must use a conflicting family of at least $t = \Omega(\log^2 n)$ witnesses n-vertex graphs.

First, this is because graphs excluding a fixed minor have pathwidth $O(\sqrt{n})$ [AST90], and constant chromatic number, due to bounded edge density [Mad67], so conditions are fulfilled to apply Theorem 6. Second, for such family $\mathcal{F}$ with $t = o(\log^2 n)$ graphs, Theorem 6 would imply that $\mathcal{U}(\mathcal{F}) = o(n \log n)$.

In order to prove Theorem 6 we need, as intermediate step, the following result:

Theorem 7 *Let $\mathcal{F}$ be a family of n-vertex p-almost equitably k-colorable graphs. If the number of graphs in $\mathcal{F}$ is $t \geqslant \max\{k^2, 811\}$, then*

$$\mathcal{U}(\mathcal{F}) \;<\; \frac{15}{14} \cdot \sqrt{t} \cdot (n - p + k) + tp \;.$$

A consequence of Theorem 7, is that an upper bound on $\mathcal{U}(\mathcal{F})$ can be derived for any graph family $\mathcal{F}$, independently from its colorability. However, unlike Theorem 7, it relies to families having $t = \Omega(n^2)$ graphs with n vertices.

Corollary 2 *Every family of $n^2 \geqslant 811$ graphs with n vertices has an induced-universal graph with less than $15n^2/7$ vertices.*

Proof. Every n-vertex graph is equitably n-colorable. So Theorem 7 can be applied to any family $\mathcal{F}$ of n-vertex graphs (with $k = n$ and $p = 0$). Assuming that the number of graphs in $\mathcal{F}$ is $t = k^2 = n^2$ graphs, we get by Theorem 7, that $\mathcal{U}(\mathcal{F}) < 15/14 \cdot \sqrt{t} \cdot (n - 0 + n) + t \cdot 0 = 15n^2/7$ as claimed. □

The proof of Theorem 7 relies on the two technical following lemmas about $A(n, 2k-2, k)$. Indeed, in the proof of Theorem 6, we will need to lower bound $A(n, 2k-2, k)$ for *every* k, and not only for prime power k as in [BSSS90, Theorem 13]. Furthermore, as already explained above Definition 1, majoring k does not necessarily keep the equitably property of colorings. An alternative to our lower bound in Lemma 2, would be the use of the well-known optimal asymptotic bound due to Erdős and Hanani [EH63, Theorem 1] (see also [AY05, Theorem 3.1] for further refinements) saying that, for each $k \geqslant 2$,

$$\lim_{n \to \infty} \frac{A(n, 2k-2, k)}{\binom{n}{2} \Big/ \binom{k}{2}} \;=\; 1 \;.$$

Unfortunately, this would introduced an uncontrolled conditions between s and k due to this limit. Nevertheless, based on [Bos38, Bus52], it is possible to construct *orthogonal arrays* $\mathrm{OA}(s^2, k, s, 2)$, a generalization of orthogonal Latin squares, for every prime power s and $k \leqslant s+1$. From them, we can derive incomplete block designs showing that $A(sk, 2k-2, k) \geqslant s^2$ (see [CD07, Part III]). For completeness, we present another simple construction of such block designs, making the universal graphs given by Theorem 6 and Theorem 7 fully constructive in polynomial time.

Lemma 2 *Let $s, k \in \mathbb{N}$ such that s is prime and $2 \leqslant k \leqslant s$. Then*

$$A(sk, 2k-2, k) \;\geqslant\; s^2 + k \cdot A(s, 2k-2, k)\,.$$

It is easy to check that if $A(s, 2k-2, k) = \binom{s}{2}\Big/\binom{k}{2}$, then $s^2 + k \cdot A(s, 2k-2, k) = s^2 + k \cdot \binom{s}{2}\Big/\binom{k}{2} = \binom{sk}{2}\Big/\binom{k}{2}$, and thus $A(sk, 2k-2, k) = \binom{sk}{2}\Big/\binom{k}{2}$ from Lemma 2.

Proof. Recall that $A(sk, 2k-2, k)$ is the maximum number of edge-disjoint copies of K_k taken in K_{sk}. To lower bound this number, we first split the sk vertices of K_{sk} into k vertex-disjoint cliques $S_1, \ldots, S_k$, each with s vertices. Then, in each S_i, we select $A(s, 2k-2, k)$ edge-disjoint copies of K_k. This is possible since each S_i is a copy of K_s and $2 \leqslant k \leqslant s$. This first process creates a collection $\mathscr{C}_1$ of $k \cdot A(s, 2k-2, k)$ pairwise edge-disjoint copies of K_k taken from K_{sk}.

In a second step, we will construct another collection $\mathscr{C}_2$ of pairwise edge-disjoint copies of K_k, all obtained by picking one vertex in each S_i. By doing this way, cliques of $\mathscr{C}_1$ and $\mathscr{C}_2$ are edge-disjoint for sure, since clique-edges of $\mathscr{C}_1$ involve endpoints inside a S_i's, whereas clique-edges of $\mathscr{C}_2$ have endpoints between distinct S_i's. In total, this creates $|\mathscr{C}_1| + |\mathscr{C}_2|$ edge-disjoint copies of K_k taken in K_{sk}, and thus it shows that $A(sk, 2k-2, k) \geqslant |\mathscr{C}_2| + k \cdot A(s, 2k-2, k)$. See Fig. 4 for an example.

It remains to show how to construct such a collection $\mathscr{C}_2$ of edge-disjoint copies of K_k such that $|\mathscr{C}_2| = s^2$.

For convenience, denote $[t] = \{0, \ldots, t-1\}$ for any positive number t. Let $u_{i,j}$ be the jth vertices of S_i, for all $i \in [k]$ and $j \in [s]$. Then, for every $(p, q) \in [s]^2$, define $C_{p,q}$ be the clique of K_{sk} induced by the vertex-set

$$V(C_{p,q}) \;=\; \Big\{u_{i,j} : i \in [k] \text{ and } j = p + iq \bmod s\Big\}\,.$$

By construction, each $C_{p,q}$ is a clique on k vertices. We set $\mathscr{C}_2 = \{C_{p,q} : (p, q) \in [s]^2\}$. Clearly, $|\mathscr{C}_2| = s^2$. It remains to show that cliques of $\mathscr{C}_2$ are edge-disjoint, or equivalently, that any two cliques of $\mathscr{C}_2$ share at most one vertex.

By way of contradiction, assume $u_{i,j}, u_{i',j'} \in V(C_{p,q}) \cap V(C_{p',q'})$ with $(i, j) \neq (i', j')$ and $(p, q) \neq (p', q')$. We consider arithmetic in the cyclic group $\mathbb{Z}/s\mathbb{Z}$. Note that since s is prime, $xy \equiv 0$ implies $x \equiv 0$ or $y \equiv 0$ for all integers x, y.

From $u_{i,j}, u_{i',j'} \in V(C_{p,q}) \cap V(C_{p',q'})$, we get the following equations on j and j':

$$\begin{aligned} j \;&\equiv\; p + iq \;&&\equiv\; p' + iq' \\ j' \;&\equiv\; p + i'q \;&&\equiv\; p' + i'q' \end{aligned}$$

If $i = i'$, then, using $j - j' \equiv (i - i')q$, we get $j - j' \equiv 0$, i.e., $j = j'$: a contradiction. Thus we have $i \neq i'$.

Using $j - j' \equiv (i - i')q \equiv (i - i')q'$, we get $(i - i')(q - q') \equiv 0$, that implies $i = i'$ or $q = q'$ since s is prime. Since $i \neq i'$, we have $q = q'$.

Using $p + iq \equiv p' + iq'$, we get $p - p' \equiv (q' - q)i \equiv 0$ since $q = q'$. This implies $p - p' \equiv 0$, i.e., $p = p'$. This is a contradiction, since we have already $q = q'$.

Therefore, the two distinct vertices $u_{i,j}, u_{i',j'} \in V(C_{p,q}) \cap V(C_{p',q'})$ do not exist, and this completes the proof. □

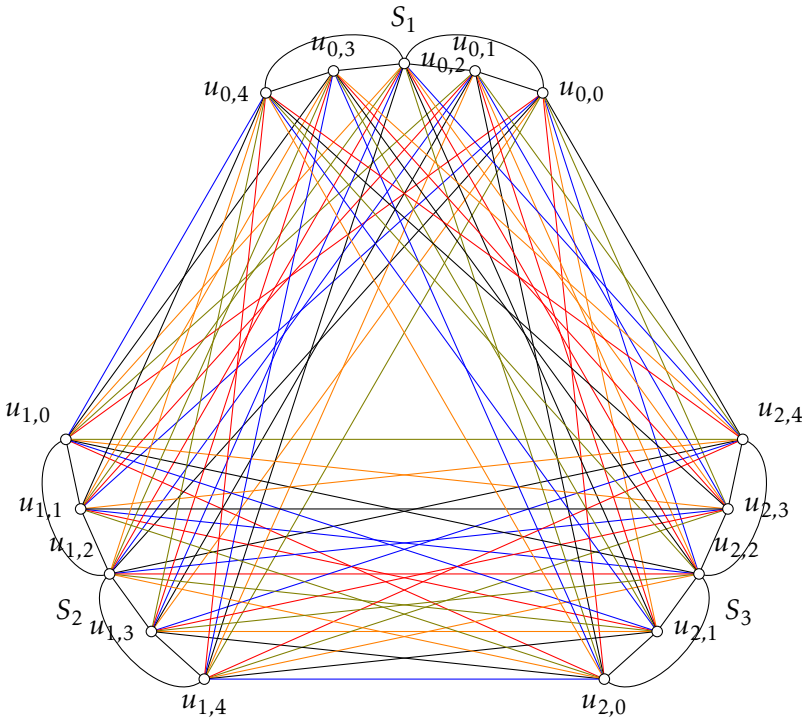

Figure 4: Illustration of the block design construction as in Lemma 2 for $s = 5$ and $k = 3$, aiming to lower bound $A(15, 4, 3)$. Note that $A(5, 4, 3) = 2$ (two K_3 sharing a vertex in K_5), and thus $A(sk, 2k - 2, k) \geqslant s^2 + k \cdot A(s, 2k - 2, k) = 25 + 3 \cdot 2 = 31$. All the edge-disjoint K_3 (triangles) in K_{15} are monochromatic, e.g. $V(C_{3,1}) = \{u_{0,3}, u_{1,4}, u_{2,0}\}$. The edges of K_{15} are not represented. According to [BSSS90, Table I-A], $A(15, 4, 3) = 35$.

Lemma 3 *For all $n, k \in \mathbb{N}$ such that $2 \leqslant k \leqslant 14/15 \cdot (n+1)/k$ and $30 \leqslant n/k$,*

$$A(n, 2k - 2, k) \geqslant \left(\frac{14}{15} \cdot \frac{n+1}{k}\right)^2 .$$

Proof. We use a generalization of Betrand's postulate. From [SGIM13, Theorem 1], for each integer $i \geqslant 2$, there is a prime in the interval $(14i, 15i)$. Since $n \geqslant 30k$, the integer $i = \lfloor n/(15k) \rfloor \geqslant 2$. Thus, there must exist a prime s such that:

$$14 \cdot \left\lfloor \frac{n}{15k} \right\rfloor < s < 15 \cdot \left\lfloor \frac{n}{15k} \right\rfloor \leqslant \frac{n}{k} .$$

In particular, $n > sk$. Since $A(n, 2k-2, k)$ is non-decreasing in n, we have $A(n, 2k-2, k) \geqslant A(sk, 2k-2, k)$. Note that[5]

$$s \geqslant 14i+1 = 14 \cdot \left\lfloor \frac{n}{15k} \right\rfloor + 1 = 14 \cdot \left\lfloor \frac{n+15k}{15k} \right\rfloor \geqslant 14 \cdot \left(\frac{n+15k+1-15k}{15k} \right) = \frac{14}{15} \cdot \frac{n+1}{k}.$$

Thus, since $k \leqslant 14/15 \cdot (n+1)/k$, we have $k \leqslant s$. We can apply Lemma 2 to get:

$$A(n, 2k-2, k) \geqslant A(sk, 2k-2, k) \geqslant s^2 \geqslant \left(\frac{14}{15} \cdot \frac{n+1}{k} \right)^2$$

as claimed. □

According to [SGIM13], the constant 14/15 that appears twice in Lemma 3 cannot be replaced by any ratio of the form $c/(c+1)$ as long as that the integer $c \leqslant 100,000,000$. By using Betrand's postulate[11], we could achieve a better condition on n/k, namely $4 \leqslant n/k$, which ultimately would decrease[12] the constant 811 in Theorem 6 and Theorem 7. However, the price of weakening the condition is a doubling factor on the upper bound on $\mathcal{U}(\mathcal{F})$, replacing the factors 15/14 and 15/7 respectively by 2 and 4.

Proof of Theorem 7. We first observe that $k \geqslant 2$, since otherwise $\mathcal{F}$ would be composed of only one graph (an independent set of n vertices), contradicting the fact that $t \geqslant 811$.

Our goal is to apply Theorem 3 to $\mathcal{F}$. Let us show that if $s = \left\lceil 15/14 \cdot k\sqrt{t} \right\rceil - 1$, then $t \leqslant A(s, 2k-2, k)$. By Lemma 3 (plugging $n = s$), $A(s, 2k-2, k) \geqslant (14/15 \cdot (s+1)/k)^2$, subject to $2 \leqslant k \leqslant 14/15 \cdot (s+1)/k$ and $30 \leqslant s/k$. So, it suffices to show that $14/15 \cdot (s+1)/k \geqslant \sqrt{t}$. This is equivalent to show that $s \geqslant 15/14 \cdot k\sqrt{t} - 1$. Thus, by choosing $s = \left\lceil 15/14 \cdot k\sqrt{t} \right\rceil - 1$, we ensure that $A(s, 2k-2, k) \geqslant t$.

The two conditions on s and k become:

$$2 \leqslant k \leqslant \frac{14}{15} \cdot \frac{\left(\left\lceil 15/14 \cdot k\sqrt{t} \right\rceil - 1 \right) + k}{k} \qquad \text{and} \qquad 30 \leqslant \frac{\left\lceil 15/14 \cdot k\sqrt{t} \right\rceil - 1}{k}$$

Recall that we have assumed $t \geqslant \max\{k^2, 811\}$. For the first condition, we have:

$$\begin{aligned} \frac{14}{15} \cdot \frac{\left(\left\lceil 15/14 \cdot k\sqrt{t} \right\rceil - 1 \right) + k}{k} &\geqslant \frac{14}{15} \cdot \frac{\left(15/14 \cdot k\sqrt{t} + k - 1 \right)}{k} \\ &\geqslant \sqrt{t} + \frac{14}{15} \cdot \left(1 - \frac{1}{k} \right) \geqslant \sqrt{t} \geqslant k \end{aligned}$$

as required.

For the second condition, the term $s/k = \left(\left\lceil 15/14 \cdot k\sqrt{t} \right\rceil - 1 \right)/k \geqslant 15/14 \cdot \sqrt{t} - 1/k$, which is increasing with $k \geqslant 2$. So, the term $15/14 \cdot \sqrt{t} - 1/2$ is at least 30 whenever $\sqrt{t} \geqslant (30 + 1/2) \cdot 14/15 \approx \sqrt{810.35}$. Thus, since $t \geqslant 811$, we have that $s/k \geqslant 30$ as required.

[11] It states that for each integer $i \geqslant 2$, there is a prime in the interval $(i, 2i)$.

[12] Following the proof of Theorem 7, the new condition would be $t \geqslant \lceil ((4+1/2)/2)^2 \rceil = 3$ instead of $t \geqslant \lceil ((30+1/2) \cdot 14/15)^2 \rceil = 811$.

From Theorem 3,

$$\begin{aligned}\mathcal{U}(\mathcal{F}) &\leqslant s\left\lceil\frac{n-p}{k}\right\rceil + tp\\ &< \left(\left\lceil 15/14\cdot k\sqrt{t}\right\rceil - 1\right)\left(\frac{n-p+k}{k}\right) + tp\\ &< \left(15/14\cdot k\sqrt{t}\right)\left(\frac{n-p+k}{k}\right) + tp\\ &< \frac{15}{14}\cdot\sqrt{t}\cdot(n-p+k) + tp\end{aligned}$$

completing the proof of Theorem 7. □

Proof of Theorem 6. Similarly to the proof of Theorem 7, we have $k \geqslant 2$ (otherwise in contradiction with $t \geqslant 811$). For technical reasons, we treat separately the case $p < 2$ from the case $p \geqslant 2$. For the former case, if $p \in \{0,1\}$, the graphs of $\mathcal{F}$ are caterpillar forests. We can use the construction of [GL22] showing that $\mathcal{U}(\mathcal{F}) \leqslant 8n$, independently of t. As $t \geqslant 811$, $(15/7)\sqrt{t}\cdot n \geqslant (15/7)\sqrt{811}\cdot n \approx 61n > 8n \geqslant \mathcal{U}(\mathcal{F})$. Therefore, the statement $\mathcal{U}(\mathcal{F}) \leqslant (15/7)\cdot\sqrt{t}\cdot n$ is trivially true if $p < 2$. So, w.l.o.g. we assume that $p \geqslant 2$.

With each $G \in \mathcal{F}$, we associate a new graph G' such that:

1. G' contains G as induced subgraph;
2. G' has $n' = 2(n-q)$ vertices, where $q = p(k-1)$; and
3. G' is equitably $2k$-colorable.

The graph G' is obtained from G plus $n-2q$ isolated vertices. Note that $n-2q \geqslant 0$ since by assumptions $p(k^2-1) \leqslant n$ and $k \geqslant 2$. Indeed, $2q = 2p(k-1) < p(2k-1) \leqslant p(k^2-1) \leqslant n$.

Clearly, G' contains G as induced subgraph and has exactly $n+n-2q = n'$ vertices.

To prove the third point, that is G' is equitably $2k$-colorable, we partition $V(G')$ into two sets S, T, each of $n-q$ vertices, and show that $G'[S]$ and $G'[T]$ are both equitably k-colorable.

To construct S, we applied to G Theorem 4 and Claim 1 to obtain a set X of $|X| = \min\{p(k-1), n-k\}$ vertices such that $G \setminus X$ is equitably k-colorable. From the conditions on (p,k,n), we get $p(k-1)+k \leqslant p(k-1)+pk \leqslant p(2k-1) \leqslant p(k^2-1) \leqslant n$. So $p(k-1) \leqslant n-k$, and thus $|X| = p(k-1) = q$.

We let $S = V(G) \setminus X$, and $T = V(G') \setminus S$. By construction, $G'[S] = G \setminus X$ has $n-q$ vertices and an equitable k-coloring into k stable sets $C_1^S, \ldots, C_k^S$ with $|C_i^S| \in \{\lfloor (n-q)/k \rfloor, \lceil (n-q)/k \rceil\}$ since we have seen that $q = |X|$.

The graph $G'[T]$ is composed of $G[X]$ plus a set I of $n-2q$ isolated vertices. Consider any k-coloring of $G[X]$ into k stable sets $C_1^X, \ldots, C_k^X$ (recall that G is k-colorable). We have $|C_i^X| \in \{0, \ldots, |X|\}$ for each i, noting that $|C_i^X| = 0$ is possible for some color i, in particular if $p = 1$.

The important remark is that the conditions on (p,k,n) imply that $|C_i^X| \leqslant |C_j^S|$ for all i, j.

Indeed, we have $|C_i^X| \leqslant |X| = q$ and $\lfloor (n-q)/k \rfloor \leqslant |C_j^S|$. Moreover,

$$\begin{aligned} & q \leqslant \lfloor (n-q)/k \rfloor \\ \Leftrightarrow \quad & q \leqslant (n-q)/k \\ \Leftrightarrow \quad & q(k+1) \leqslant n \\ \Leftrightarrow \quad & p(k-1)(k+1) \leqslant n \\ \Leftrightarrow \quad & p(k^2-1) \leqslant n\,. \end{aligned}$$

Now, $|C_i^X| \leqslant |C_j^S|$ for all i, j implies that we can color the vertices of I, and add $|C_i^S| - |C_i^X| \geqslant 0$ vertices taken from I to C_i^X resulting in a new color class of size exactly $|C_i^S|$. This process defines a k-coloring for $G'[T]$ whose color classes have the same color distribution as the k-coloring of $G'[S]$. Since $|S| = |T|$ and the k-coloring of $G'[S]$ is equitable, the k-coloring of $G'[T]$ must be equitable too.

Now, we apply Theorem 7 to the family $\mathcal{F}' = \{G' \colon G \in \mathcal{F}\}$ composed of all the graphs G' constructed from $G \in \mathcal{F}$ as above. Note that $\mathcal{F}'$ is composed of t graphs, each with n' vertices, and that are equitably k'-colorable with $k' = 2k$. Since $t \geqslant \max\{4k^2, 811\}$, we have that $t \geqslant \max\{k'^2, 811\}$ as required to apply Theorem 7. Note that $k - q = k - p(k-1) \leqslant k - 2(k-1) = 2 - k \leqslant 0$, because we have $k, p \geqslant 2$.

Each graph $G \in \mathcal{F}$ appears as induced subgraph of some $G' \in \mathcal{F}'$. Therefore, by Theorem 7, we get:

$$\mathcal{U}(\mathcal{F}) \leqslant \mathcal{U}(\mathcal{F}') < \frac{15}{14} \cdot \sqrt{t} \cdot (n' + k') \leqslant \frac{15}{14} \cdot \sqrt{t} \cdot (2(n-q+k)) \leqslant \frac{15}{7} \cdot \sqrt{t} \cdot n\,.$$

This completes the proof of Theorem 6. □

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