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Procedia Computer Science 00 (2025) 000–000

Procedia Computer Science

www.elsevier.com/locate/procedia

XIII Latin American Algorithms, Graphs, and Optimization Symposium (LAGOS 2025)

# Lower Bounds for Induced-Universal Graphs

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**Abstract**

We give a series of new lower bounds on the minimum number of vertices required by a graph to contain every graph of a given family as induced subgraph. In particular, we show that this induced-universal graph for $n$-vertex planar graphs must have at least $10.52n$ vertices. We also show that the number of conflicting graphs to consider in order to beat this lower bound is at least 137. In other words, any family of less than 137 planar graphs of $n$ vertices has an induced-universal graph with less than $10.52n$ vertices, stressing the difficulty in beating such lower bounds. Similar results are developed for other graph families, including but not limited to, trees, outerplanar graphs, series-parallel graphs, $K_{3,3}$-minor free graphs. As a byproduct, we show that any family of $t$ graphs of $n$ vertices having small chromatic number and sublinear pathwidth, like any proper minor-closed family, has an induced-universal graph with less than $^{15}/_{7}\sqrt{t}\cdot n$ vertices. If $t \geqslant n^2$, the bound actually holds for any family of $t$ graphs. Our results are achieved by making a bridge between equitable colorings, combinatorial designs, and path-decompositions.



*Keywords:* planar graphs; universal graphs; bounded pathwidth graphs; equitable coloring; combinatorial designs

## 1. Introduction

Universality plays an important role in Graph Theory and Computer Science. For the famous Traveling Salesperson Problem, [19] showed that every set $S$ of $n$ points in a metric space has some universal tour, computable in polynomial time, that approximates, up to poly-logarithmic factor, the best tour of every subset of $S$ by taking the induced sub-tour of this universal ordering. In graph theory, the celebrated Excluded Grid Theorem of Robertson and Seymour [31] (which states that graphs excluding as minor a fixed planar graph have bounded treewidth), relies on the fact that every planar graph is a minor of some relatively small grid. So, excluding such a grid as minor implies excluding as minor every sufficiently small planar graph. Therefore, rather than considering individually each graph of a given graph family, it is much simpler to manipulate one single graph (here a grid), whose properties is close enough to those of the graphs of the family (here the property is planarity). This grid-minor universality has been extended by [12], by showing that for every $n$ and $g$, there is some fixed graph of Euler genus $g$ and with only $O(g^2(n+g)^2)$ vertices that is minor-universal for all $n$-vertex graphs of Euler genus $g$. Here the size of the minor-universal graph is an important

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parameter, and its polynomial dependency in $n$ and $g$ is the crux for the very recent polynomial bound for the Graph Minor Structure Theorem [17].

*Induced-universal graphs.* In this paper, we focus on universality under induced-subgraph containment. More precisely, a graph $U$ is an *induced-universal graph* for a graph family $\mathcal{F}$ if every graph of $\mathcal{F}$ is isomorphic to an induced-subgraph of $U$. Here, the term "graph family" must be considered in a broad sense: it can contain a finite set of graphs (like two bounded size graphs, or two trees with $n$ vertices) or a countably finite set of graphs (like the class of all planar graphs). We denote by $\mathcal{U}(\mathcal{F})$ the smallest number of vertices of an induced-universal graph for $\mathcal{F}$, and by $\mathcal{F}_n$ the sub-family composed only of $n$-vertex graphs of $\mathcal{F}$ (if any).

For the family $\mathcal{G}$ of all graphs, Alon [1] showed that $\mathcal{U}(\mathcal{G}_n) = (1 + o(1)) \cdot 2^{(n-1)/2}$. For the family $\mathcal{P}$ of all planar graphs, a result of [10, 14] implies that $\mathcal{U}(\mathcal{G}_n) \leqslant n \cdot 2^{O(\sqrt{\log n})}$. Lastly, for the family $\mathcal{T}$ of all trees, [3] showed that $\mathcal{U}(\mathcal{T}_n) \leqslant cn$, where $c \gg 768$ is a rather large constant[1]. The results for trees and planar graphs come from the design of *adjacency labeling scheme* also called *implicit representation*, which can be viewed as an alternative definition of induced-universal graphs with algorithmic applications [20, 4].

Whereas for all graphs, the first order of $\mathcal{U}(\mathcal{G}_n)$ is well-established, the situation is very different for planar graphs and trees. Up to the best of our knowledge, the only non-trivial better-than-$n$ lower bound is $\mathcal{U}(\mathcal{P}_n) \geqslant 11\lfloor n/6 \rfloor \approx 1.83n$, coming from the family of paths and cycles [11, Claim 1]. For trees, the lower bound is even weaker. A folklore bound $\mathcal{U}(\mathcal{T}_n) \geqslant 3\lfloor n/2 \rfloor$ can be obtained by considering a family of two trees that can only share $\lceil n/2 \rceil$ vertices: a star (a depth-1 tree) and a path, each with $n$ vertices. These both lower bounds are ad-hoc techniques based on the impossibility of coexisting a certain number of "conflicting graphs" in a single graph that is too small.

*On the difficulty of proving lower bounds.* It is perhaps worth mentioning that it can be very difficult to prove strict lower bounds on the size of universal objects for specific combinatorial properties. For instance, consider *universal planar point sets*, i.e., a fixed set of points such that each $n$-vertex planar graph has a straight-line drawing on it. The best upper bound is $2n^2$ (an $2n \times n$-grid), and the current best lower bound has been improved only recently to $1.29n$ for $n$ large enough [32], a large gap. Interestingly, [32] showed that for $n = 11$, no universal planar point sets of exactly $n$ points exist. In other words, the lower bound is at least $n+1$ for $n = 11$. Proving an $(n+1)$-lower bound, even for small $n$, is not so easy. Technically, they have considered (computer-assisted) a family of 49 "conflicting" graphs, each with $n = 11$ vertices, which cannot be simultaneously drawn on any set of $n$ points. They show that more than 36 conflicting graphs are required. In other words, for $n = 11$, in order to beat the trivial $n$-lower bound any conflicting family has size at least 37. It is only conjectured that the $n$-lower bound can be beaten for every $n \geqslant 11$. There are even conjectures made about the size of any conflicting family w.r.t the trivial $n$-lower bound [7, (3), p. 129].

For induced-universal graphs for trees, it is not clear if a lower bound as low as say $2.1n$ can be achieved. As previously said, the folklore $3\lfloor n/2 \rfloor$-lower bound for trees is achieved by considering a conflicting family of two caterpillars[2] on $n$ vertices: a star and a path. Naively, one may believe that by considering a conflicting family of three trees (instead of two caterpillars) may allow us to easily beat this folklore $3n/2$-lower bound. Unfortunately, one of our results implies that $\mathcal{U}(\mathcal{F}) \leqslant 3n/2 + O(\log n)$ for any family $\mathcal{F}$ composed of three trees with $n$ vertices. So, at least 4 trees need to be considered in any conflicting family w.r.t. the $3n/2$-lower bound. Even worse, we show that in order to beat $2.1n$, we will need to consider at least 7 trees. Clearly, a lower bound proof involving 7 trees is far more complicated than one involving 2 trees, as pairwise intersections of the trees in an induced-universal graph solution may be hard to control.

*Our contributions.* First, we use some classical Information-Theoretic lower bound to improve the current lower bounds based on conflicting families. This concerns trees, planar and some other families (see Table 1 and Table 3 columns $c$). We also develop another lower bound for specific conflicting families containing small cliques (see Table 2). The bound is actually optimal for unions of cliques of bounded size. This captures the folklore $3n/2$-lower bound for trees (cliques of size at most two), and the $11n/6$-lower bound for paths and cycles (cliques of size at most

[1] It is not given explicitly in [3]. Their involved construction is based on a solution for caterpillars that contains at least $12 \cdot 2^5 n = 384n$ vertices. Extending the solution for trees requires to at least double the caterpillar solution. An estimate of $10^5$ or $10^6$ for $c$ seems to be closer to reality.

[2] Recall that a caterpillar is a tree in which the nodes of degree at least 2 induced a path. They are also exactly the maximal graphs of pathwidth 1.

three). As intermediate step, it gives a simple $25n/12$-lower bound for planar graphs by considering cliques of size at most four. Using our Information-Theoretic lower bound, the bound for planar graphs is improved to $10.52n$.

Second, we use block designs and special vertex-coloring to construct small induced-universal graphs for any family $\mathcal{F}$ with $t$ graphs of $n$ vertices. The result depends on two more integral parameters on $\mathcal{F}$: $p$ and $k$. More precisely, we show in Theorem 3 that, if every graph of $\mathcal{F}$ has a set of $p$ vertices whose removal leaves a graph that is *equitably $k$-colorable* (a $k$-coloring with same color-class sizes), then

$$\mathcal{U}(\mathcal{F}) \leqslant s \cdot \frac{n}{k} + tp\,. \tag{1}$$

The number $s = s(t,k)$ is related to some block designs and the Coding-Theoretic Function as defined in Section 3.1. This is the smallest integer $s$ such that $\{1,\dots,s\}$ contains at least $t$ subsets of size $k$ that pairwise intersect in at most one element. Actually, we show in Theorem 4 that parameter $p$ is at most the pathwidth of any graph in $\mathcal{F}$ times the chromatic number $k$ of $\mathcal{F}$.

For instance, for any family $\mathcal{F}$ composed of $t = 3$ trees with $n$ vertices, Eq. (1) implies $\mathcal{U}(\mathcal{F}) \leqslant 3n/2 + O(\log n)$, because $k = 2$, $p = O(\log n)$ (trees are bipartite of logarithmic pathwidth), and because one can check that $s = 3$ in this case. For $\mathcal{F}$ composed of $t = 6$ trees, we have $s = 4$, and thus $\mathcal{U}(\mathcal{F}) \leqslant 2n + O(\log n)$. If $\mathcal{F}$ is composed of $t = 20$ planar graphs, $\mathcal{U}(\mathcal{F}) \leqslant 4n + O(\sqrt{n})$, because in that case $k = 4$, $p = O(\sqrt{n})$, and $s = 16$. The fact that $s(20,4) = 16$ requires non-trivial block designs. From above, at least 7 conflicting trees would be required in order to beat a hypothetical $2.1n$-lower bound for trees, as $\mathcal{U}(\mathcal{F}) \leqslant 2n + O(\log n)$ for $t = 6$ trees. Actually, we have shown a lower bound of $1.626n$ for tree families, and $\mathcal{U}(\mathcal{F}) \leqslant 3n/2 + O(\log n)$ for $t = 3$ trees. Thus, at least 4 conflicting trees are required in order to beat our $1.626n$-lower bound (cf. the first line of Table 1). For planar graph families, $k = 4$, $p = O(\sqrt{n})$, and $s = 42$ if $t = 136$. So, at least 137 conflicting graphs are required in order to beat our $10.52n$-lower bound, since for 136 graphs Eq. (1) provides an upper bound of only $42 \cdot n/4 + O(\sqrt{n}) = 10.5n + O(\sqrt{n})$.

This framework applies to several families of graphs, and the series of new lower bounds we obtained, on $\mathcal{U}$ and conflicting family size, are summarized in columns $c$ and $t$ of Table 1.

| $\mathcal{F}$ | $c$ | $t$ | $k$ | $A(\lceil ck\rceil - 1, 2k-2, k)$ |
|---|---|---|---|---|
| forests | 1.626 | 4 | 2 | $A(3,2,2) = 3$ |
| outer-planar | 3.275 | 13 | 3 | $A(9,4,3) = 12$ |
| series-parallel | 3.850 | 17 | 3 | $A(11,4,3) = 16$ |
| $K_5^-$-minor-free[3] | 6.264 | 51 | 4 | $A(25,6,4) = 50$ |
| planar | 10.520 | 137 | 4 | $A(42,6,4) = 136$ |
| $K_{3,3}$-minor-free | 10.521 | 124 | 5 | $A(52,8,5) \in \{123,124\}$[4] |

Table 1. New lower bounds given by Theorem 2 for various family $\mathcal{F}$ of the form $\mathcal{U}(\mathcal{F}_n) \geqslant cn - o(n)$. Here $t$ is a lower bound on the size of a conflicting family needed to beat the corresponding $cn$-lower bound. From Theorem 5, we have $t = 1 + A(\lceil ck\rceil - 1, 2k-2, k)$, where $k$ is the chromatic number of $\mathcal{F}$, and $A$, related to block designs, is the Coding-Theoretic Function defined in Section 3.1.

Motivated by the quest for an $\Omega(n \log n)$-lower bound for planar graphs, we show that a conflicting family w.r.t. to this lower bound (if it exists) must have a size of at least $\Omega(\log^2 n)$. More generally, we show in Theorem 6 that any family $\mathcal{F}$ of $t$ graphs having a small chromatic number and sublinear pathwidth, like any proper minor-closed family as planar or bounded genus graphs, verifies $\mathcal{U}(\mathcal{F}_n) < {}^{15}/_{7}\sqrt{t} \cdot n$.

*Organization of the paper.* Section 2 presents the union-of-clique lower bound, and the Information-Theoretic lower bound. Section 3 gives our construction, as in Eq. (1), using block designs, and its applications to conflicting families. Section 4 presents our upper bound in $O(\sqrt{t} \cdot n)$.

[3] $K_5^-$ is the graph $K_5$ minus one edge.
[4] According to Andries E. Brouwer's web pages, it is not known whether $A(52,8,5) = 123$ or 124.

Due to space limitation, proofs and all technical lemmas are omitted. There are available in the full paper [13].

## 2. Lower Bounds

### 2.1. Union of cliques

The first lower bound is based on the union of small cliques. The bounds presented in Theorem 1 are optimal and generalize previously known lower bounds: the folklore lower bound of $\lfloor 3n/2 \rfloor$ for acyclic graphs of maximum degree two, and the lower bound of $11\lfloor n/6 \rfloor$ for graphs of maximum degree at most two [11, Claim 1]. We remark for this latter that our lower bounds are slightly better. It actually matches ours only for $n$ multiple of six. E.g., if $n = 6i + 5$ for integer $i$, $11\lfloor n/6 \rfloor = 11i$ whereas our bound gives $11i + 8$.

**Theorem 1.** *Let $\mathcal{K}_{n,k}$ be the family of graphs with at most $n$ vertices composed of a disjoint union of cliques each with at most $k$ vertices. Then, $\mathcal{U}(\mathcal{K}_{n,k}) = \sum_{i=1}^{k} \lfloor n/i \rfloor \geqslant (n+1)\ln(k+1) - k$.*

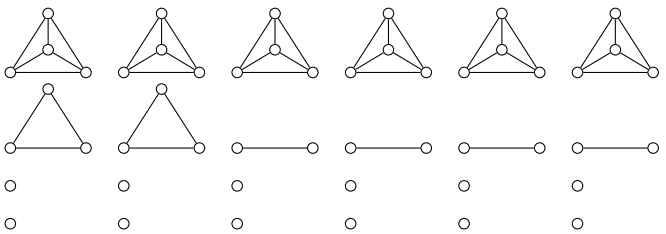

Fig. 1. The minimum induced-universal graph for $\mathcal{K}_{24,4}$, the family of graphs with at most 24 vertices that are union of disjoint cliques with at most 4 vertices. Its has $\mathcal{U}(\mathcal{K}_{24,4}) = 24 \cdot (1 + 1/2 + 1/3 + 1/4) = 50$ vertices.

Note that graphs in $\mathcal{K}_{n,2}$ are forests (it is actually a matching), graphs in $\mathcal{K}_{n,3}$ have maximum degree at most two and thus are outer-planar, and $\mathcal{K}_{n,4}$ are planar. Thus, for these families, for infinitely many $n$ (for $n$ divisible by $k!$), we have respectively $3n/2$, $11n/6$, and $25n/12$ as lower bounds on the number of vertices for their minimum induced-universal graphs (see Table 2). However, for forests, outer-planar and planar graphs, the next lower bounds, as presented in Theorem 2, will be better (see also Table 3).

| $\mathcal{F}$ | lower bounds on $\mathcal{U}(\mathcal{F}_n)$ |
|---|---|
| caterpillar forests $\supset \mathcal{K}_{n,2}$ | $3n/2$ |
| union of cycles and paths $\supset \mathcal{K}_{n,3}$ | $11n/6$ |
| planar $\supset \mathcal{K}_{n,4}$ | $25n/12$ |
| union of cliques with $\leqslant k$ vertices $= \mathcal{K}_{n,k}$ | $(n+1)\ln(k+1) - k$ |
| union of cliques $= \mathcal{K}_{n,n}$ | $(n+1)\ln(n+1) - n$ |

Table 2. Summary of lower bounds derived from Theorem 1.

### 2.2. Information-Theoretic lower bound

Given an infinite graph family $\mathcal{F}$, the *unlabeled constant-growth* of $\mathcal{F}$ is the following number (if it exists),

$$g(\mathcal{F}) \;=\; \limsup_{n\to\infty} (u_n)^{1/n}$$

where $u_n$ counts the number of unlabeled graphs of $\mathcal{F}$ having $n$ vertices.

**Theorem 2.** *Let $\mathcal{F}$ be a infinite graph family with unlabeled constant-growth at least*[5] $g > e/2$. *For every* $0 < \varepsilon < g/e - 1/2$, *and for infinitely many* $n$, $\mathcal{U}(\mathcal{F}_n) > (c-\varepsilon)n$, *where* $c$ *is solution of* $c^c/(c-1)^{c-1} = g$. *We have* $c \geqslant g/e + 1/2$. *Moreover, if* $\lim_{n\to\infty}(u_n)^{1/n}$ *does exist, then* $\mathcal{U}(\mathcal{F}_n) \geqslant cn$ *for every large enough* $n$.

At first glance, the condition "for infinitely many $n$" in the lower bound of Theorem 2 is not satisfactory, as we may expected stronger condition like "for every $n$, $\mathcal{U}(\mathcal{F}_n) \geqslant (c-\varepsilon)n$" or "$\mathcal{U}(\mathcal{F}_n) \geqslant cn - o(n)$". However, we observe that there are graph families where such a stronger condition cannot be true. This is the case of cubic planar graphs since there are no cubic graphs with an odd number of vertices. Nevertheless, the lower bound $\mathcal{U}(\mathcal{F}_n) \geqslant cn - o(n)$ holds for all the families considered in Table 3, as $\lim_{n\to\infty}(u_n)^{1/n}$ exists for all of them.

Now, using known results on counting graphs for various families, and combined with Theorem 2, we get the following set of new lower bounds (cf. Table 3):

| $\mathcal{F}$ | $g$ | refs | $c$ |
|---|---|---|---|
| caterpillar forests[6] | 2 | [18] | 1.293 |
| forests | 2.9557 | [30] | 1.626 |
| outer-planar | 7.5036 | [5] | 3.275 |
| series-parallel | 9.0733* | [6] | 3.850 |
| $K_5^-$-minor-free | 15.65* | [16] | 6.264 |
| planar | 27.2268* | [16] | 10.520 |
| bounded genus | 27.2268* | [26] | 10.520 |
| $K_{3,3}$-minor-free | 27.2293* | [15] | 10.521 |

Table 3. Lower bounds for various family $\mathcal{F}$ of the form $\mathcal{U}(\mathcal{F}_n) \geqslant cn - o(n)$, where $c$ is the solution of $g = c^c/(c-1)^{c-1}$ and where $g$ is a lower bound on the unlabeled constant-growth of $\mathcal{F}$. All figures in the numbers are correct (not rounded). Starred numbers correspond to the *labeled* constant-growth of $\mathcal{F}$ defined as $\limsup_{n\to\infty}(\ell_n/n!)^{1/n}$, where $\ell_n$ counts the number of labeled graphs of $\mathcal{F}$ having $n$ vertices, a lower bound on the unlabeled constant-growth of $\mathcal{F}$.

The labeled constant-growth of other families can be founded in [16, 27, 29]. An intriguing question, we left open, is to know whether there are hereditary families $\mathcal{F}$ with unlabeled constant-growth $g$ for which the lower bound given by Theorem 2 is tight.

## 3. Almost Equitable Coloring

The main idea in our construction of small induced-universal graph in Theorem 3 is to reuse large stable sets found in each graph of the family. For this purpose, one need these stable sets have similar size.

Recall that a $k$-coloring is *equitable* if the number of vertices in two color classes differ by at most one. In other words, if $G$ has an equitable $k$-coloring, then $V(G)$ has a partition into stable sets $C_1, \ldots, C_k$ such that $|C_i| \in \{\lceil n/k \rceil, \lfloor n/k \rfloor\}$ for each $i$, where $n = |V(G)|$. Note that $|C_i| = 0$ for some $i$ implies $|C_j| \in \{0, 1\}$ for all $j$.

It is well-known that any $n$-vertex graph with maximum degree $\Delta$ has an equitable $(\Delta+1)$-coloring, as proved by Hajnal and Szemerédi (see [21] for a shorter proof), that can be computed in time $O(\Delta n^2)$ [23]. It is conjectured that in fact every connected graph is equitably $\Delta$-colorable [28], if it is neither a clique nor an odd cycle. This has been confirmed for many graph classes. In particular, this is true if $\Delta \leqslant 4$ [22].

We point out that there are graphs with equitable $k$-coloring but no equitable $(k+1)$-coloring, e.g. a 2-coloring of $K_{3,3}$. There are also bipartite graphs with no equitable 2-coloring, for instance $K_{p,q}$ with $q > p+1$. Even worst for $p = 1$, $K_{1,q}$ has an equitable $k$-coloring if and only if $k \geqslant 1 + q/2$ (cf. [28]). However, by removing $p$ vertices, we can obtain a graph that has an equitable 2-coloring, and even so an equitable 1-coloring.

This leads to the following extension of equitable coloring that will be the crux for our main result. See Fig. 2 for an illustration.

[5] Here $e = \exp(1) = 2.7182818284....$

[6] Improved by Theorem 1.

**Definition 1.** *A graph $G$ is* $p$-almost equitably $k$-colorable *if there exists a set $X$ of at most $p$ vertices such that $G \setminus X$ has an equitable $k$-coloring.*

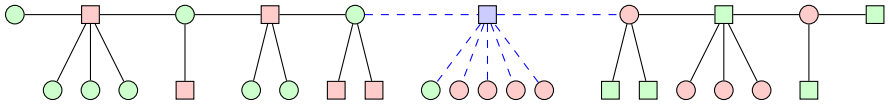

Fig. 2. A caterpillar of 29 vertices with a 2-coloring that is not equitable (18 circle vs. 11 squared vertices). In fact, it has no equitable 2-coloring, but it has a 1-almost equitable 2-coloring (14 green and 14 red vertices) obtained by removing the squared vertex in the middle of the path.

So, equitably $k$-colorable graphs, i.e., the graphs having an equitably $k$-coloring, are exactly those that are 0-almost equitably $k$-colorable.

### *3.1. Block design based construction*

Theorem 3 below relies on the well-known Coding-Theoretic Function $A(n, d, w)$ that is the maximum number of binary words of $n$ bits that are pairwise at distance[7] at least $d$ and of weight[8] $w$. For instance, with the bijection between binary words of length $n$ to subsets of $\{1, \ldots, n\}$, $A(n, 2, 2)$ can be seen as the number of distinct pairs of integers taken from $\{1, \ldots, n\}$, since to be at distance at least two, binary words of weight two can share at most one 1. Therefore, $A(n, 2, 2) = \binom{n}{2}$. More generally, for each integer $k \geqslant 2$, $A(n, 2k-2, k)$ is the maximum number of edge-disjoint copies of $K_k$ taken in $K_n$. Such clique packing is also called *block design* or *incomplete block design*, and is part of the Combinatorial Design Theory [9].

Not all the values of $A(n, d, w)$ are known, and we refer to [8, 9] for best known bounds on $A(n, d, w)$. However, by counting edges of the cliques, it is easy to see that $A(n, 2k-2, k) \leqslant \binom{n}{2}/\binom{k}{2}$. The equality holds if and only if a Steiner System $S(2, k, n)$ exists (for instance, see [8, Theorem 7]). We note that Reed-Solomon codes over a finite field of prime power order are an efficient way to achieve such a bound for suitable pairs $(n, k)$.

By combining block designs and almost equitable coloring, we can show:

**Theorem 3.** *Let $s, t, k \in \mathbb{N}$ such that $t \leqslant A(s, 2k-2, k)$. For every family $\mathcal{F}$ with $t$ graphs, each being $p$-almost equitably $k$-colorable with $n$ vertices,*

$$\mathcal{U}(\mathcal{F}) \leqslant s\left\lceil \frac{n-p}{k} \right\rceil + tp\,.$$

Note that the smallest number $s = s(t, k)$ such that $t \leqslant A(s, 2k-k, k)$ can be seen as the smallest integer $s$ such that $\{1, \ldots, s\}$ contains at least $t$ subsets of size $k$ that pairwise intersect in at most one element. Given the numbers $A(s, 2k-2, k)$ and their corresponding clique packing, the induced-universal graph $U_{\mathcal{F}}$ for $\mathcal{F}$ given by Theorem 3 has an explicit construction. Furthermore, all embedded graphs of $\mathcal{F}$ in $U_{\mathcal{F}}$ are pairwise edge disjoint, as only vertices are shared into some common stable sets. It follows that $U_{\mathcal{F}}$ is also an induced-universal graph for any edge-colored graph of $\mathcal{F}$. For an induced-universal graph of the family of all edge-colored cliques with $r$ colors, we refer to [24].

### *3.2. Conflicting family*

An application of the construction in Theorem 3 is that it can give a lower bound on the size of a conflicting family w.r.t. some $cn$-lower bound. For this purpose, we will use Theorem 4 making a link between pathwidth[9] and almost equitable coloring.

[7] I.e., the Hamming distance.
[8] I.e., the number of 1's in the binary word.
[9] Recall that a graph $G$ has pathwidth at most $p$ if it is a subgraph of some interval graphs with maximum clique size $p+1$.

**Theorem 4.** *Every $k$-colorable graph of pathwidth at most $p$ is $p(k-1)$-almost equitably $k$-colorable.*

**Theorem 5.** *Let $\mathcal{F}$ be a family of $k$-colorable graphs with $n$ vertices and pathwidth $o(n/k)$. If $\mathcal{U}(\mathcal{F}) \geqslant cn - o(n)$ for some constant $c$, then $\mathcal{F}$ contains more than $A(\lceil ck \rceil - 1, 2k-2, k)$ graphs, for every large enough $n$.*

To illustrate Theorem 5, consider any family $\mathcal{F}_n$ of bipartite graphs with sublinear pathwidth. From Theorem 5, if $\mathcal{U}(\mathcal{F}_n) \geqslant 2.1n - o(n)$, then $\mathcal{F}_n$ must contains more than $A(\lceil 2.1 \cdot 2 \rceil - 1, 2, 2) = A(4,2,2) = 6$ graphs. In particular, any conflicting family w.r.t. a $2.1n$-lower bound for trees must contains at least 7 trees.

Now, we can combine the lower bounds on $c$ collected from Table 3, with Theorem 5. By this way, we obtain in Table 4 the lower bounds on the size of conflicting families. Note that all families considered therein have bounded chromatic number and sublinear pathwidth.

| $\mathcal{F}$ | $c$ | $t$ | $k$ | $A(\lceil ck \rceil - 1, 2k-2, k)$ |
|---|---|---|---|---|
| forests | 1.626 | 4 | 2 | $A(3,2,2) = 3$ |
| outer-planar | 3.275 | 13 | 3 | $A(9,4,3) = 12$ |
| series-parallel | 3.850 | 17 | 3 | $A(11,4,3) = 16$ |
| $K_5^-$-minor-free[10] | 6.264 | 51 | 4 | $A(25,6,4) = 50$ |
| planar | 10.520 | 137 | 4 | $A(42,6,4) = 136$ |
| $K_{3,3}$-minor-free | 10.521 | 124 | 5 | $A(52,8,5) \in \{123, 124\}$[11] |

Table 4. Lower bounds on the size $t$ of a conflicting families w.r.t. a $cn$-lower bound.

## 4. Towards Super-Linear Lower Bounds

The motivation of this part is mostly linked to the search for lower bounds for families of $n$-vertex graphs for which one suspect a super-linear lower bound, say $\lambda(n) \cdot n$ where $\lambda(n)$ is non-constant. We show that, for many of those families, namely those with small chromatic number and sublinear pathwidth, the size of any conflicting family w.r.t. a $\lambda(n) \cdot n$-lower bound cannot be linear in $\lambda(n)$. As shown in the next theorem, it must be at least quadratic.

**Theorem 6.** *Let $\mathcal{F}$ be a family of $n$-vertex $k$-colorable graphs of pathwidth at most $p$ such that $p(k^2-1) \leqslant n$. If the number of graphs in $\mathcal{F}$ is $t \geqslant \max\{4k^2, 811\}$, then $\mathcal{U}(\mathcal{F}) < {}^{15}/_{7}\sqrt{t} \cdot n$.*

A consequence of Theorem 6 is that any $\Omega(n \log n)$-lower bound proof for planar graphs (if it exists), and more generally for fixed minor-closed family of graphs, must use a conflicting family of at least $t = \Omega(\log^2 n)$ witnesses $n$-vertex graphs. First, this is because graphs excluding a fixed minor have pathwidth $O(\sqrt{n})$ [2], and constant chromatic number, due to their low edge density [25]. The conditions are therefore fulfilled for the application of Theorem 6. Second, for such family $\mathcal{F}$ with $t = o(\log^2 n)$ graphs, Theorem 6 would imply that $\mathcal{U}(\mathcal{F}) = o(n \log n)$.

In order to prove Theorem 6 we need, as intermediate step, the following result:

**Theorem 7.** *Let $\mathcal{F}$ be a family of $n$-vertex $p$-almost equitably $k$-colorable graphs. If the number of graphs in $\mathcal{F}$ is $t \geqslant \max\{k^2, 811\}$, then $\mathcal{U}(\mathcal{F}) < {}^{15}/_{14}\sqrt{t} \cdot (n - p + k) + tp$.*

A consequence of Theorem 7, is that an upper bound on $\mathcal{U}(\mathcal{F})$ can be derived for any graph family $\mathcal{F}$, independently from its colorability. However, unlike Theorem 7, it relies to families having $t = \Omega(n^2)$ graphs with $n$ vertices.

**Corollary 1.** *Every family of $t \geqslant n^2 \geqslant 811$ graphs with $n$ vertices has an induced-universal graph with less than ${}^{15}/_{7}\sqrt{t} \cdot n$ vertices.*

[10] $K_5^-$ is the graph $K_5$ minus one edge.
[11] According to Andries E. Brouwer's web pages, it is not known whether $A(52,8,5) = 123$ or 124.

## References


[1] Alon, N., 2017. Asymptotically optimal induced universal graphs. Geometric and Functional Analysis 27, 1–32. doi:10.1007/s00039-017-0396-9.

[2] Alon, N., Seymour, P.D., Thomas, R., 1990. A separator theorem for graphs with an excluded minor and its applications, in: 22nd Annual ACM Symp. on Theory of Computing (STOC), ACM Press. pp. 293–299. doi:10.1145/100216.100254.

[3] Alstrup, S., Dahlgaard, S., Bæk Tejs Knudsen, M., 2017. Optimal induced universal graphs and adjacency labeling for trees. J. of the ACM 64, No. 27, pp. 1–22. doi:10.1145/3088513.

[4] Alstrup, S., Kaplan, H., Thorup, M., Zwick, U., 2019. Adjacency labeling schemes and induced-universal graphs. SIAM Journal on Discrete Mathematics 33, 116–137. doi:10.1137/16M1105967.

[5] Bodirsky, M., Fusy, É., Kang, M., Vigerske, S., 2007a. Enumeration and asymptotic properties of unlabeled outerplanar graphs. Electronic J. of Combinatorics 14, #R66.

[6] Bodirsky, M., Giménez, O., Kang, M., Noy, M., 2007b. Enumeration and limit laws for series-parallel graphs. European J. of Combinatorics 28, 2091–2105. doi:10.1016/j.ejc.2007.04.011.

[7] Brass, P., Cenek, E., Duncan, C.A., Efrat, A., Erten, C., Ismailescu, D.P., Kobourov, S.G., Lubiw, A., Mitchell, J.S., 2007. On simultaneous planar graph embeddings. Computational Geometry 36, 117–130. doi:10.1016/j.comgeo.2006.05.006.

[8] Brouwer, A.E., Shearer, J.B., Sloane, N.J.A., Smith, W.D., 1990. A new table of constant weight codes. IEEE Transactions on Information Theory 36, 1334–1380. doi:10.1109/18.59932.

[9] Colbrun, C.J., Dinitz, J.H., 2007. Handbook of Combinatorial Designs. Disc. Math. and its Applications, Chapman and Hall/CRC. doi:10.1201/9781420010541.

[10] Dujmović, V., Esperet, L., Gavoille, C., Joret, G., Micek, P., Morin, P., 2021. Adjacency labelling for planar graphs (and beyond). J. of the ACM 68, No. 42, pp. 1–33. doi:10.1145/3477542.

[11] Esperet, L., Labourel, A., Ochem, P., 2008. On induced-universal graphs for the class of bounded-degree graphs. Information Processing Letters 108, 255–260. doi:10.1016/j.ipl.2008.04.020.

[12] Gavoille, C., Hilaire, C., 2023. Minor-Universal Graph for Graphs on Surfaces. Technical Report 2305.06673v1 [cs.DM]. arXiv. doi:10.48550/arXiv.2305.06673.

[13] Gavoille, C., Jacques, A., 2025. Lower Bounds for Induced-Universal Graphs. Technical Report 2508.11585v1 [math.CO]. arXiv. URL: http://arxiv.org/abs/2508.11585.

[14] Gawrychowski, P., Janczewski, W., 2022. Simpler adjacency labeling for planar graphs with B-trees, in: 25th Symp. on Simplicity in Algorithms (SOSA), ACM-SIAM. pp. 24–36. doi:10.1137/1.9781611977066.3.

[15] Gerke, S., Giménez, O., Noy, M., Weißl, A., 2008. The number of graphs not containing $K_{3,3}$ as a minor. Electronic J. of Combinatorics 15. doi:10.37236/838.

[16] Giménez, O., Noy, M., 2009. Counting planar graphs and related families of graphs, in: Surveys in Combinatorics, Cambridge University Press. pp. 169–210. doi:10.1017/CBO9781107325975.008.

[17] Gorsky, M., Seweryn, M.T., Wiederrecht, S., 2025. Polynomial Bounds for the Graph Minor Structure Theorem. Technical Report 2504.02532v1 [math.CO]. arXiv. doi:10.48550/arXiv.2504.02532. to appear in FOCS '25.

[18] Harary, F., Schwenk, A.J., 1973. The number of caterpillars. Disc. Math. 6, 359–365. doi:10.1016/0012-365X(73)90067-8.

[19] Jia, L., Lin, G., Noubir, G., Rajaraman, R., Sundaram, R., 2005. Universal approximations for TSP, Steiner tree, and set cover, in: 37th Annual ACM Symp. on Theory of Computing (STOC), ACM Press. pp. 386–395. doi:10.1145/1060590.1060649.

[20] Kannan, S., Naor, M., Rudich, S., 1992. Implicit representation of graphs. SIAM J. on Disc. Math. 5, 596–603. doi:10.1137/0405049.

[21] Kierstead, H.A., Kostochka, A.V., 2008. A short proof of the Hajnal-Szemerédi theorem on on equitable colouring. Combinatorics, Probability and Computing 17, 265–270. doi:10.1017/S0963548307008619.

[22] Kierstead, H.A., Kostochka, A.V., 2012. Every 4-colorable graph with maximum degree 4 has an equitable 4-coloring. J. of Graph Theory 71, 31–48. doi:10.1002/jgt.20630.

[23] Kierstead, H.A., Kostochka, A.V., Mydlarz, M., Szemerédi, E., 2010. A fast algorithm for equitable coloring. Combinatorica 30, 217–224. doi:10.1007/s00493-010-2483-5.

[24] Kouekam, C., 2021. Induced universal graphs for edge-colored complete graphs. Master thesis. Karlsruhe Institute of Tech., Dept. of Informatics Institute of Theoretical Informatics. URL: http://i11www.iti.kit.edu/_media/teaching/theses/ma-kouekam-21.pdf.

[25] Mader, W., 1967. Homomorphieeigenschaften und mittlere Kantendichte von Graphen. Math. Ann. 174, 265–268. doi:10.1007/BF01364272.

[26] McDiarmid, C.J., 2008. Random graphs on surfaces. J. of Combinatorial Theory, Series B 98, 778–797. doi:10.1016/j.jctb.2007.11.006.

[27] McDiarmid, C.J., Kurauskas, V., 2012. Random graphs containing few disjoint excluded minors. Random Structures and Algorithms 44, 240–268. doi:10.1002/rsa.20447.

[28] Meyer, W., 1973. Equitable coloring. The American Mathematical Monthly 80, 920–922. doi:10.1080/00029890.1973.11993408.

[29] Noy, M., Requilé, C., Rué, J., 2020. Further results on random cubic planar graphs. Random Structures and Algorithms 56, 892–924. doi:10.1002/rsa.20893.

[30] Otter, R., 1948. The number of trees. Ann. of Math. 49, 583–599. doi:10.2307/1969046.

[31] Robertson, N., Seymour, P.D., Thomas, R., 1994. Quickly excluding a planar graph. J. of Combinatorial Theory, Series B 62, 323–348. doi:10.1006/jctb.1994.1073.

[32] Scheucher, M., Schrezenmaier, H., Steiner, R., 2020. A note on universal point sets for planar graphs. J. of Graph Algorithms and Applications 24, 247–267. doi:10.7155/jgaa.00529.