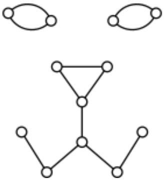
MINOR-UNIVERSAL GRAPH FOR GRAPHS ON SURFACE

Claire HILAIRE

with Cyril GAVOILLE

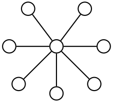


Graph minors

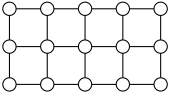
H is a minor of G if H can be obtained from G by

- taking a subgraph
- contracting edges

H



G



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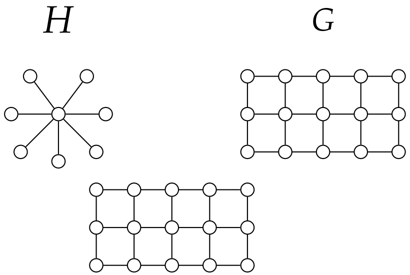

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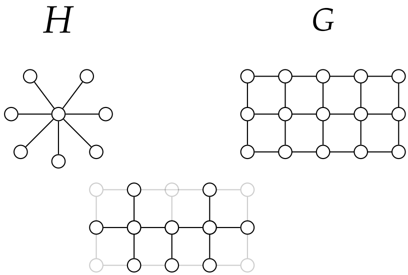

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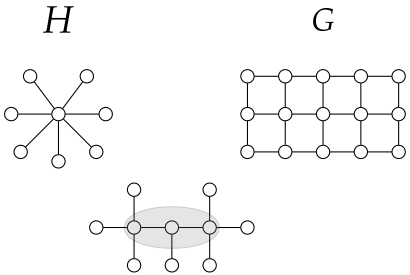

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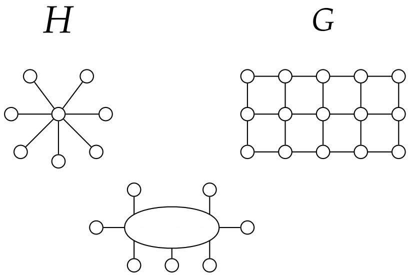

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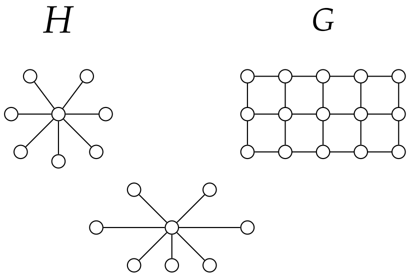
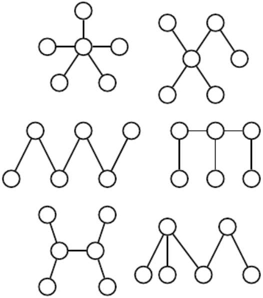


Minor-universal graph

Let $\mathcal{F}$ be a family of finite graphs.
U is **minor-universal** for $\mathcal{F}$ if for every $G \in \mathcal{F}$, G is a minor of U.



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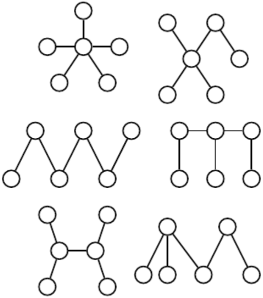

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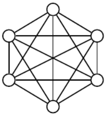

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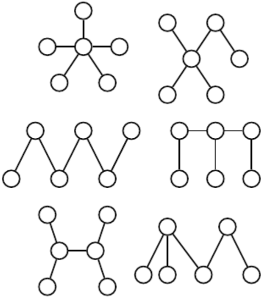

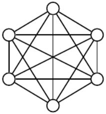

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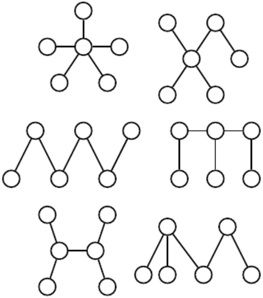

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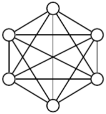
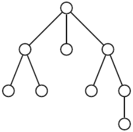
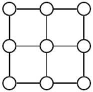


none



universal tree



universal grid

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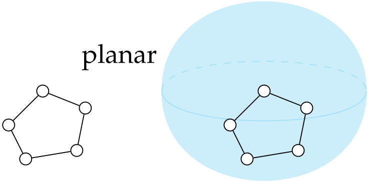
→ $O(n^2)$ with the $2n \times 2n$-grid [RST94]

Graphs on surfaces



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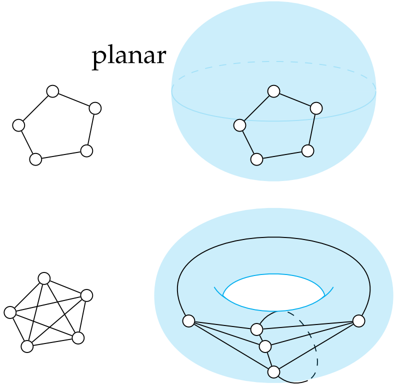

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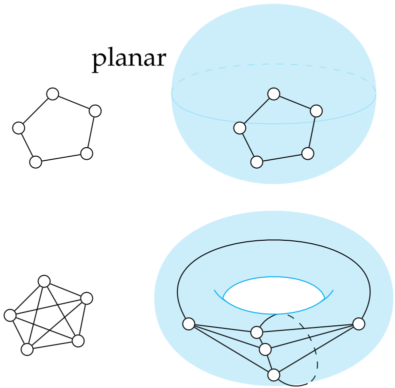
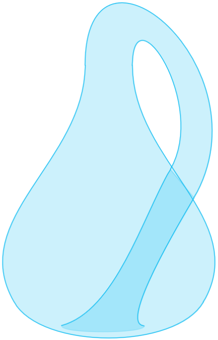


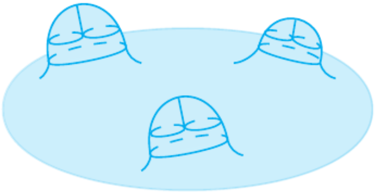


Classification of surfaces

Every connected surface without boundary is homeomorphic to either:

- An **orientable** surface of Euler genus $2g \geqslant 0$:

- A **non-orientable** surface of Euler genus $g \geqslant 0$:



here $g = 3$

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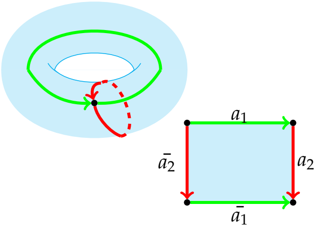
Gavoille and H. (2023+)

For every n and every surface Σ of Euler genus $g \geqslant 1$, there is a graph embedded on Σ with $O(g^2(n+g)^2)$ vertices minor-universal for the n-vertex graphs embeddable on Σ.

Polygonal schema for surfaces



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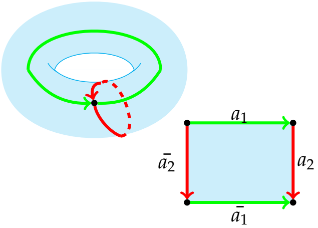

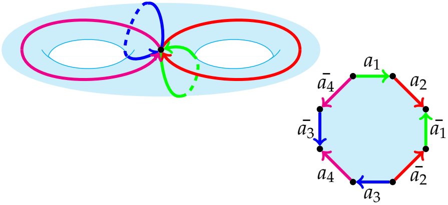

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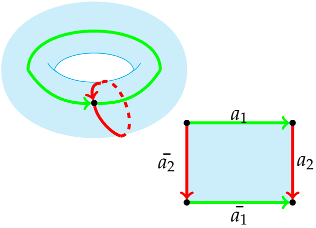
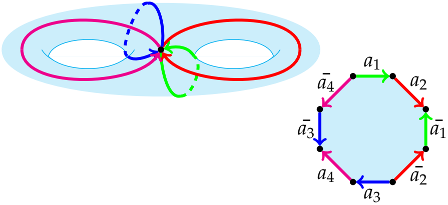


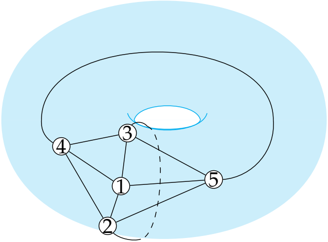


Classification Theorem

Every compact, connected surface of Euler genus $g \geqslant 1$ is homeomorphic to a polygonal surface given by one of the following **canonical signatures** σ:

- **Orientable:** $a_1 a_2 \bar{a}_1 \bar{a}_2 \ldots a_{g-1} a_g \bar{a}_{g-1} \bar{a}_g$
- **Non-orientable:** $a_1 a_1 \ldots a_g a_g$

Polygonal embedding for graphs



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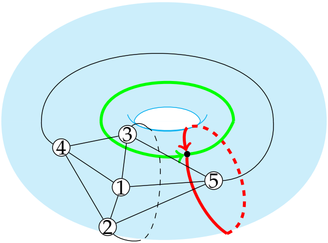

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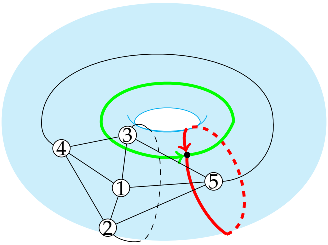

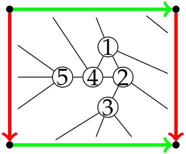

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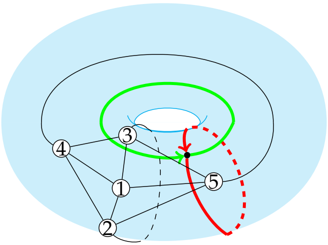

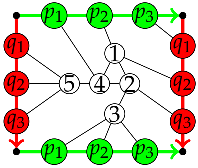

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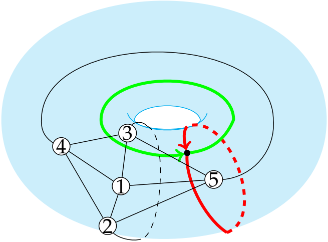

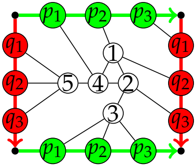

G has a **polygonal embedding** characterized by:

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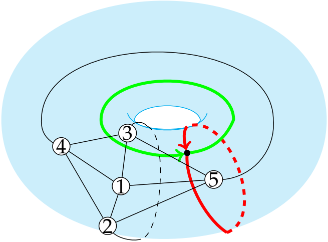

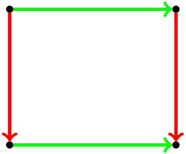

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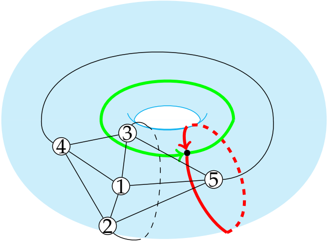

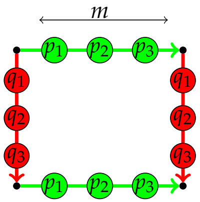

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- at most m external vertices on each side ($p_1, p_2, p_3, q_1, q_2, q_3$).

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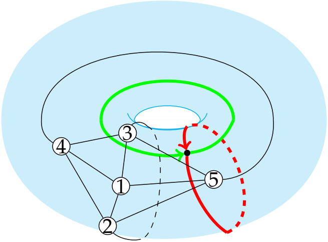

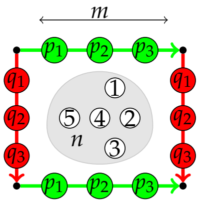

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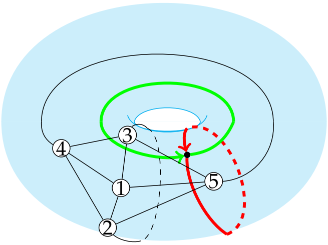

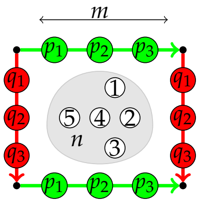

G has a **polygonal embedding** of type $P_\sigma(m, n)$:

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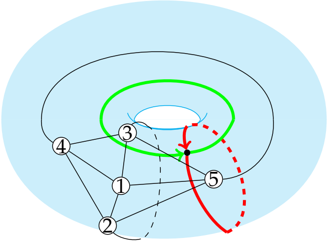

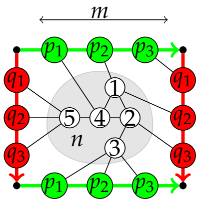

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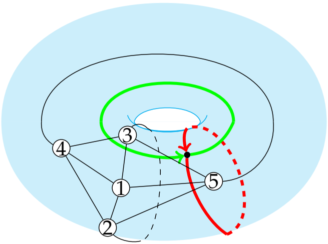

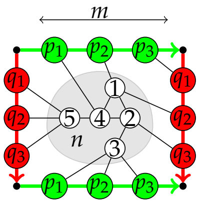

depends only on g and orientability

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$O(n+g)$ [LPVV01,FHdM22]

Sketch of the proof

Gavoille and H. (2023+)

For every n and every surface Σ of Euler genus $g \geqslant 1$, there is a graph embedded on Σ with $O(g^2(n+g)^2)$ vertices minor-universal for the n-vertex graphs embeddable on Σ.

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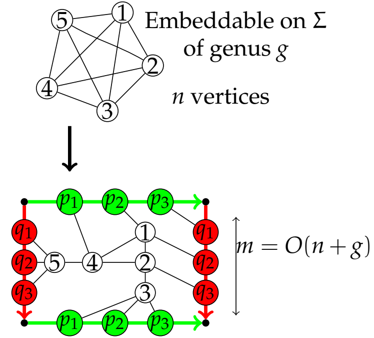

$P_\sigma(m,n)$ with $|\sigma| = 2g$

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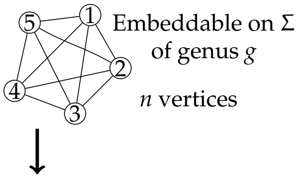
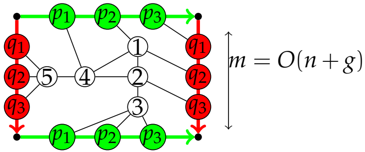
Technical theorem.

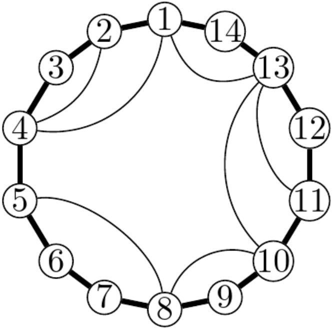
$\forall \sigma, m, n$, there is a graph with a polygonal embedding $P_\sigma(m+2n, |\sigma|^2(m+2n)^2)$, minor-universal for the graphs with a polygonal embedding $P_\sigma(m,n)$.

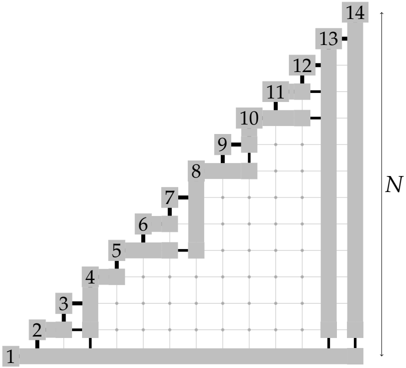




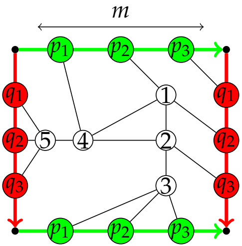
$P_\sigma(m,n)$ with $|\sigma| = 2g$

Planar graphs: main step





Step 1/2: getting an outerplanar-like embedding



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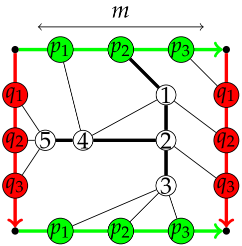

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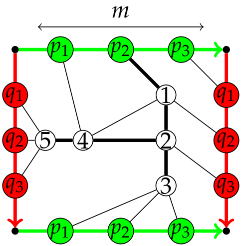
Step 1/2: getting an outerplanar-like embedding



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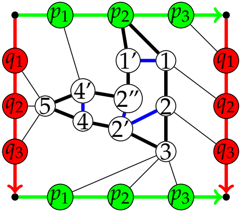

$P_\sigma(m, 2n-1)$

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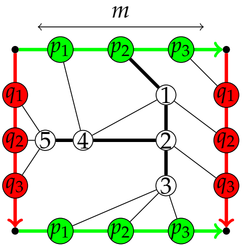

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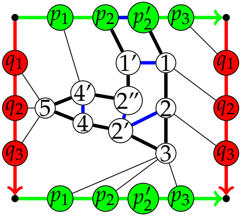

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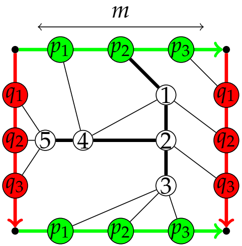

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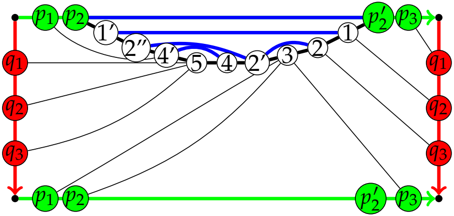

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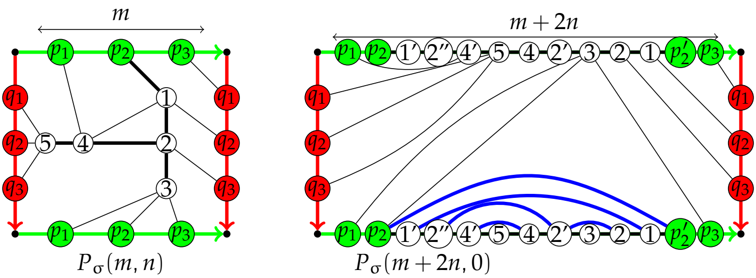
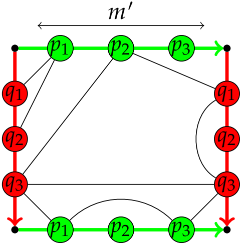
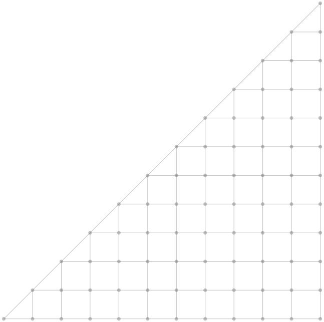


step 2/2: grid-like minor-universal graph





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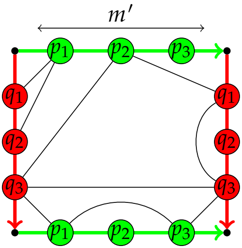

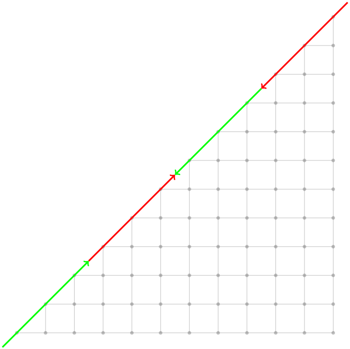

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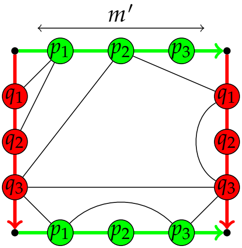

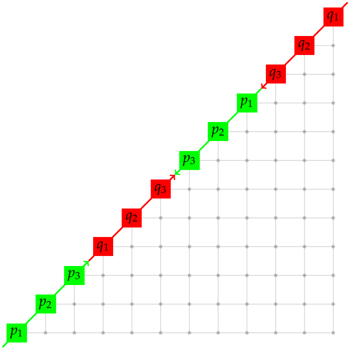

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Thank you!