

From fault tolerant spanners to multipath spanners

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Part I: Spanners

General background

$G = (V, E)$ some weighted graph, $\#V = n$, $\#E = m$.
Potentially $m = n^2$. Can we spare some edges while keeping:

- connectivity?
- some distance property?

Spanners

Introduced by Peleg and Schäffer in 89 for undirected graphs.

Let $d(u, v)$ be the distance: length of smallest path between u and v .

Definition

H covering subgraph of G is an (α, β) -spanner of G if for any $u, v \in V$,

$$d_H(u, v) \leq \alpha \cdot d_G(u, v) + \beta$$

(α, β) (or only α if $\beta = 0$) is called the **stretch**.

- A **multiplicative** spanner is a spanner where $\beta = 0$.
- An **additive** spanner is a spanner where α is as small as possible, possibly 1.

What counts for a spanner?

Three elements to determine the quality:

- Number of edges: less is better.
- The **stretch**: smaller is better.
- Computation time.

For some stretch, the less edges the better.

Why spanners?

Fundamental object related to:

- Compact routing
- Distance oracles
- Distributed distance guessing

One can view them as a possible extension of spanning trees.

Some spanners

- $(2k - 1, 0)$ -spanner with $O(n^{1+1/k})$ edges.
- $(1, 2)$ -spanner with $O(n^{3/2})$ edges.
- $(1, 6)$ -spanner with $O(n^{4/3})$ edges.
- $(k, k - 1)$ -spanner with $O(n^{1+1/k})$ edges.
- $(1 + \epsilon, 4)$ with $O(1/\epsilon \cdot n^{4/3})$ edges.

Probably optimal.

Part II: Multipath spanners

Goals for improvement

We would like to:

- Improve bandwidth
- Be tolerant to faulty vertices/links
- Minimize delays

Multipath spanners

The standard spanner definition is oblivious to the metric used.

Definition 1

$d^p(a, b)$ is the minimum cost of a subgraph containing p vertex-disjoint paths between a and b .

We plug d^p into the definition:

Definition 2

H is a p -multipath (α, β) -spanner of G if $\forall u, v :$

$$d_H^p(u, v) \leq \alpha \cdot d_G^p(u, v) + \beta$$

Question

Can we construct such things ?

Bad news

We have bad news :

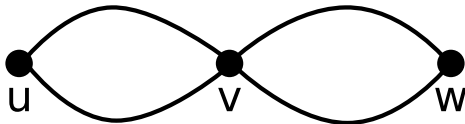


Figure: $d^2(u, v)$ & $d^2(v, w)$ finite, yet $d^2(u, w) = \infty$

And most of spanner algorithms make use of triangular inequality.

Fault-tolerant spanners

Close cousin built for vertex fault tolerance

Definition

A f -fault-tolerant spanner is a spanner that can tolerate f faults and still be a spanner on the remaining graph.

Question

Is a $p - 1$ -fault tolerant s -spanner a p -multipath spanner ?

Answer

In weighted graphs : **NO** if it is not a spanner of the same graph with all edges weights equal to 1.

Example

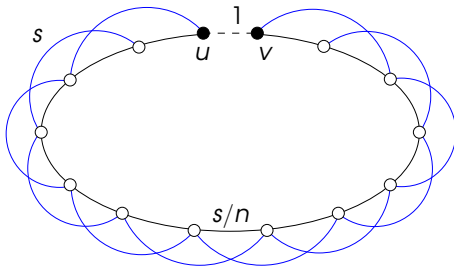


Figure: Cycle of n hops, weight s/n . Blue paths of weight s . Stretch unbound for d^2 but bounded for 1-fault.

Node disjoint spanners

Theorem

Theorem 1

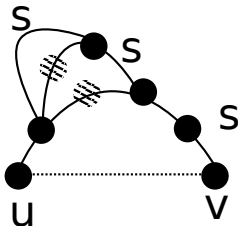
A s -hop $p - 1$ -fault tolerant spanner is a p -multipath $sp \cdot O(1 + p/S)^s$ -spanner.

Bonus

It is buildable in the *LOCAL* model in no additional time.

Idea

Do every possible fault on a path, and recursively.



That explains why we need bounded hop spanners.

Node disjoint spanners

Another theorem

Theorem 2

Every graph has a spanner of stretch $(2, O(1))$ and $O(n^{3/2})$ edges for the d^2 metric.

Idea

The proof is inspired by the $(2, 1)$ spanner algorithm: remove highly connected nodes, and replace them with an efficient structure.

Tools

Balls

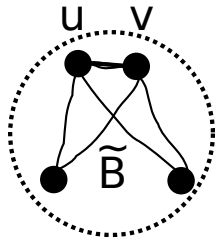
We define a new type of balls:

Definition

$\tilde{B}(uv, r)$: all x such that $d^2(uv, x) \leq r$.

Implication

$x, y \in \tilde{B}(uv, r) \Rightarrow d^2(x, y)$ bounded.



Algorithm

- 1 While $\exists \tilde{B}(uv, 4) \cap (N(u) \cup N(v)) > \sqrt{n}$ do:
 - 1 Add 2-tree $\cup \text{BFS}(u, 2) \cup \text{BFS}(v, 2)$
 - 2 $G = G \setminus B \cap N \dots$
- 2 $H = H \cup G$

Must check:

- Number of edges
- Stretch

Number of edges

Believe me

Useful result:

Lemma

A graph with every $\tilde{B}(uv, 2k) \cap N(u) \leq n^{1/k}$ has $O(n^{1+1/k})$ edges.

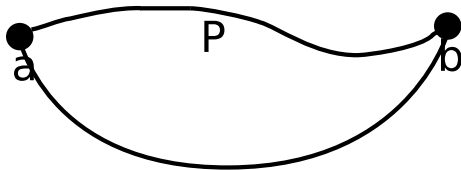
And:

- “While” executes itself n/\sqrt{n} times.
- “While” adds $O(n)$ edges at each step.
- Apply lemma to remaining graph G before step 4 and $k = 2$.

So H has $O(n^{3/2})$ edges.

Stretch

Let two nodes a, b , and a cycle P which realize $d^2(a, b)$.



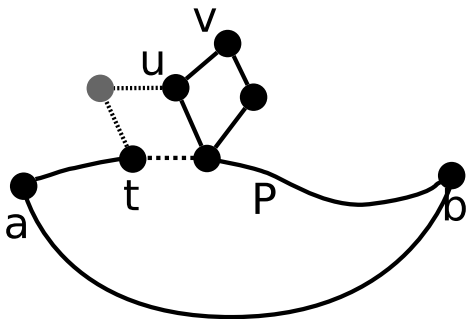
Two cases are possible:

- The path is in H : nothing to prove
- It is not: some edge is missing. Take the first one.

Stretch

Sketch

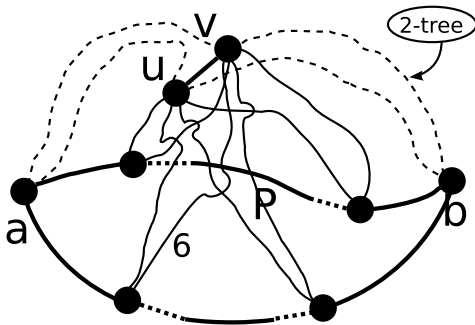
- One of the vertices t is in vicinity of the root edge of some $\tilde{B}(uv, 4)$
- Because of *BFS* it is covered by another node to u
- $d^2(t, u) \leq 6$



Stretch

Sketch (continued)

It goes the same for every nodes of edges missing. The closest from a and b are the important ones.



We use these cycles to bound $d^2(a, uv)$ and $d^2(b, uv)$ in G , and so in H because of the 2-tree.

We use menger theorem to finally bound $d^2(a, b)$ in H .

Conclusion

Summary

- We showed that fault-tolerant spanners existed, and can be built in a local setting.
- We showed how to construct a $(2, O(1))$ -bipath-spanner.

What next ?

- factor 2 mandatory for 2-paths ?
- Other special constructions ?
- Improve the stretch of the general result ?