

On the Treewidth of Planar Minor Free Graphs

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Outline

- 1 Introduction
- 2 Preliminary definitions
- 3 Results
- 4 Proof

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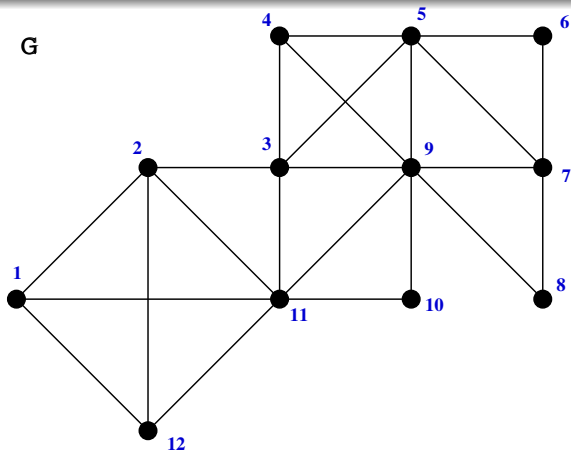
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What is a Tree-Decomposition?

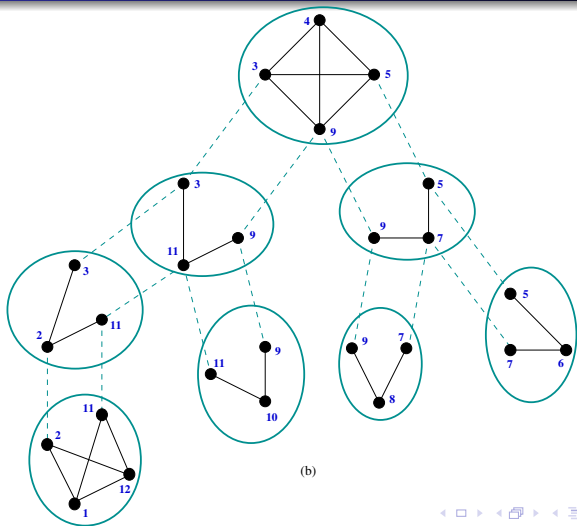
A *tree-decomposition* of a graph G is a tree \mathcal{T} whose nodes, called *bags*, are subsets of $V(G)$, and such that:

- 1 $\bigcup_{X \in V(\mathcal{T})} X = V(G)$;
- 2 $\forall \{u, v\} \in E(G), \exists X \in V(\mathcal{T})$ such that $u, v \in X$; and
- 3 $\forall u \in V(G)$, the set of bags containing u induces a subtree of \mathcal{T} .

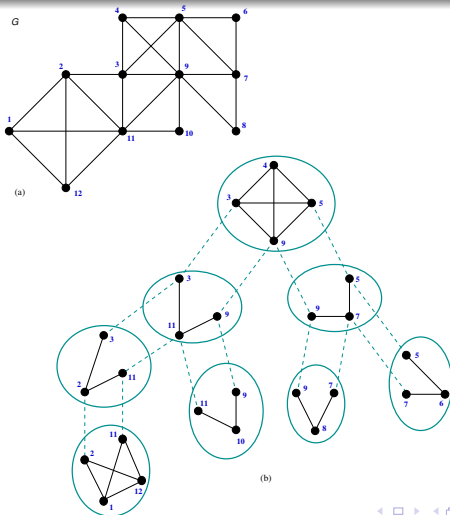
Example



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Tree-Width for a graph G

- 1 The **Width** of a tree-decomposition \mathcal{T} is $\max_{X \in \mathcal{T}} |X| - 1$.
- 2 The **Tree-Width** of G is the minimum of the **Width** over all tree-decompositions of G .

Some Results

[Arnborg *et al.* - 1987] The problem of determining whether the tree-width of a graph is k is NP-complete.

[But there are linear time algorithms for each fixed k .]

[Feige *et al.* - 2008] The best polynomial time approximation algorithm achieves $O(\sqrt{\log k})$ performance ratio.

[For tree-width k graphs.]

[Kloks and Bodlaender - 1992] For planar graphs, approximation algorithms with performance ratio 1.5 do exist.

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Interest of studying tree-width

- One of the most general and effective technique for designing efficient graph algorithms.
- Many optimization problems on graphs, including NP-hard ones, can be solved for graphs of small treewidth.
 - See [Arnborg - 1985, Arnborg et all. - 1989, Bodlaender - 1996, Courcelle - 1990]
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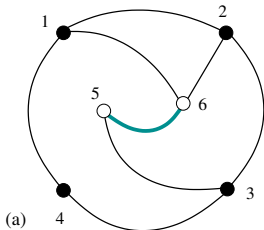
What is a minor?

- A **minor** of G is a subgraph of a graph obtained from G by edge contraction.
- Exemple : By contracting edge $\{ 5, 6 \}$ in G , then we have a new graph who is a minor of G .

A graph G **excludes** a graph H as minor (G is H -minor free) if H is not a minor of G .

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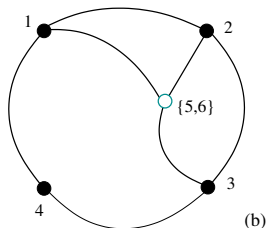
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A Drawing of a graph G

Maps each vertex to a point of the plane and each edge to a simple open Jordan curve between its endpoints.

- A $p \times q$ -grid-drawing of G is a drawing where vertices of the G have integer coordinates in the $p \times q$ grid.

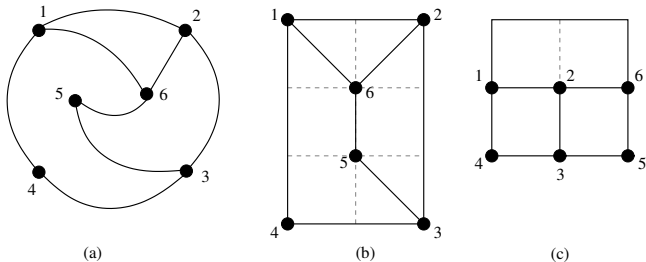


Figure: (a) A planar graph G ; (b) a 4×3 -grid **straight-line** drawing of G ; and (c) a **poly-line** orthogonal 3×3 -grid drawing of G .

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Theorem 1

The treewidth of every planar graph G excluding as minor a graph H having a poly-line $p \times q$ -grid drawing is $O(p\sqrt{q})$.

Corollary 1

Let $H \in \{\mathcal{C}_{2,r}, \mathcal{C}_{2,r}^, K_{2,r}\}$. The treewidth of every planar graph excluding H as minor is $O(\sqrt{r})$.*

Because $\mathcal{C}_{2,r}$ and its dual $\mathcal{C}_{2,r}^*$ have $4 \times r$ -grid drawings, and that $K_{2,r}$ is a minor of $\mathcal{C}_{2,r}^*$.

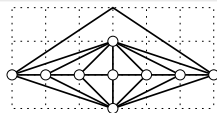
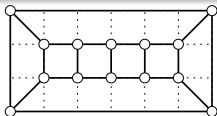
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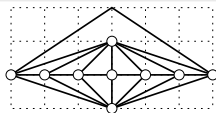
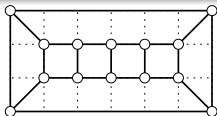
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By combining **Theorem 1** and [Biedl - 2014, Theorem 9], it can easily be demonstrated that :

- The treewidth of every planar graph excluding as minor an r -vertex outerplanar graph of pathwidth k is $O(k\sqrt{r})$.
- This bound can be slightly improved as follows:

Proposition 1

Every planar graph excluding as minor an r -vertex outerplanar graph of pathwidth k has treewidth at most $O(\sqrt{r \log r})$.

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Proof of Theorem 1

Recall the statement:

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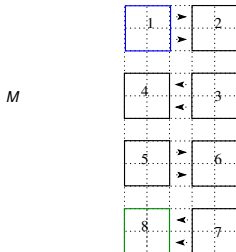
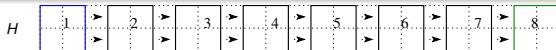
- 1 First, we give the structure of the Proof.
- 2 Second, we give the proof of one useful lemmas.

Structure of the Proof of Theorem 1

We start with a simple lemma:

Lemma 1

The $p \times q$ grid is a minor of the $\lfloor (2 + \sqrt{2})\sqrt{pq} \rfloor \times \lfloor (2 + \sqrt{2})\sqrt{pq} \rfloor$ grid.

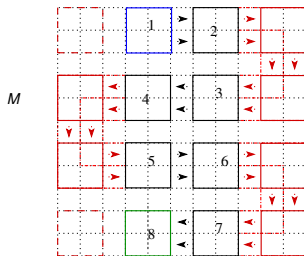
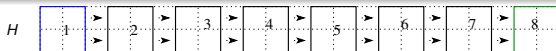


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The following lemma is useful :

Lemma 2

Every graph H having a poly-line $p \times q$ -grid drawing is minor of the $p \times (3pq)$ grid.

Thanks to Lemma 1, H is a minor of $H' = O(p\sqrt{q}) \times O(p\sqrt{q})$ grid.

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- Then by Lemma 3(i), the treewidth of G is $O(p\sqrt{q})$, which completes the proof of Theorem 1.

Lemma 3 (Gu et al 2012, Robertson et al. 1994)

Every planar graph excluding a planar graph \bar{H} as minor has treewidth at most:

- $9r/2 - 4$, if \bar{H} is the $r \times r$ grid with $r \geq 2$; and
- $9r - 22$, if \bar{H} has $r \geq 3$ vertices.

Conclusion

We establish a connection between the treewidth of a planar graph G excluding as minor a graph H and the ability of poly-line grid-drawing of H with small height.

One of the consequences of our main result is that :

- the treewidth of every planar excluding such graph H is $O(\sqrt{r})$ where $r = |V(H)|$, which is optimal.
- if H is outerplanar, then the bound slightly increases to $O(\sqrt{r \log r})$.
- We leave open
 - The question of the **optimality of this bound**.

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Figure: Cyril Gavaille



Figure: Youssou Dieng

Thank you!