



# On the Tree-Width of Planar Graphs

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## Abstract

We prove that every planar graph  $G$  of tree-length  $\ell$  has a tree-decomposition for which every bag is the union of at most 10 shortest paths of length  $O(\ell)$ . As a consequence, the tree-width of  $G$  is bounded by  $O(\ell)$ , generalizing the linear local tree-width result of planar graphs, since the tree-length of a graph does not exceed its diameter. Such a tree-decomposition can be computed in polynomial time without any prior decomposition of the input graph.

*Keywords:* tree-width, tree-length, planar graphs, tree-decomposition

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## 1 Introduction

*Tree-decomposition* is one of the most general and effective techniques for designing efficient algorithms. Many classes of graphs can be decomposed in a tree-like fashion, allowing us to solve difficult optimization problems [1,2,4,5]. The problem of determining whether the tree-width of a graph is  $k$  is NP-complete [3], but there are linear time algorithms for each fixed  $k$ . The best polynomial time approximation algorithm achieves  $O(\sqrt{\log k})$  performance

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ratio for tree-width  $k$  [7]. For planar graphs, approximation algorithms with performance ratio 1.5 do exist [8], and more generally with performance ratio  $f(H)$  for  $H$ -minor free graphs [7].

Tree-width relies to the size of a bag in a tree-decomposition. The notion of *tree-length*, related to the maximum distance of two vertices in a bag in a tree-decomposition, has been introduced in [6] for solving different problems.

In this paper, we give a tree-decomposition of planar graphs of tree-length  $\ell$  for which every bag is the union of at most 10 shortest paths of length  $O(\ell)$ . Such a decomposition can be constructed in polynomial time, without any prior decomposition, since the tree-length of any graph can be approximated within a constant factor [6]. It follows that the tree-width of a planar graph of tree-length  $\ell$  is  $O(\ell)$ . Note that this does not hold for many other families of graphs, for the complete graphs for instance. This refines the well-known fact that the tree-width of a planar graph of diameter  $D$  is  $O(D)$ , since the tree-length is no more than  $D$ .

Due to space constraint, proofs have been removed.

## 2 Preliminaries

A *tree-decomposition* of a graph  $G$  is a tree  $\mathcal{T}$  whose vertices, called *bags*, are subsets of vertices of  $G$  such that:

- (i) for every vertex  $u$  of  $G$ , there exists a bag  $X$  such that  $u \in X$ ;
- (ii) for every edge  $\{u, v\}$  of  $G$ , there exists a bag  $X$  such that  $u, v \in X$ ;
- (iii) for every vertex  $u$  of  $G$ , the set of bags containing  $u$  induces a sub-tree of  $\mathcal{T}$ .

The *width* of a tree-decomposition  $\mathcal{T}$  is  $\max_{X \in \mathcal{T}} |X| - 1$ , and the *length* of  $\mathcal{T}$  is  $\max_{X \in \mathcal{T}} \max_{u, v \in X} d_G(u, v)$ , where  $d_G(u, v)$  denotes the length of a shortest path between  $u$  and  $v$  in  $G$ . The *tree-width* and the *tree-length* of  $G$  are respectively the minimum of the width and of the length over all tree-decompositions of  $G$ .

A tree-decomposition is *reduced* if there is no two bags  $X$  and  $Y$  in the decomposition such that  $X \subseteq Y$ . It is easy to reduce a tree-decomposition by contracting successively the edges  $\{X, Y\}$  of the tree such that  $X \subseteq Y$ .

A *private part* of a bag  $X$  in  $\mathcal{T}$  is a non-empty subset  $S \subseteq X$  such that  $S \cap Y = \emptyset$  for each bag  $Y \neq X$  of  $\mathcal{T}$ . Note that a reduced tree-decomposition contains necessarily a bag with a private part, a degree-1 bag for instance.

A subtree  $T$  rooted at  $r$  in  $G$  is *k-monotone* if for every  $v \in T$ , the path in  $T$  from  $v$  to  $r$  is composed of at most  $k$  shortest paths in  $G$ .

For technical reasons, we shall use specific tree-decompositions where each bag has a private part and with a particular spanning tree (so in particular induced a connected subgraph). More precisely:

**Definition 2.1** A *planted tree-decomposition* of a graph  $G$  is a pair  $(\mathcal{T}, \mathcal{S})$ , where  $\mathcal{T}$  is a tree-decomposition of  $G$ , and  $\mathcal{S} = \{T_X\}_{X \in \mathcal{T}}$  is a family trees of  $G$  such that, for every bag  $X \in \mathcal{T}$ ,  $T_X$  is rooted in a private part of  $X$  and  $V(T_X) = X$ . Moreover, if every tree of  $\mathcal{S}$  is  $k$ -monotone and of depth  $\leq h$ , we say that  $(\mathcal{T}, \mathcal{S})$  is  $k$ -monotone of depth  $h$ .

It is clear that every connected graph has a planted tree-decomposition. One can choose a unique bag  $X = V(G)$  with a shortest path spanning tree. It is not difficult to see that there are graphs (e.g., cycles) for which any planted tree decomposition has width arbitrarily larger than the tree-width of the graph. So forcing connectiveness of bags do not preserve width in general up to a constant factor. However, this is true for the length.

**Theorem 2.2** *Given a tree-decomposition of length  $\ell$  for a graph  $G$ , one can compute in polynomial time a planted tree-decomposition for  $G$  that is 2-monotone of depth  $3\ell/2$ .*

### 3 Main Result

In this section we prove our main result, namely, that the tree-width of a planar graph  $G$  of tree-length  $\ell$  is  $O(\ell)$ .

**Definition 3.1 (diamond)** Let  $(\mathcal{T}, \{T_X\}_X)$  be a planted tree-decomposition of a graph  $G$ , let  $\{X, Y\}$  be an edge of  $\mathcal{T}$ , and let  $x, y$  be the roots of  $T_X$  and  $T_Y$  respectively. We denote by  $T_{XY}$  the minimal sub-tree of  $T_X$  rooted at  $x$  and spanning  $X \cap Y$ . A *diamond* w.r.t.  $(X, Y)$  is the subgraph composed of  $T_{XY} \cup T_{YX}$ .

Let us sketch our decomposition scheme.

By Theorem 2.2,  $G$  has a planted tree-decomposition  $\mathcal{T}$  of depth  $O(\ell)$ . From  $\mathcal{T}$  we construct a tree-decomposition where each bag is either a diamond between two adjacent bags, or a subset of a private part of a bag, called *half-diamond*. Then, this tree-decomposition is successively refined by splitting each diamond (or half-diamond) into pieces. We show that this is doable until getting a structure composed of at most 5 paths of length  $\leq \ell$  and 5 paths of length  $\leq \ell/2$ . This leads to the following result:

**Theorem 3.2** *Every planar graph having a tree-decomposition of length  $\ell$  has a tree-decomposition, computable in polynomial time, for which every bag is the union of at most 10 shortest paths in the graph of length  $O(\ell)$ .*

**Corollary 3.3** *The tree-width of a planar graph of tree-length  $\ell$  is  $O(\ell)$ .*

## 4 Conclusion

We leave open the question of generalizing the result to other families of graphs, in particular higher genus graphs. More precisely, prove or disprove that every bounded genus graph of tree-length  $\ell$  has a tree-decomposition in which every bag is the union of a finite number of shortest paths of length  $O(\ell)$ .

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