



# On the Path Separability of Planar Graphs

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## Abstract

It is known that every weighted planar graph with  $n$  vertices contains three shortest paths whose removal halves the graph into connected components of at most  $n/2$  vertices. Whether this property remains true with the use of two shortest paths only is an open problem.

We show that two shortest paths are enough for a large family of planar graphs, called face-separable, composed of graphs for which every induced subgraph can be halved by removing the border of a face in some suitable embedding of the subgraph. We also show that this non-trivial family of graphs includes unbounded treewidth graphs.

*Keywords:* treewidth, planar graphs, path-separable

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## 1 Motivation and Background

To solve a problem we often use data decompositions to simplify. The problem is solved on the different parts. Then, we merge the different results to obtain the desired result on the whole data. In Graph Theory, we try to divide graphs

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into connected components by the use of suitable separators, e.g., of small size. In this study we are interested to separators composed of shortest paths in the graph, and that halves its number of vertices. This leads to the notion of  $k$ -path separability, introduced in [1].

**Definition 1.1** A weighted graph  $G$  with  $n$  vertices is  $k$ -path separable if there exists a subgraph  $S$ , called  $k$ -path separator, such that:

- $S = P_0 \cup P_1 \cup \dots$ , where each subgraph  $P_i$  is the union of  $k_i$  minimum cost paths in  $G \setminus \bigcup_{j < i} P_j$  where  $\sum_i k_i \leq k$ ; and
- every connected component of  $G \setminus S$  (if any) is  $k$ -path separable and has at most  $n/2$  vertices.

It has been shown in [1,4] that  $k$ -path separators can be efficiently used to solve many “Object Location Problems” including small-worldization, distance and reachability oracles, compact routing.

Interestingly, treewidth- $k$  graphs are  $\lceil (k+1)/2 \rceil$ -path separable. However, the class of bounded path separability graphs is strictly larger than the one of bounded treewidth, since planar graphs, that have unbounded treewidth, are 3-path separable. This comes from the well-known fact that every planar graph having a depth- $h$  rooted tree has a tree-decomposition where each bag consists of 3 paths of the tree starting from to the root (see [2][pp. 305] and [4] for instance), and so is of treewidth at most  $3h$ . More generally,  $H$ -minor free graphs are  $k$ -path separable for some constant  $k = f(H)$  [1]. The proof is based on the deep Robertson-Seymour’s decomposition [3].

The 2-path separability of planar graphs is an open question. In this paper we show that a large class of planar graphs, called face-separable, is indeed 2-path separable. We also show that the family of face-separable graphs contains unbounded treewidth graphs.

## 2 Face-Separable Graphs

A subset  $S$  of vertices of a graph  $G$  is a *face-separator*, if each connected component of  $G \setminus S$  has at most  $|V(G)|/2$  vertices, and if there exists a plane embedding of  $G$  such that  $S$  is the border of a face.

**Definition 2.1** A graph is *face-separable* if every induced subgraph has a face-separator.

By definition, outerplanar graphs are face-separable, since the outerface contains all vertices of the graph, thus is a face-separator. We will see that

the family of face-separable graphs includes less specific graphs, like the subdivisions of any outerplanar graph (Proposition 2.3), the subdivisions of  $K_4$ , and even includes some unbounded treewidth planar graphs (Theorem 2.4).

A  $k$ -path separator  $S$  is *strong* if  $S$  consists of one set of shortest paths, i.e.,  $S = P_0$  in Definition 1.1. The graph is then called *strongly  $k$ -path separable*. Obviously, every strongly  $k$ -path separable graph is  $k$ -path separable. From the above discussion, planar graphs are strongly 3-path separable. However, as shown in [1], there are  $K_6$ -minor free graphs with  $n$  vertices that are 2-path separable, but without any strong  $k$ -path separator with  $k < \frac{1}{3}\sqrt{n}$ .

**Theorem 2.2** *Every face-separable weighted graph is strongly 2-path separable.*

Roughly speaking, we root a shortest-path spanning tree at a vertex of the face-separator. Then, we show that there must exist two vertices of the face-separator whose paths in the tree leading to the root form a strong 2-path separator  $S$ . Note that these two paths are not necessarily part of the face-separator. The connected components of  $G \setminus S$ , where  $G$  is the input graph, are also face-separable (by definition), so we can recurse on them, and show that  $G$  is strongly 2-path separable.

**Proposition 2.3** *Every subdivision of any outerplanar graph is face-separable.*

This can be proved by showing that every subdivision  $G$  of any outerplanar graph has a tree-decomposition where each bag consists of the vertices of the border of a face. And, it is well-known that any tree-decomposition has a bag whose removal leaves in  $G$  only connected components of at most  $|V(G)|/2$  vertices.

Subdivisions of outerplanar graph have treewidth at most two. Nevertheless, the family of face-separable graphs is a much wider. It contains unbounded treewidth graphs, as shown in Theorem 2.4.

**Theorem 2.4** *For every  $n$ , there is a face-separable graph with at most  $n$  vertices whose treewidth is  $\Omega(\log \log n)$ .*

The proof is based on the construction of a graph called  $G_p$ , for integral  $p \geq 1$ . It has treewidth  $k = p - O(\log \log p)$  because we can show it contains a minor  $k \times k$  grid, and the number of vertices of  $G_p$  is  $n < 2^{2^p}$ . In other words, the treewidth of  $G_p$  is at least  $\log \log n - O(\log^{(4)} n)$ .

Graph  $G_p$  is composed of a spanning tree  $T_p$  of depth  $p$  where each vertex of depth  $i < p$  has exactly  $d(i)$  children, for some function  $d$  defined later. Furthermore, for each depth  $i$ , a path linking all depth- $i$  vertices is added

to  $T_p$  to form  $G_p$ . Let us denote by  $L(i)$  the number of depth- $i$  vertices in  $T_p$ . The values  $L(i)$  and  $d(i)$  obey to the following induction:  $L(0) = 1$  and  $L(i) = L(i - 1) \cdot d(i - 1)$ , where  $d(i) = \sum_{j=0}^i L(j)$ . The first values of  $L(i)$  and  $d(i)$  are given in the table hereafter, and  $G_4$  is depicted on Fig. 1.

$i$	0	1	2	3	4	5	...
$L$	1	1	2	8	96	10368	...
$d$	1	2	4	12	108	10464	...

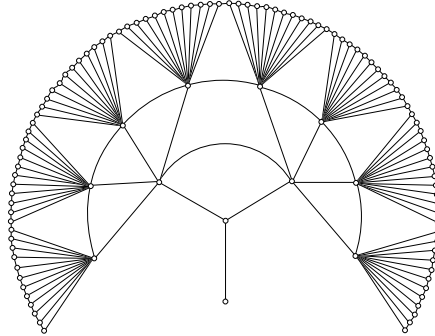


Fig. 1. The graph  $G_4$  with 108 vertices.

To prove Theorem 2.4, we show that every subgraph  $H$  of  $G_p$  contains a face-separator. One property we use is that in  $G_p$ , the number  $d(i)$  of children for a vertex of depth  $i$  is at least the vertex number of the graph induced by  $T_{i-1}$ . The key point is that  $H$  is either outerplanar, or there must exist a vertex  $v$  of depth  $i$  in  $T_p$  such that all its children belongs to the border of the outerface of  $H$ . In the first case,  $H$  is trivially face-separable. In the second one, using the property on  $d(i)$ , we derive that at least half the vertices of  $H$  lie on the outerface.

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