

# CONNECTIVITY CHECK IN 3-CONNECTED PLANAR GRAPHS WITH OBSTACLES

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# FORBIDDEN-SET ROUTING

- [Goal] Route informations from point  $u$  to point  $v$  and each node must act locally.
- We have to be aware of failures that can happen at any time.
- [Question] Is  $u$  and  $v$  connected in the network with failures ?  
How to check the connectivity of  $u$  and  $v$  locally ?
- [Solution] Assign each node a label that must be computed once.
- But we cannot represent the whole graph in each node (space constraints).
- [Optimistic] We want “short” labels (say of (poly)logarithmic size) in each node.

## 1 PRELIMINARIES

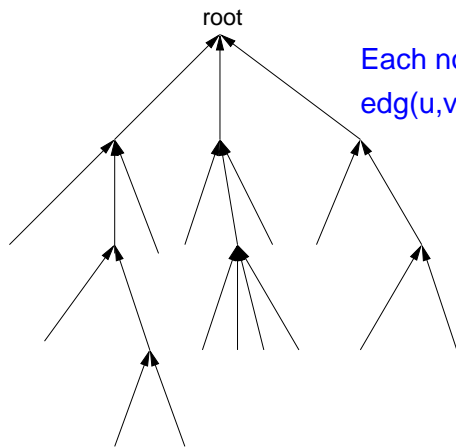
## 2 Plane Graphs

## 3 Representation by Unary Functions

# LABELING SCHEME

- Let  $P(x_1, \dots, x_p, Y_1, \dots, Y_q)$  be a graph property.
- An  $f(n)$ -labeling scheme for property  $P$  in a class  $\mathcal{C}$  of  $n$ -vertex graphs is a pair of algorithms  $(\mathcal{A}, \mathcal{B})$  such that:
  - ▶ For all  $G \in \mathcal{C}$ ,  $\mathcal{A}$  constructs a labeling  $J: V(G) \rightarrow \{0, 1\}^*$  such that  $|J(x)| \leq f(n)$  for each  $x \in V(G)$ .
  - ▶  $\mathcal{B}$  checks whether  $G$  satisfies  $P(a_1, \dots, a_p, U_1, \dots, U_q)$  by using  $J(a_1), \dots, J(a_p), J(U_1), \dots, J(U_q)$  where  $J(U) = \{J(x) \mid x \in U\}$ .

- A  $2 \log(n)$ -labeling scheme for adjacency in the class of forests.



Each node  $u$  store its father  $p(u)$   
 $\text{edg}(u,v)$  iff  $u=p(v)$  or  $v = p(u)$

- The construction can be extended to planar graphs (union of 3 forests).

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- We let  $d(x, y, X)$  denote the distance between  $x$  and  $y$  in the graph  $G \setminus X$  (subgraph of  $G$  induced by  $V(G) - X$ ).
- Courcelle and Twigg have proved that  $d(x, y, X)$  admits an  $O(\log^2(n))$ -labeling scheme in the class of graphs of bounded clique-width (STACS'07).
- Any property expressible in monadic second order logic admits an  $O(\log(n))$ -labeling scheme in the class of graphs of bounded clique-width (Courcelle and Vanicat 2003).
- Any property expressible in first order logic with set arguments admits an  $O(\log(n))$ -labeling scheme in the class of graphs that are locally tree-decomposable (Courcelle et al. 2008).



# CONNECTIVITY QUERY

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- For  $u, v \in V(G)$ ,  $X \subseteq V(G) - \{u, v\}$  we let  $Conn(u, v, X)$  mean:  
 *$u$  and  $v$  are in the same connected component in  $G \setminus X$ .*

## MAIN THEOREM

There exists an  $O(\log(n))$ -labeling scheme for  $Conn(u, v, X)$  in the class of 3-connected planar graphs.

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## REMARK

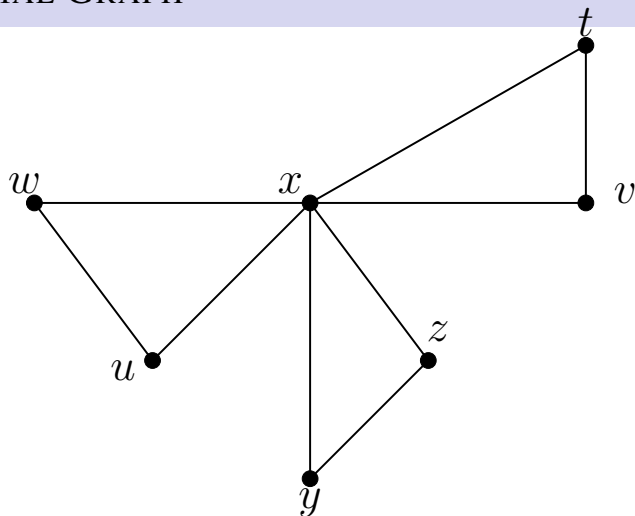
We have extended the main theorem to **all planar graphs**, but we will only give the proof for 3-connected planar graphs.

1 Preliminaries

2 **PLANE GRAPHS**

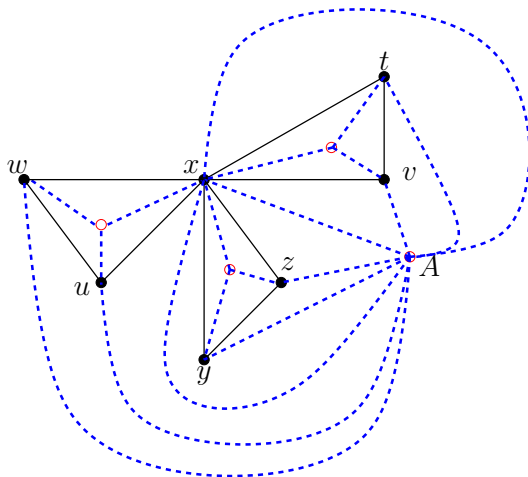
3 Representation by Unary Functions

# RADIAL GRAPH



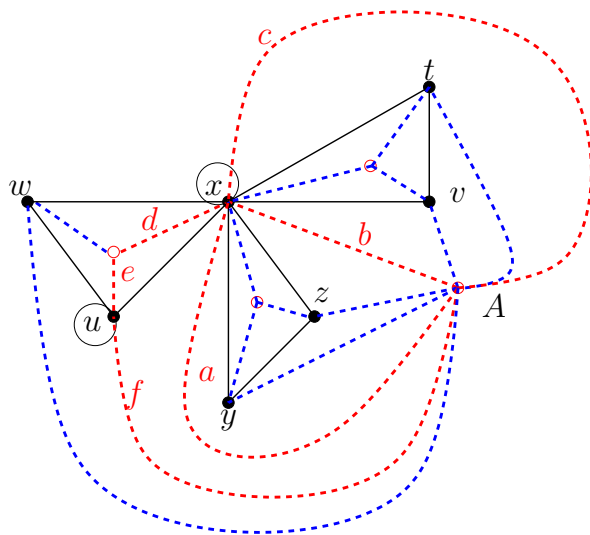
Plane Graph  $G = (V, E, F)$

# RADIAL GRAPH



$$G^+ = (V \cup F, \underbrace{E \cup \{\{f, x\} \mid x \in V, f \in F \text{ and } x \text{ incident with } f\}}_{E'})$$

# BARRIER WITH RESPECT TO EMBEDDING



$X = \{x, u\}$ ,  $Bar(X, \mathcal{E}^+) = \text{Red Lines}$

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Let  $\mathcal{E}^+ = (p, s)$  be an embedding of  $G^+$ . We let

$$\text{Bar}(X, \mathcal{E}^+) = \{s(e) \mid e \in \text{Bar}(X)\}$$

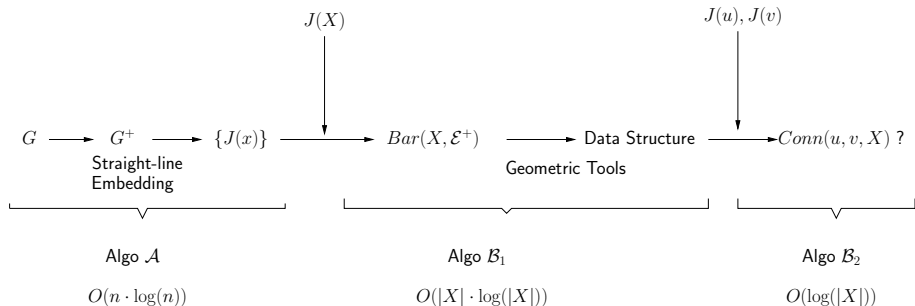
where  $\text{Bar}(X)$  is the set

$$\left\{ \{f, x\} \in E' \mid x \in X \text{ and there exists } y \in X \text{ such that } \{f, y\} \in E' \right\}.$$

## REMARK

In the sequel, We let  $\mathcal{E}^+$  be a straight-line embedding in order to use geometric tools.

# OVERVIEW OF THE ALGORITHMS



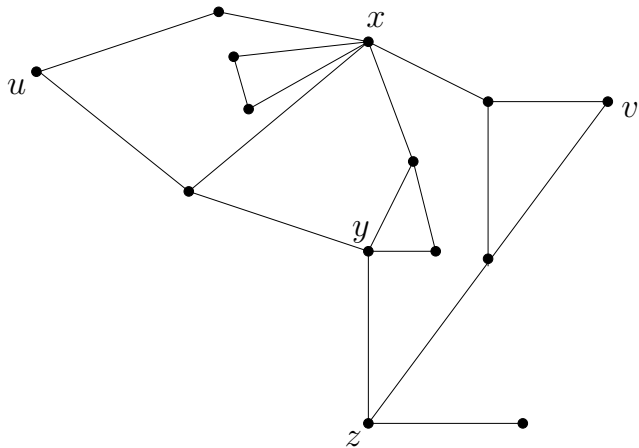


# PROPOSITION 1

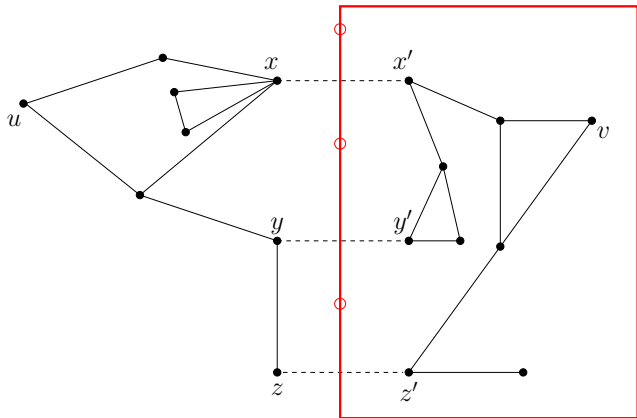
## PROPOSITION 1

Let  $\mathcal{E}^+ = (p, s)$  be an embedding of  $G^+$ . For every  $X \subseteq V$  and  $u, v \in V - X$ ,  $u$  and  $v$  are separated by  $X$  if and only if  $p(u)$  and  $p(v)$  are separated by  $\text{Bar}(X, \mathcal{E}^+)$ .

# PROOF SKETCH

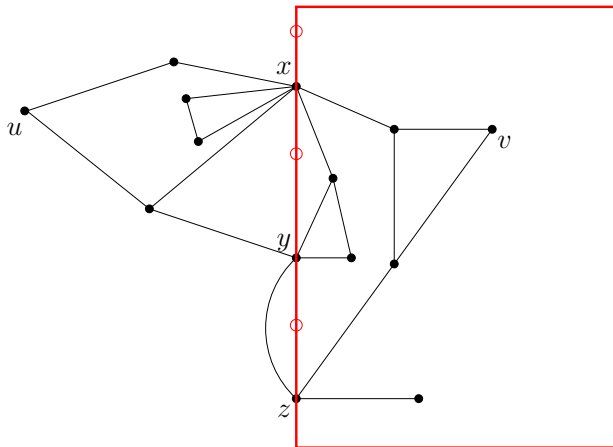


# PROOF SKETCH



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Red lines =  $\text{Bar}(X, \mathcal{E}^+)$ ,  $X = \{x, y, z\}$  and separates  $u$  and  $v$



# WHY LABELS OF VERTICES IN $X$

- Each connectivity request is composed of two vertices  $x, y \in V(G)$  and of a set of vertices  $X \subseteq V(G)$  to be removed.
- By **Proposition 1**,  $x$  and  $y$  are separated by  $X$  iff  $p(x)$  and  $p(y)$  are separated by  $Bar(X, \mathcal{E}^+)$ .
- We cannot store in the labels of each vertex all the possible barriers.
- To have a chance to get an  $O(\log(n))$ -labeling scheme, we will specify  $Bar(X, \mathcal{E}^+)$  in the labels of vertices in  $X$ .

1 Preliminaries

2 Plane Graphs

3 REPRESENTATION BY UNARY FUNCTIONS

# REPRESENTATION OF ADJACENCY BY UNARY FUNCTIONS

## LEMMA 1 (FOLKLORE)

There exists a  $4 \log(n)$ -labeling scheme for adjacency in planar graphs.

## PROOF.

- Edge set  $E(G)$  can be partitioned into 3 forests  $E_1, E_2$  and  $E_3$ .
- For each forest  $E_i$  there exists a  $2 \log(n)$ -labeling scheme for adjacency (each vertex  $x$  has label  $(x, g_i(x))$ ) **Forests**.
- for each  $x$  we let  $J(x) = (x, g_1(x), g_2(x), g_3(x))$  (of size  $4 \log(n)$ ).
- Then  $x$  and  $y$  are adjacent in  $G$  iff

$$x \neq y \wedge \left( \bigvee_{1 \leq i \leq 3} x = g_i(y) \vee y = g_i(x) \right).$$

# SAME FACE PROPERTY

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Let  $G = (V, E, F)$  be a plane graph. Two vertices  $x$  and  $y$  **satisfy the same face property** if there exists at least one face incident with  $x$  and  $y$ .



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## PROPOSITION 2

There exists a  $15 \log(n)$ -labeling scheme for same face property in connected plane graphs.

## PROOF OF PROPOSITION 2(1)

- We let  $g_i^+$  for  $i = 1, 2, 3$  be the 3 partial functions representing adjacency in  $G^+$ .
- $x$  and  $y$  respects the same-face property iff:

$$\bigvee_{1 \leq i, j \leq 3} g_i^+(x) = g_j^+(y) \in F \quad (1a)$$

$$\vee \bigvee_{1 \leq i, j \leq 3} g_i^+(x) \in F \wedge g_j^+(g_i^+(x)) = y \quad (1b)$$

$$\vee \bigvee_{1 \leq i, j \leq 3} g_i^+(y) \in F \wedge g_j^+(g_i^+(y)) = x \quad (1c)$$

$$\vee \exists f \in F. \left( \bigvee_{1 \leq i, j \leq 3} g_i^+(f) = x \wedge g_j^+(f) = y \right). \quad (1d)$$

## PROOF OF PROPOSITION 2(2)

CONDITION (1A)

$g_i^+(x) \in F$  is replaced by

$g'_i(x) = \text{if } g_i^+(x) \in F \text{ then } g_i^+(x) \text{ else undefined.}$

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### CONDITION (1B)-(1C)

$g_i^+(x) \in F \wedge g_j^+(g_i^+(x)) = y$  is replaced by:

$g'_{i,j} = \text{if } g_i^+(x) \in F \text{ and } g_j^+(g_i^+(x)) \text{ is defined then } g_j^+(g_i^+(x))$   
else undefined.

# PROOF OF PROPOSITION 2(3) $\Rightarrow$ (1D)

## CONDITION (1D)

$$\exists f \in F \left( \bigvee_{1 \leq i, j \leq 3} g_i^+(f) = x \wedge g_j^+(f) = y \right)$$

## PROOF OF PROPOSITION 2(3) $\Rightarrow$ (1D)

### CONDITION (1D)

$$\exists f \in F \left( \bigvee_{1 \leq i, j \leq 3} g_i^+(f) = x \wedge g_j^+(f) = y \right)$$

### ELIMINATION OF THE EXISTENTIAL QUANTIFICATION

- Let  $H = (V(G), E(H))$  where  $xy \in E(H)$  iff  $g_i^+(f) = x$  and  $g_j^+(f) = y$  for some  $i, j \in [3]$ .
- $H$  is planar because we can draw such an edge in a drawing of  $G$  by adding to each face of  $G$  at most 3 edges.
- Let  $h_1, h_2$  and  $h_3$  be the adjacency functions of  $H$  (Lemma 1).
- It is clear that (1d) can be replaced by

$$\bigvee_{1 \leq i \leq 3} h_i(x) = y \vee h_i(y) = x.$$

# LABELING EACH VERTEX

## LABEL OF $x$

- For each  $x$  we let

$$J(x) = \left( (g'_i(x))_{i \in [3]}, (h_i(x))_{i \in [3]}, (g'_{i,j}(x))_{i,j \in [3]} \right).$$

- It is clear that  $|J(x)| \leq 15 \log(n)$ .
- By [Equation \(1\)](#) we can answer adjacency and same face property between  $x$  and  $y$  just by looking at labels  $J(x)$  and  $J(y)$ .

# LABELING SCHEME FOR FACE SELECTION

## FACE SELECTION PROBLEM

Let  $G = (V, E, F)$  be a 3-connected plane graph. For every  $x, y \in V$  we let  $Faces(x, y)$  mean the **selection** of the at most **two faces incident with  $x$  and  $y$** . We call it the **face selection problem**.



# LABELING SCHEME FOR FACE SELECTION

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## PROPOSITION 3

There exists a  $27 \log(n)$ -labeling scheme for face selection problem in 3-connected plane graphs.

# PROOF OF PROPOSITION 3(1)

- We let  $g_i^+$  for  $i = 1, 2, 3$  be the 3 partial functions representing adjacency in  $G^+$ .
- $x$  and  $y$  respects the same-face property iff:

$$\bigvee_{1 \leq i, j \leq 3} g_i^+(x) = g_j^+(y) \in F \quad (1a)$$

$$\vee \bigvee_{1 \leq i, j \leq 3} g_i^+(x) \in F \wedge g_j^+(g_i^+(x)) = y \quad (1b)$$

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$$\vee \exists f \in F. \left( \bigvee_{1 \leq i, j \leq 3} g_i^+(f) = x \wedge g_j^+(f) = y \right) \quad (1d).$$

## PROOF OF PROPOSITION 3(2)

- 1 Either the two faces incident with  $x$  and  $y$  verify conditions (1a)-(1c).
- 2 Then we are done since we can select them by Proposition 2 with labels of size  $15 \log(n)$ .
- 3 Or at least one verifies condition (1d).
- 4 Hence either  $h_i(x) = y$  or  $h_i(y) = x$  (6 cases mutually exclusive).  
We let:

$h_{i,j}^+(x) = \text{Face}_j(x, h_i(x))$  if  $h_i(x)$  and  $\text{Face}_j(x, h_i(x))$  are defined,

$h_{i,j}^-(x) = \text{Face}_j(h_i(x), x)$  similarly.

## PROOF OF PROPOSITION 3(3)

- It is clear that a face  $f$  satisfy (1d) if and only if

$$\bigvee_{i,j} f = h_{i,j}^+(x) \wedge y = h_i(x) \vee \bigvee_{i,j} f = h_{i,j}^-(y) \wedge x = h_i(y).$$

- For each  $x$  we let  $J(x)$  be the label computed by Proposition 2. We let

$$L(x) = \left( J(x), h_{i,j}^+(x), h_{i,j}^-(x) \right).$$

- It is clear that  $L(x)$  and  $L(y)$  allow to select the at most two faces incident with  $x$  and  $y$ .

# PROOF OF THE MAIN THEOREM

- Let  $G = (V, E)$  be a 3-connected planar graph.
- Construct an embedding of  $G$  (**unique up to homeomorphism**).
- Let  $G^+$  be the radial graph obtained from  $G = (V, E, F)$ .
- Let  $\mathcal{E}^+ = (p, s)$  be a straight-line embedding of  $G^+$ .
- Since each component of the label in  $L(x)$ ,  $x \in V$ , is a vertex of  $G^+$  (a vertex of  $G$  or a face-vertex of  $G$ ), we can assume that they are represented by their coordinates in  $\mathcal{E}^+$ .
- For each  $x$  we let:

$$D(x) = (p(x), L(x))$$

- Given  $D(u)$  and  $D(v)$ , recover  $p(u)$  and  $p(v)$ .
- By **Proposition 3**, if  $x$  and  $y$  in  $X$  respects the same face property we can recover the at most two faces which are adjacent to them by using  $L(x)$  and  $L(y)$ .
- Since  $\mathcal{E}^+$  is a straight-line embedding, we can thus construct  $Bar(X, \mathcal{E}^+)$ .
- By **Proposition 1**,  $u$  and  $v$  are separated by  $X$  iff  $p(u)$  and  $p(v)$  are separated by  $Bar(X, \mathcal{E}^+)$ .
- How to decide if  $p(u)$  and  $p(v)$  are separated by  $Bar(X, \mathcal{E}^+)$  ?  
**By using geometric tools.**

# POINT LOCATION PROBLEM

What we want is:

*Given  $Bar(X, \mathcal{E}^+)$ , a set of straight-line segments, say if  $p(u)$  and  $p(v)$  are separated by  $Bar(X, \mathcal{E}^+)$ .*

# POINT LOCATION PROBLEM

What we want is:

*Given  $\text{Bar}(X, \varepsilon^+)$ , a set of straight-line segments, say if  $p(u)$  and  $p(v)$  are separated by  $\text{Bar}(X, \varepsilon^+)$ .*

## THEOREM 2 (TARJAN ET AL.)

Let  $Y$  be a set of  $m$  straight-line segments. One can construct in time  $O(m \log(m))$  a data structure of size  $O(m)$  and given any points  $u$  and  $v$  in the plane test in  $O(\log(m))$  if  $u$  and  $v$  are separated by  $\cup Y$ .



# CONCLUSION

- 1 We can extend the construction to deletions of edges by subdividing each edge of  $G$  and by letting for each edge  $e = uv$   $D(e) = (D(u), D(v))$ .
- 2 **Proposition 3** can be extended to graphs such that each pair of vertices is incident with a bounded number of faces.
- 3 By using logical tools from Courcelle and Vanicat-  **$O(\log(n))$ -labeling scheme for MSO queries in trees**- we can construct an  $O(\log(n))$ -labeling scheme for  $Conn(u, v, X)$  on planar graphs.
- 4 Can we do the same for bounded-genus graphs ?
- 5 What about solving  $d(u, v, X)$  (distance with failures) in planar graphs ?