Recognizing Bipartite Incident-Graphs of Circulant Digraphs

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Abstract. Knödel graphs and Fibonacci graphs are two classes of bipartite incident-graph of circulant digraphs. Both graphs have been extensively studied for the purpose of fast communications in networks, and they have deserved a lot of attention in this context. In this paper, we show that there exists an $O(n \log^5 n)$-time algorithm to recognize Knödel graphs, and that the same technique applies to Fibonacci graphs. The algorithm is based on a characterization of the cycles of length six in these graphs (bipartite incident-graphs of circulant digraphs always have cycles of length six). A consequence of our result is that none of the Knödel graphs are edge-transitive, apart those of $2^k - 2$ vertices. An open problem that arises in this field is to derive a polynomial-time algorithm for any infinite family of bipartite incident-graphs of circulant digraphs indexed by their number of vertices.

Keywords: graph isomorphism, circulant graphs, chordal rings, broadcasting, gossiping.

1 Introduction

So-called Knödel graphs and Fibonacci graphs have been used by Knödel [12], and Even and Monien [7] (see also [5,13,18]), respectively, for the purpose of performing efficient communications in networks. More precisely, consider a network of n nodes, and assume that communications among the nodes proceed by a sequence of synchronous calls between neighboring vertices. A *round* is defined as the set of calls performed at the same time. Knödel on one hand, and Even and Monien on the other hand, were interested in computing the minimum number of rounds necessary to perform a all-to-all broadcasting, also called *gossiping*, between the nodes (see [8,9,11] for surveys on gossiping and related problems). The communication constraints assume that a call involves exactly two neighboring nodes, and that a node can communicate to at most one neighbor at a

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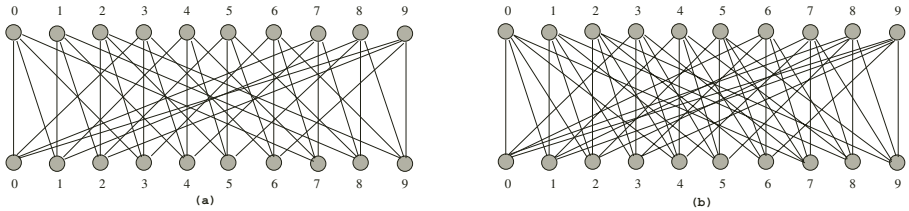

Fig. 1. A Knödel graph of 20 vertices (a), and a Fibonacci graph of 20 vertices (b).

time. Knödel considered the 2-way mode (full-duplex) in which the two nodes involved in the same call can exchange their information in one round, whereas Even and Monien considered the 1-way mode (half-duplex) in which the information can flow in one direction at a time, that is, during the call between x and y, only y can receive information from x, or x can receive information from y, not both. Under these hypotheses, it was shown in [12] that, for n even, one cannot perform gossiping in less that $\lceil \log_2 n \rceil$ rounds in the 2-way mode, and that there are graphs, called here Knödel graphs, that allow gossiping to be performed in $\lceil \log_2 n \rceil$ rounds. Even and Monien have shown in [7] that, for n even, one cannot perform gossiping in less than $2 + \lceil \log_\varrho \frac{n}{2} \rceil$ rounds in the 1-way mode, where $\varrho = \frac{1+\sqrt{5}}{2}$, and that there are graphs, called here Fibonacci graphs, that allow gossiping to be performed in that number of rounds (up to an additive factor of one).

Knödel graphs and Fibonacci graphs are bipartite graphs $G = (V_1, V_2, E)$ of $2n$ vertices. Each partition has n vertices labeled from 0 to $n-1$. In the Knödel graphs, there is an edge between $x \in V_1$ and $y \in V_2$ if and only if there exists $i \in \{0, 1, \ldots, k\}$ such that $y = x + 2^i - 1 \pmod n$, $k = \lfloor \log_2 n \rfloor$. In the Fibonacci graphs, there is an edge between $x \in V_1$ and $y \in V_2$ if and only if there exists $i \in \{0, 1, \ldots, k\}$ such that $y = x + F(i+1) - 1 \pmod n$, $k = F^{-1}(n) - 1$, where $F(i)$ denotes the ith Fibonacci number ($F(0) = F(1) = 1$, and $F(i) = F(i-1) + F(i-2)$ for $i \geq 2$) and $F^{-1}(n)$ denotes the integer i for which $F(i) \leq n < F(i+1)$. Both graphs are Cayley graphs on the dihedral group, and thus they are vertex-transitive. See Figure 1 for an example of a Knödel graph and a Fibonacci graph. Note that graphs on Figure 1 look pretty dense but Knödel graphs and Fibonacci graphs are of degree $O(\log n)$.

Knowing whether a graph G of n nodes allows gossiping to be performed optimally, that is in $\lceil \log_2 n \rceil$ rounds in the 2-way mode, and in (about) $\lceil \log_\varrho n \rceil$ rounds in the 1-way mode, is NP-complete [5]. In particular there are graphs that are not isomorphic to Knödel graphs (resp. Fibonacci graphs), and that allow gossiping to be performed optimally in the 2-way mode (resp. 1-way mode). In this paper, we want to recognize Knödel graphs and Fibonacci graphs. In other words, given a graph G, we want to know whether G is isomorphic to a Knödel

graph of the same order, or to know whether G is isomorphic to a Fibonacci graph of the same order.

A closely related topic deals with circulant digraphs. Recall that a digraph is circulant if nodes can be labeled so that the adjacency matrix is circulant, that is node x has $k+1$ out-neighbors $x + a_i \pmod n$ for $i = 0, \ldots, k$ for some k and some constants a_i's independent of x. Circulant digraphs are Cayley digraphs over $\mathbb{Z}_n$. Ponomarenko has given in [17] a polynomial-time algorithm to decide whether a given tournament is a circulant digraph (a tournament is a digraph obtained by giving an orientation to the edges of a complete graph). More recently, Muzychuk and Tinhofer [15] have shown that one can decide in polynomial-time whether a digraph of prime order is circulant. Deciding whether two circulant digraphs are isomorphic is also a difficult problem. Ádám [1] conjectured that two circulant digraphs are isomorphic if and only if the generators of one digraph can be obtained from the generators of the other digraph via a product by a constant. This conjecture is wrong [6] although it holds in many cases. For instance, Alspach and Parsons [2] have proved that Ádám's conjecture is true for values of n such as the product of two primes (see also [3,14,16]).

Nevertheless, even if they are closely related to circulant digraphs, Knödel graphs and Fibonacci graphs are not circulant graphs but bipartite incident-graphs of circulant digraphs, and they are thus sometimes called bi-circulant graphs. (The bipartite incident-graph of a digraph $H = (V, A)$ is a graph $G = (V_1, V_2, E)$ such that $V_1 = V_2 = V$, and for any $x_1 \in V_1$, and $x_2 \in V_2$, $\{x_1, x_2\} \in E \Leftrightarrow (x_1, x_2) \in A$. Note that two non isomorphic digraphs H and H' can yield isomorphic bipartite incident-graphs, e.g., $H = (\{u, v\}, \{(u, u), (v, v)\})$ and $H' = (\{u, v\}, \{(u, v), (v, u)\})$.) It is unknown whether there exists a polynomial-time algorithm to decide whether a given graph is isomorphic to a given circulant digraph or a given incident-graph of a circulant digraph. This is why we have studied specific algorithms for the case of Knödel graphs and Fibonacci graphs.

The paper is organized as follows. In Section 2, we study the form of solutions of equations involving powers of two. The characterization of these solutions allows us to recognize Knödel graphs in $O(n \log^5 n)$ time, as shown in Section 3. This algorithm is optimal up to a polylogarithmic factor since Knödel graphs have $\Theta(n \log n)$ edges. Section 4 concludes the paper with some remarks on bipartite incident-graphs of circulant digraphs defined by an arbitrary increasing sequence $(g_i)_{i \geq 0}$ of integers, including Fibonacci graphs.

2 Preliminary Results

Let $H = (V, A)$ be a circulant digraph of n vertices and of generators $g_0, \ldots, g_k$, and let $G = (V_1, V_2, E)$ be the corresponding bipartite incident-graph. By 6-*cycle*, we mean an elementary cycle of length six.

Lemma 1. *There is a* 6*-cycle in* G *if and only if one can find a sequence of six generators*

$$(g_{i_0}, g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}, g_{i_5})$$

and $\alpha \in \{0,1,2\}$ *such that*

$$\begin{cases} g_{i_0} + g_{i_2} + g_{i_4} = g_{i_1} + g_{i_3} + g_{i_5} + \alpha\, n \\ g_{i_j} \neq g_{i_{j+1 \pmod 6}} \text{ for any } j \in \{0,1,2,3,4,5\} \end{cases} \tag{1}$$

Proof. Let $(u_0, u_1, u_2, u_3, u_4, u_5)$ be a 6-cycle in G (all the u_i's are pairwise distinct), and assume without loss of generality that $u_0 = 0 \in V_1$. We have:

$$\begin{cases} u_1 = u_0 + g_{i_0} \\ u_1 = u_2 + g_{i_1} + \alpha_1\, n \\ u_3 = u_2 + g_{i_2} + \alpha_2\, n \\ u_3 = u_4 + g_{i_3} + \alpha_3\, n \\ u_5 = u_4 + g_{i_4} + \alpha_4\, n \\ u_5 = u_0 + g_{i_5} \end{cases}$$

where $\alpha_i \in \{-1, 0\}$ and $g_{i_j} \neq g_{i_{j+1 \pmod 6}}$ for any $j \in \{0,1,2,3,4,5\}$ since the cycle uses six different edges. Therefore we have

$$g_{i_0} + g_{i_2} + g_{i_4} = g_{i_1} + g_{i_3} + g_{i_5} + \alpha\, n$$

where $\alpha = (\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4)$, that is $-2 \leq \alpha \leq 2$ by definition of the α_i's. By possibly swapping even and odd g's, we get the claimed result with $0 \leq \alpha \leq 2$.

Conversely, let $(g_{i_0}, g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}, g_{i_5})$ be such that

$$\begin{cases} g_{i_0} + g_{i_2} + g_{i_4} = g_{i_1} + g_{i_3} + g_{i_5} + \alpha\, n \\ g_{i_j} \neq g_{i_{j+1 \pmod 6}} \text{ for any } j \in \{0,1,2,3,4,5\} \end{cases}$$

with $\alpha \in \{0,1,2\}$. Then let $u_0 \in V_1$, and let

$$\begin{cases} u_1 = u_0 + g_{i_0} \bmod n \\ u_2 = u_1 - g_{i_1} \bmod n \\ u_3 = u_2 + g_{i_2} \bmod n \\ u_4 = u_3 - g_{i_3} \bmod n \\ u_5 = u_4 + g_{i_4} \bmod n \\ u_6 = u_5 - g_{i_5} \bmod n \end{cases}$$

We have

$$u_6 = u_0 + (g_{i_0} + g_{i_2} + g_{i_4}) - (g_{i_1} + g_{i_3} + g_{i_5}) \bmod n.$$

Since

$$g_{i_0} + g_{i_2} + g_{i_4} = g_{i_1} + g_{i_3} + g_{i_5} \pmod n,$$

we get $u_6 = u_0$, and therefore $(u_0, u_1, u_2, u_3, u_4, u_5)$ is a cycle of length six in G. This cycle is elementary because $g_{i_j} \neq g_{i_{j+1 \pmod 6}}$ for any $j \in \{0,1,2,3,4,5\}$. □

From Lemma 1, any bipartite incident-graph of a circulant digraph has 6-cycles since $g_{i_0} = g_{i_3}$, $g_{i_1} = g_{i_4}$, and $g_{i_2} = g_{i_5}$ is a solution of Equation 1. We will solve Equation 1 to characterize 6-cycles of Knödel graphs and Fibonacci

graphs, and to identify the possible generators of a candidate to be a Knödel graph or a Fibonacci graph.

Let us start first with Knödel graphs. Let $x_0, x_1, x_2, x_3, x_4, x_5$ be six integers in $\{0, \ldots, k\}$, and let n be any integer such that $2^k \leq n < 2^{k+1}$. From Lemma 1, we are interested in computing the solutions of the equation

$$\begin{cases} 2^{x_0} + 2^{x_2} + 2^{x_4} = 2^{x_1} + 2^{x_3} + 2^{x_5} + \alpha\, n \\ x_i \neq x_{i+1} \pmod 6 \text{ for any } i \in \{0,1,2,3,4,5\} \end{cases} \tag{2}$$

where $\alpha \in \{0, 1, 2\}$.

Lemma 2. *For $\alpha = 0$, Equation 2 has four types of solutions:* $(x_0, x_1, x_2, x_3, x_4, x_5) =$

a) $(\gamma, \gamma'', \gamma', \gamma, \gamma'', \gamma')$ $\quad \gamma, \gamma', \gamma'' \in \{0, \ldots, k\}$ $\gamma \neq \gamma', \gamma' \neq \gamma'', \gamma'' \neq \gamma$
b) $(\gamma, \gamma', \gamma, \gamma', \gamma' + 1, \gamma + 1)$ $\quad \gamma, \gamma' \in \{0, \ldots, k-1\}$ $\gamma \neq \gamma'$
c) $(\gamma, \gamma + 1, \gamma, \gamma', \gamma' + 1, \gamma')$ $\quad \gamma, \gamma' \in \{0, \ldots, k-1\}$ $\gamma \neq \gamma'$
d) $(\gamma, \gamma', \gamma, \gamma + 1, \gamma' + 1, \gamma')$ $\quad \gamma, \gamma' \in \{0, \ldots, k-1\}$ $\gamma \neq \gamma'$

up to cyclic permutations[1] *of the x_i's.*

Proof. The case x_0, x_2, x_4 pairwise distinct generates the first type of solutions. Assume $x_0 = x_2 = \gamma$ and $x_4 = \gamma' \neq \gamma$. There is an impossibility to solve Equation 2 if $\gamma = \gamma' - 1$ because it would imply either $x_0 = x_1$ or $x_0 = x_5$. If $\gamma \neq \gamma' - 1$, we get a solution if two x_{2i+1}'s are both equal to $\gamma' - 1$, and the third one is equal to $\gamma + 1$. This generates the three last types of solutions by changing $\gamma' - 1$ into γ'. □

The solutions of Lemma 2 induces cycles of length 6. These 6-cycles have their edges labeled by *dimensions* as illustrated on Figure 2 (the generator $2^i - 1$ induces edges in dimension i). However, cycle (b) and cycle (d) are isomorphic (just travel (b) clockwise and (d) counterclockwise from the black node), that is the second and the fourth types of solutions induces the same labeled cycle. In the following, only cycles (a), (b), and (c) will be considered.

Notation. The number of blocks of consecutive 1's in the binary representation of n will be denoted by $B_1(n)$.

For instance $B_1((1101100111010)_2) = 4$, $B_1((100)_2) = 1$, $B_1((101)_2) = 2$, and $B_1((0)_2) = 0$. Integers of the form

$$n = (\overline{1}\,\underbrace{00\ldots00}\,\overline{1}\,\underbrace{00\ldots00}\,\overline{0}\overbrace{11\ldots11}\,\underline{0}\overbrace{11\ldots11}\,\underline{0}\overbrace{11\ldots11}\,\underline{1}\underbrace{00\ldots00})_2$$

satisfy $B_1(n) = 5$, and there is a solution to Equation 2 for $\alpha = 1$ with x_1, x_3, x_5 equal to the underlined bit-positions, and x_0, x_2, x_4 equal to the over-lined bit-positions. We have the following lemma:

[1] A permutation σ of p symbols is a cyclic permutation if $\sigma(x_1, x_2, \ldots, x_p) = (x_2, \ldots, x_p, x_1)$.

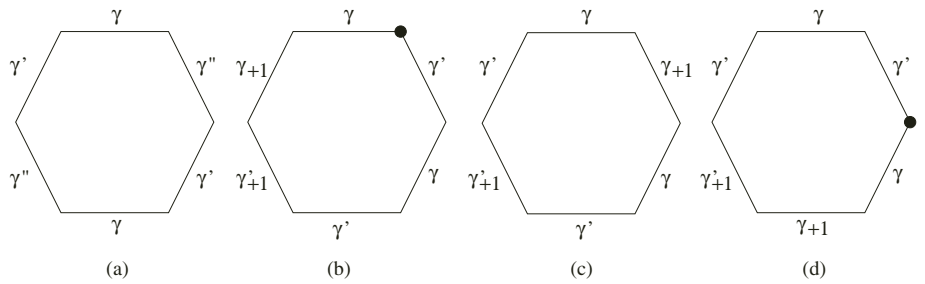

Fig. 2. The four types of solutions of Equation 2 for $\alpha = 0$.

Lemma 3. *If $B_1(n) \geq 6$, then Equation 2 has no solution for $\alpha \neq 0$.*

Proof. If $B_1(n) \geq 6$, then the sum of n and three powers of two cannot result in the sum of three powers of two. Indeed, the binary representation of $2^{x_0}+2^{x_2}+2^{x_4}$ has at most three 1-entries, that is $B_1(2^{x_0}+2^{x_2}+2^{x_4}) \leq 3$. On the other hand, for every n, and every y, $B_1(n+2^y) \geq B_1(n)-1$, and thus $B_1(n+2^{x_1}+2^{x_3}+2^{x_5}) \geq B_1(n) - 3$. Moreover, for n such that $B_1(n) \geq 6$, the binary representation of $n+2^{x_1}+2^{x_3}+2^{x_5}$ has at least four 1-entries. This completes the proof for $\alpha = 1$. The result holds for $\alpha = 2$ too because $B_1(2n) = B_1(n)$. □

Lemma 3 has an important consequence that is, for most of the integers n, Knödel graphs of order $2n$ have 6-cycles only of the form given on Figure 2. There are orders however for which Equation 2 has solutions for $\alpha \neq 0$. This is typically the case for $n = 2^k$. Actually, this special case deserves a particular interest motivated by the simplicity of the solution.

Lemma 4. *If $n = 2^k$ then every 4-cycle of the Knödel graph of order $2n$ is a labeled cycle of type*

$$(k, \gamma - 1, \gamma, \gamma - 1) \text{ for } \gamma \in \{1, \ldots, k\}.$$

Proof. By similar arguments as in the proof of Lemma 1, one can check that 4-cycles exist if and only if there exist four generators $g_i = 2^{x_i} - 1$, $0 \leq i \leq 3$, $x_i \neq x_{i+1} \pmod 4$, satisfying $2^{x_0} + 2^{x_2} = 2^{x_1} + 2^{x_3} + \alpha n$ for $\alpha \in \{0, 1\}$. The equation $2^{x_0} + 2^{x_2} = 2^{x_1} + 2^{x_3}$ has no solution for $x_i \neq x_{i+1} \pmod 4$. Therefore, 4-cycles exist only for solutions of $2^{x_0} + 2^{x_2} = 2^{x_1} + 2^{x_3} + 2^k$, that is for $x_0 = k$, and $x_1 = x_3 = x_2 - 1$, $x_2 \in \{1, \ldots, k\}$. Thus the solution is $(x_0, x_1, x_2, x_3) = (k, \gamma - 1, \gamma, \gamma - 1)$, up to a square-cyclic permutation[2] of the x_i's. □

3 Recognizing Knödel Graphs

In order to recognize Knödel graphs, we use the following basic property:

[2] A permutation σ of p symbols is a square-cyclic permutation if $\sigma(x_1, x_2, x_3 \ldots, x_p) = (x_3, \ldots, x_p, x_1, x_2)$.

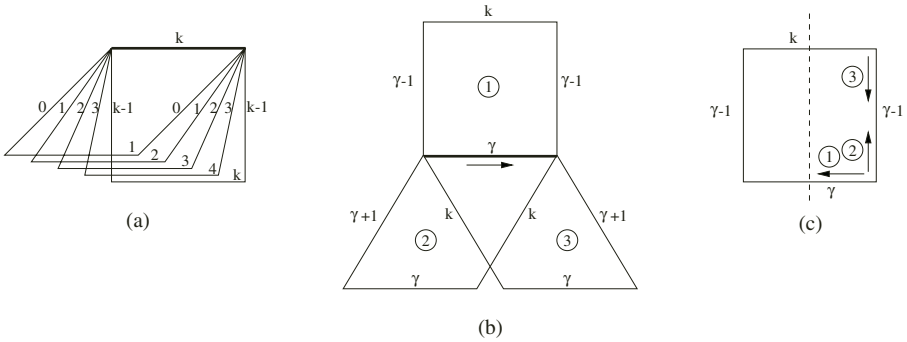

Fig. 3. The 4-cycles in a Knödel graph of order $2 \cdot 2^k$.

Lemma 5. *Let G be a graph whose some edges are colored by 0 or 1. G is a Knödel graph whose edges in dimension 0 and 1 are colored 0 and 1 respectively if and only if:*

1. *A path P using colors 0 and 1, alternatively, from an arbitrary node is Hamiltonian; and*
2. *Assuming the jth node of P is labeled $(j+1 \bmod 2, \lfloor j/2 \rfloor)$, $j \geq 0$, we have, for every i and x, there is an edge connecting $(0,x)$ with $(1, x+2^i-1 \bmod n)$, and no extra edges.*

Proof. Let $G = (V_1, V_2, E)$ be a Knödel graph whose nodes are labeled (i,x), $i = 0,1$ and $x \in \{0,\ldots,n-1\}$ so that $V_1 = \{(0,x), x \in \{0,\ldots,n-1\}\}$, and $V_2 = \{(1,x), x \in \{0,\ldots,n-1\}\}$. Assume that the edges in dimension 0 and 1 of G are colored 0 and 1 respectively. Then the path P is Hamiltonian. Let u be the starting point of P. Since a Knödel graph is vertex-transitive, one can assume w.l.g. that $u = (1,0)$. Therefore, the labeling induced by the path corresponds to the connections of a Knödel graph.

Conversely, let G be a graph whose some edges are colored by 0 or 1. Let P be a path using colors 0 and 1, alternatively, from an arbitrary node of G. Assume P is Hamiltonian, label the vertices according to the rule of the second property, and assume the connection rule fulfills. Then G is a Knödel graph by definition. □

Let us start with $n = 2^k$, $k \geq 2$. From Lemma 4, we know that there is only one type of labeled 4-cycle in a Knödel graph of order $2n$, namely $(k, \gamma-1, \gamma, \gamma-1)$, for $\gamma \in \{1,\ldots,k\}$. Therefore, for any edge of dimension k, there are k 4-cycles using that edge (see Figure 3(a)). For any edge of dimension $k-1$, there are two 4-cycles using that edge (see Figure 3(b) where cycle 2 and cycle 3 are the same). For any edge of dimension γ, $\gamma \in \{1,\ldots,k-2\}$, there are three 4-cycles using that edge (see Figure 3(b)). Finally, for any edge of dimension 0, there are two 4-cycles using that edge (the cycle 1 in Figure 3(b) does not exist for $\gamma = 0$). This counting yields the following corollary of Lemma 4:

Corollary 1. *There exists an $O(n \log^3 n)$-time algorithm to recognize Knödel graphs of order $2n = 2^{k+1}$, $k \geq 1$.*

Proof. Assume $k \geq 4$ (note that for any $k \leq k_0 = O(1)$, recognizing Knödel graphs of order 2^{k+1} can be done in constant time). Given an input graph $G = (V, E)$, we count the number $C_4(e)$ of 4-cycles passing through any edge $e \in E$. From the counting of the number of 4-cycles passing through an edge in a Knödel graph, if there is an edge e such that $C_4(e) \neq 2$, $C_4(e) \neq 3$, and $C_4(e) \neq k$ then G is not a Knödel graph. Otherwise, every edge e with $C_4(e) = 2$ is a candidate to be an edge of dimension 0 or dimension $k-1$, and every edge e with $C_4(e) = k$ (recall $k \neq 2$ and $k \neq 3$) is a candidate to be an edge of dimension k. From Figure 3(b), edges in dimension $\gamma = k-1$ are edges e such that $C_4(e) = 2$ and included in a 4-cycle containing an edge of dimension k which is not adjacent to e in this cycle. This allows to distinguish dimensions 0 and $k-1$. From dimensions 0 and k, one can identify dimension 1 by considering all paths of type $0, k, 0$. The end vertices of each such path are connected by an edge of dimension 1 (see Figure 3(a)). Color the edge of dimension 0 and 1 of G by colors 0 and 1, respectively, and check the conditions of Lemma 5.

This algorithm has a cost of $O(n \log^3 n)$ because, for every edge e, counting the number of 4-cycles using that edge takes at most a time of $O(\log^2 n)$ assuming node-adjacency testing in constant time. (The two end-vertices of every edge are both adjacent to $O(\log n)$ nodes. Thus, by testing all the possible edges between these nodes, one can determine the 4-cycles in $O(\log^2 n)$ time.) Checking the conditions of Lemma 5 takes $O(n \log n)$ time. □

Let us carry on our study by considering integers n such that $B_1(n) \geq 6$. In this case, one can apply Lemmas 2 and 3, and we are dealing with the four types of cycles of Figure 2 (recall that cycles (b) and (d) are isomorphic). Figure 2(a) implies that, for any edge of dimension γ, $\gamma \in \{0, \ldots, k\}$, there are $k(k-1)$ 6-cycles of type (a) using that edge.

The contribution of cycles of type (b) in Figure 2 to each dimension is more difficult to calculate. We proceed as for the 4-cycles studied in the case $n = 2^k$. The counting for $n = 2^k$ can be formalized as follows. Consider again Figure 3. On Figure 3(c), there are four edges (whose one is of a fixed dimension, dimension k), and two possible senses of travel, clockwise and counterclockwise. For any γ such that $1 \leq \gamma \leq k-1$, there are potentially four positions for γ. However, only three of them are valid because $\gamma \neq k$. Moreover, once the position of γ has been fixed, we have only two ways to travel around the cycle. This fact gives six possible labeled cycles for any edge to belong to. However, only three of these 4-cycles are distinct because there is a symmetry along the axis perpendicular to dimension k. This symmetry reduces the number of travels by a factor of 2: for each travel of Figure 3(c), there is a corresponding travel in Figure 3(b) starting in the direction indicated by the arrow.

Coming back to the 6-cycle of Figure 2(b), there are six edges, and two possible directions (clockwise and counterclockwise). Thus we get twelve possibilities to travel along the edges of a labeled 6-cycle. However, we actually get only

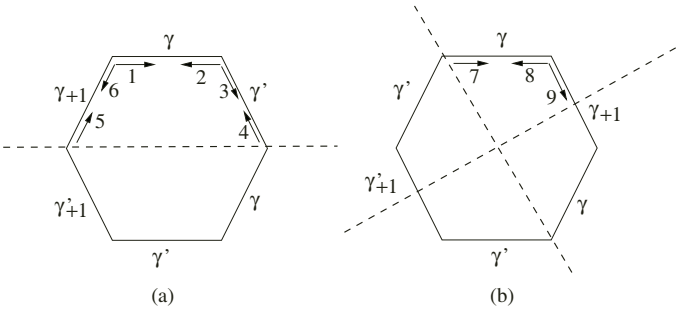

Fig. 4. (a) The six possible travels of the cycle of type (b) in Figure 2. (b) The three possible travels of the cycle of type (c) in Figure 2.

six possibilities for cycle (b) of Figure 2 because of a symmetry along the axis parallel to γ, γ' (see Figure 4(a)).

These six possibilities are:

$$\left.\begin{array}{l} 1.\ (\gamma \quad \gamma' \quad \gamma \quad \gamma' \quad \gamma'+1\ \gamma+1) \\ 2.\ (\gamma\ \gamma+1\ \gamma'+1 \quad \gamma' \quad \gamma \quad \gamma') \\ 3.\ (\gamma \quad \gamma' \quad \gamma \quad \gamma+1\ \gamma'+1 \quad \gamma') \\ 4.\ (\gamma \quad \gamma' \quad \gamma'+1\ \gamma+1 \quad \gamma \quad \gamma') \end{array}\right\} \gamma, \gamma' \in \{0, \ldots, k-1\}, \gamma \neq \gamma';$$

and

$$\left.\begin{array}{l} 5.\ (\gamma\ \gamma-1\ \gamma'\ \gamma-1\ \gamma'\ \gamma'+1) \\ 6.\ (\gamma\ \gamma'+1\ \gamma'\ \gamma-1\ \gamma'\ \ \gamma-1) \end{array}\right\} \gamma \in \{1, \ldots, k\}, \gamma' \in \{0, \ldots, k-1\}, \gamma' \neq \gamma-1.$$

Note that solutions 1 and 3 are the same if $\gamma' = \gamma + 1$, solutions 2 and 4 are the same if $\gamma' = \gamma + 1$, solutions 3 and 4 are the same if $\gamma' = \gamma - 1$, and solutions 5 and 6 are the same if $\gamma' = \gamma - 2$.

We do the same analysis for 6-cycles of type (c) in Figure 2. The counting uses the fact that Figure 2(c) is symmetric along the axis perpendicular to the edges $\gamma+1$ and $\gamma'+1$, and along the axis parallel to $\gamma+1, \gamma'+1$ (see Figure 4(b)). Therefore, we get three new possibilities:

$$\left.\begin{array}{l} 7.\ (\gamma\ \gamma+1 \quad \gamma \quad \gamma'\ \gamma'+1 \quad \gamma') \\ 8.\ (\gamma \quad \gamma' \quad \gamma'+1\ \gamma' \quad \gamma \quad \gamma+1) \end{array}\right\} \gamma, \gamma' \in \{0, \ldots, k-1\}, \gamma \neq \gamma';\ \text{and}$$

$$9.\ (\gamma\ \gamma-1\ \gamma'\ \gamma'+1\ \gamma'\ \gamma-1), \gamma \in \{1, \ldots, k\}, \gamma' \in \{0, \ldots, k-1\}, \gamma' \neq \gamma-1.$$

Note that solutions 1 and 7 are the same if $\gamma' = \gamma + 1$, solutions 2 and 7 are the same if $\gamma' = \gamma - 1$, and solutions 3 and 7 are the same if $\gamma' = \gamma + 1$. Similarly solutions 1 and 8 are the same if $\gamma' = \gamma - 1$, solutions 2 and 8 are the same if $\gamma' = \gamma + 1$, and solutions 4 and 8 are the same if $\gamma' = \gamma + 1$. Finally, solutions 5 and 9 are the same if $\gamma' = \gamma - 2$, and solutions 6 and 9 are the same if $\gamma' = \gamma - 2$.

Now we can count the number of 6-cycles using a given edge of a Knödel graph.

Lemma 6. *Assume $B_1(n) \geq 6$, and let e be an edge of a Knödel graph of order $2n$ with $k \geq 4$. Let $C_6(e)$ denote the number of 6-cycles using edge e. We have:*

$$C_6(n) = \begin{cases} k^2 + 5k - 10 \text{ if } e \text{ is of dimension } 0 \\ k^2 + 8k - 16 \text{ if } e \text{ is of dimension } 1 \\ k^2 + 8k - 18 \text{ if } e \text{ is of dimension } \gamma,\ 2 \leq \gamma \leq k-2 \\ k^2 + 8k - 14 \text{ if } e \text{ is of dimension } k-1 \\ k^2 + 2k - 5 \quad \text{if } e \text{ is of dimension } k \end{cases}$$

Proof. Let e be an edge of dimension γ, $0 \leq \gamma \leq k$. Let us count the contribution to that edge of the nine travels identified in Figure 4. We get $6k-10$ cycles for dimension 0, $9k-16$ cycles for dimension 1, $9k-18$ cycles for dimension γ, $2 \leq \gamma \leq k-2$, $9k-14$ cycles for dimension $k-1$, and $3k-5$ cycles for dimension k. Then add $k(k-1)$ cycles from the solutions of type (a) in Figure 2 to get the claimed result. □

Corollary 2. *There exists an $O(n \log^5 n)$-time algorithm to recognize Knödel graphs of order $2n$, for all n such that $B_1(n) \geq 6$.*

Proof. Assume $k \geq 4$. From Lemma 6, one can identify edges of dimensions 0 and 1 in $O(n \log^5 n)$-time. In time $O(n \log n)$, one can check the conditions of Lemma 5. □

Theorem 1. *There exists an $O(n \log^5 n)$-time algorithm to recognize Knödel graphs of any order.*

Sketch of the proof. (The complete proof is given in the full paper [4].) From Corollaries 1 and 2, the theorem holds for n power of two, or n such that $B_1(n) \geq 6$. Thus, assume that n satisfies $B_1(n) < 6$, $n \neq 2^k$. Assume moreover that $n \neq 2^{k+1} - 1$ (this latter case deserves to be treated separately because the Knödel graph with $n = 2^{k+1} - 1$ is edge-transitive [10]). The key argument is that, for almost all such n, if $C_6(e)$ denotes the number of 6-cycles passing through an edge e of a Knödel graph of order n, then $C_6(e) \neq C_6(e')$ for any e and e' of dimensions 0 and k respectively, and $C_6(e) \neq C_6(e')$ for any e dimension 0 or k, and any e' of dimension i, $i \neq 0$ and $i \neq k$.

The difficulty of the proof comes from the fact that, if $B_1(n) < 6$, then Equation 2 has solutions for $\alpha \neq 0$, and proceeding to a precise counting of the number of 6-cycles passing through the edges of a Knödel graph is tricky. Anyway, we were able to prove that dimension 0 and dimension k can be identified by such counting. From the knowledge of the edges of dimension 0 and k, we have shown that one can determine the set of edges of dimension 1. Then it remains only to check the conditions of Lemma 5.

To summarize, the algorithm is the following:

Algorithm *Recognize*.
`Input:` a regular graph $G = (V, E)$ of $2n$ vertices, and degree $k = \lfloor \log_2 n \rfloor$;
`Output:` tell whether or not G is isomorphic to the Knödel graph K of order $2n$.

Phase 1. For every $i \in \{0, \ldots, k\}$, compute p_i = number of 6-cycles passing through any edge of dimension i of K;
Phase 2. For every $e \in E$, compute $C_6(e)$ = number of 6-cycles passing through the edge e of G;
Phase 3. Identify the two sets $S_0 \subset E$, and $S_1 \subset E$, of edges of G that are possibly edges of dimension 0 and 1 in K, respectively;
Phase 4. Check the conditions of Lemma 5.

The first phase has a cost of $O(log^5 n)$, assuming node-adjacency testing in constant time. The second phase has a cost at most $O(n \log^5 n)$ because the degree of every vertex is $O(\log n)$. The third and the fourth phase cost $O(n \log n)$ time.

To prove the correctness of Algorithm *Recognize*, the difficult part is to prove that Phase 3 is doable. This is formaly proved in the complete version of the proof (see [4]). The case of Knödel graphs of order $2^{k+2} - 2$ deserves a particular attention due to the edge-transitivity of such graphs. We have proved that one can identify dimensions 0 and k by counting the number of 6-cycles traversing a path of length 3. □

Corollary 3. *If $n \neq 2^{k+1} - 1$, then Knödel graphs of order $2n$ are non edge-transitive. (From [10], Knödel graphs of order $2^{k+2} - 2$ are edge-transitive.)*

Proof. The proof of correctness of Algorithm *Recognize*, given in [4], uses the fact that, for $n \neq 2^{k+1} - 1$, the number of 6-cycles using edges of different dimensions is not the same. □

4 Conclusion and Further Research

In this paper, we have shown that Knödel graphs can be recognized in $O(n \log^5 n)$ time. The same result holds for Fibonacci graphs (see [4]). The natural question arising in this field is to ask for which sequences g_i the same result holds. Let us formalize this question. Given an infinite and increasing sequence of integers $\Gamma = (g_i)_{i \geq 0}$, consider the sequence $(G_n^\Gamma)_{n \geq 0}$ of circulant digraphs of order n such that, for any $n \geq 0$, G_n^Γ has generators $g_0, g_1, \ldots, g_k$ where k is the largest integer such that $g_k \leq n - 1$ (in other words, $g_k \leq n - 1 < g_{k+1}$). Then let $(H_n^\Gamma)_{n \geq 0}$ be the corresponding sequence of bipartite incident-graphs of order $2n$. The problem is the following:

Problem 1.
INSTANCE: An integer n, and a graph G of $2n$ vertices;
QUESTION: Is G isomorphic to H_n^Γ?

Note that the sequence Γ is fixed in Problem 1, and thus that this problem is different from the problem of deciding whether a graph is isomorphic to the bipartite incident-graph of a circulant digraph, or to decide whether two bipartite incident-graphs of circulant digraphs are isomorphic. These two latter problems

are known to be difficult, even if they might be simpler than the problem of deciding whether a graph is a Cayley graph.

We have seen that Problem 1 can be solved polynomially if $\Gamma = (2^i - 1)_{i \geq 0}$ or if $\Gamma = (F(i+1)-1)_{i \geq 0}$. The question is: for which family the techniques used to solve the problem for Knödel graphs and Fibonacci graphs can be extended, and how? Actually, as soon as we know how to solve Equation 1, then we are able to enumerate the 6-cycles, and to use the same techniques as the techniques used for Knödel graphs and Fibonacci graphs. We let as an open problem the characterization of the sequences Γ for which Problem 1 can be solved using the same techniques as for Knödel graphs and Fibonacci graphs.

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