

# Smaller Induced-Universal Graph for Planar Graphs

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## EXTENDED ABSTRACT

A graph  $\mathcal{U}$  is an induced-universal graph for a given family of graphs  $\mathcal{F}$  if every graph of  $\mathcal{F}$  is isomorphic to an induced-subgraph of  $\mathcal{U}$ . A challenging question is to construct induced-universal graphs of smallest size, i.e., with the minimum number of vertices. We construct an induced-universal graph of size  $n^{4/3+o(1)}$  for the family of all  $n$ -vertex planar graphs. This is the first sub-quadratic bound for planar graphs. Our construction generalizes to the family of graphs of Euler genus at most  $g$ . The size of the universal graph is the same, up to a lower-order term depending on  $g$ .

Our construction is based on a shorter implicit representation of planar graphs. We show how to assign to each label of any planar graph a label of  $\frac{4}{3} \log n + O(\log \log n)$  bits such that adjacency between any two vertices can be decided by a fixed procedure that examines only their labels.

## 1 Introduction

In standard graph representations, such as with adjacency lists and matrices, vertex identifiers do not play any particular role with respect to the structure of the graph: they are essentially just pointers in the data structure. On the contrary, *implicit representation* associates with each vertex of the graph more information so that adjacency, for instance, can be efficiently determined from these pieces of information, without the need of any global data-structure. For instance, if  $G$  is an interval graph with  $n$  vertices, one can associate with each vertex  $u$  some interval  $I(u) \subseteq [1, 2n]$  with integer endpoints so that  $u, v$  are adjacent if and only if  $I(u) \cap I(v) \neq \emptyset$ . Clearly, no adjacency lists or matrices are required anymore. Although  $G$  may have a quadratic number of edges, such an implicit representation uses  $2 \log n + O(1)$  bits per vertex<sup>1</sup>, regardless of its degree. Compact representations have several advantages, not only for the memory storage, but also from algorithmic perspectives. For instance, given a succinct representation, BFS traversal can be done in  $O(n)$  time for interval graphs, even if they may have  $\Omega(n^2)$  edges. Speedups due to succinct representations are ubiquitous in the design of algorithms and data structures.

Formally introduced by Peleg [20], *informative labeling schemes* present a way to formalize implicit representations of graphs. For a given function  $\Pi$  defined on pairs of vertices of a graph from some given family of graphs, an informative labeling scheme has two components: an *encoding algorithm* that associates with each vertex a piece of information (label); and a *decoding algorithm* that computes  $\Pi(u, v, G)$ , the value of  $\Pi$  applied on vertices  $u, v$  of the graph  $G$ . The input of the decoding algorithm consists solely of the labels of  $u$  and of  $v$ , with no other information provided. So, finding an implicit representation of a graph  $G$  can be restated as computing an *adjacency labeling scheme* for  $G$ , that is, an informative labeling scheme where  $\Pi(u, v, G)$  is TRUE if and only if  $u, v$  are adjacent in  $G$ .

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This work is a part of project TOTAL (Michał Pilipczuk) that have received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 677651). It is also partially funded by the French ANR projects ANR-16-CE40-0023 (DESCARTES) and ANR-17-CE40-0015 (DISTANCIA).

<sup>1</sup>Throughout this extended abstract, we denote by  $\log n$  the binary logarithm of  $n$ .



## 2 Planar Graphs

Planar graphs are perhaps the most studied family of graphs in this area, due to the wide variety of their implicit representations. To mention just a few, planar graphs are contact graphs of circles [19], of 3D boxes [23], of triangles [11], and more recently, of L-shapes [17]. They also have 1-string representations [9]. Each of these representations leads to a labelling scheme where each vertex can be encoded using a label consisting of  $O(\log n)$  bits, independent of its degree.

The first explicit bound on the label length, given by Kannan et al. [18], was  $4 \lceil \log n \rceil$  bits. The representation in terms of dimension-3 posets due to Schnyder [22] actually implies a  $3 \lceil \log n \rceil$  bit labeling, and similar bounds can be derived from polynomial sized universal graphs (cf. related work below). By significantly improving the labeling scheme for bounded treewidth graphs, namely from  $O(k \log n)$  [18] to  $\log n + O(k \log \log n)$ , Gavoille and Labourel [16] showed that partitioning the edges of a planar graph into two bounded treewidth subgraphs, rather than into three forests, leads to a shorter representation: with labels consisting of  $2 \log n + O(\log \log n)$  bits. This is the currently best upper bound for planar graphs, see also Table 2 for the best results for sub-families of planar graphs.

Graph families (with $n$ vertices)	Upper bound (label length in bits)	References
maximum degree-2	$\log n + O(1)$	[8, 14, 1]
caterpillars	$\log n + O(1)$	[7]
bounded degree trees	$\log n + O(1)$	[10]
bounded depth trees	$\log n + O(1)$	[15]
trees	$\log n + O(1)$	[4]
bounded degree outerplanar	$\log n + O(1)$	[10, 2]
outerplanar	$\log n + o(\log n)$	[16]
bounded treewidth planar	$\log n + o(\log n)$	[16]
maximum degree-4 planar	$\frac{3}{2} \log n + o(\log n)$	[2]
bounded degree planar	$2 \log n + O(1)$	[10]
planar	$2 \log n + o(\log n)$	[16]
diameter- $d$ planar	$\log n + \log d + o(\log n)$	<b>[new]</b>
planar	$\frac{4}{3} \log n + o(\log n)$	<b>[new]</b>

Table 1: State-of-the-art for adjacency labeling schemes on planar graphs and some sub-families. All the second-order terms quoted above as  $o(\log n)$  are in  $O(\log \log n)$ . The bounds from references [10, 14, 8] come from induced-universal graphs, whereas all the others come from labeling schemes. The only known lower bound for planar graphs is  $\log n + \Omega(1)$ .

## 3 Our Contribution

As shown in Table 2, in this work we present a new labeling scheme for planar graphs that uses labels of length bounded<sup>2</sup> by  $\frac{4}{3} \log n$ . Note that this not only improves the previously best known bound of  $2 \log n$  for general planar graphs [16], but even the refined bound of  $\frac{3}{2} \log n$  for the case of planar graphs of maximum degree 4 [2]. Our contribution is actually three-fold.

<sup>2</sup>For brevity, in this informal exposition we ignore terms of lower order  $o(\log n)$ .

First, we design a labeling scheme with labels of length  $\log n + \log d$  if the graph has diameter at most  $d$ . This parametrized bound is never worse than the currently best known bound of  $2 \log n$ , because we always have  $d < n$ . Our scheme is based on a recent decomposition theorem, which states that the vertices of a planar graph can be partitioned into geodesics (shortest paths) so that contracting every geodesic to a single vertex turns the graph into a graph of constant treewidth. This was first proved by Pilipczuk and Siebertz in [21], and then refined by Dujmović et al. [13] as follows: the geodesics in the partition can be selected from any fixed BFS forest of the graph. This statement was used in [13] to prove that planar graphs have bounded queue number and in [12] to prove that they have bounded nonrepetitive chromatic number, which resolved two long-standing open questions in graph theory. Thus, in this work we provide another application of the result of Dujmović et al.: a construction of shorter labeling schemes for planar graphs.

The second contribution is the main one: a labeling scheme for planar graphs that uses labels of length  $\frac{4}{3} \log n$ . To achieve this, we combine the scheme for planar graphs of low diameter with the layering technique. Essentially, we compute a decomposition of the graph into strips of depth bounded by some parameter  $d \in \mathbb{N}$ . Strips are separated by borders whose union is a graph on  $O(n/d)$  vertices and of constant treewidth. Using the results of [16], for this border graph we can compute a labeling with labels of length  $\log(n/d)$ . On the other hand, the low-diameter result provides a scheme for the strips with labels of length at most  $\log n + \log d$ . At this point, superposing these two schemes gives no improvement, because vertices appearing at the borders of strips have to inherit labels from both labelings:  $\log n + \log d$  from the labeling of the strips and  $\log n/d$  from the labeling of the border, which sums up to  $2 \log n$ . However, by revisiting the scheme for graphs of bounded treewidth we are able to show that for vertices at the borders of strips, the labeling for strips can use much shorter labels: only of length  $\log(n/d)$  instead of  $\log n + \log d$ . Hence, the combined labels of border vertices are of length at most  $2 \log(n/d)$ , implying that every vertex receives a label of length bounded by

$$\max \{ \log n + \log d, 2 \log(n/d) \}.$$

This expression is minimized for  $d = n^{1/3}$  and then evaluates to  $\frac{4}{3} \log n$ , the desired bound.

The third contribution is a generalization of the previous technique to graphs of bounded Euler genus. Namely, for every fixed  $g \in \mathbb{N}$ , we construct a labelling scheme for graphs of Euler genus at most  $g$  that uses labels of length at most  $\frac{4}{3} \log n$ .

In all our labeling schemes, we can compute the labeling of all the vertices of the graph in polynomial time, while the adjacency can be determined from the labels in constant time.

## 4 Connections with Universal Graphs

It has been observed in [18] that the design of labeling schemes with short labels is tightly connected with the construction of small induced-universal graphs. Recall that a graph  $\mathcal{U}$  is induced-universal for a given set of graphs  $\mathcal{S}$  if every graph  $G \in \mathcal{S}$  is isomorphic to some induced subgraph of  $\mathcal{U}$ . Then graphs from  $\mathcal{S}$  admit a labeling scheme with  $k$ -bit labels if and only if  $\mathcal{S}$  has an induced-universal graph  $\mathcal{U}$  with at most  $2^k$  vertices, see [18]. Thus, our new labeling scheme provides an explicit construction of an induced-universal graph for  $n$ -vertex planar graphs that has  $n^{4/3+o(1)}$  vertices, improving upon the previously best known bound of  $n^{2+o(1)}$ , derived from [16].

Therefore, we proved that the minimum possible number of vertices of an induced-universal graph for  $n$ -vertex planar graphs lies between  $\Omega(n)$  and  $n^{4/3+o(1)}$ . The search for optimum bounds on the sizes of induced-universal graphs is a well-studied topic in graph theory, see for example the recent developments for general  $n$ -vertex graphs [3, 5] and for  $n$ -vertex trees [4]. We refer readers interested in this topic to the recent survey of Alstrup et al. [6].

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